A SUMMARY OF MY RESEARCH

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My area of research is low-dimensional topology and related areas, including geometric group theory. I have summarised below my research.

Some of my earlier papers are motivated by the long-standing question, now part of the geometrisation conjecture, that any finite group action on S^3 is conjugate to a linear action. My more recent work has focused on the topology of smooth 4-manifolds, widely considered (along with symplectic geometry) one of the most important areas in Geometry/Topology for the foreseeable future. Another question on which I have worked, and expect to work for some years to come, is to understand the borderline between *hard* and *soft* geometry.

I also have a few other results, in 3-manifold topology as well as group theory and high-dimensional topology.

1. Cobordisms and Reidemeister torsions of homotopy lens spaces

Geom. Topol. ***5*** *(2001) 109-125.*

This paper is motivated by the contrast between the conjecture that free finite group actions on S^3 are conjugate to linear ones and the fact that in higher dimensions there are non-linear actions.

Free actions of cyclic groups (with odd, prime order) on S^n for $n \geq 5$ are classified by two invariants - the Reidemeister torsion and the multi-signature. Of these the Reidemeister torsion suffices in distinguishing between linear actions (i.e., lens spaces). The Reidemeister torsion is an invariant of the simple-homotopy type, and determines the simple-homotopy type for such manifolds.

Thus, a prediction of the geometrisation conjecture is that 3-dimensional homotopy lens spaces M whose Reidemeister torsion coincides with that of a lens space $L(p,q)$ must also have the same multi-signature as $L(p,q)$. Equivalently, M must be s-cobordant to $L(p,q)$.

This is the main result of this paper.

Theorem 1.1. *Suppose M^3 is a 3-manifold with a simple-homotopy equivalence $f\colon M^3 \to L(p,q)$. Then M^3 is s-cobordant with $L(p,q)$.*

While this does not show that M is a lens space, it may be useful in studying finite group actions on S^3.

The methods used include classical 3-manifold topology, surgery theory and Freedman's theorems on topological 4-manifolds. The interaction between this is perhaps another point of interest.

2. The pq-condition for 3-manifold groups

Proc. Amer. Math. Soc. ***129*** *(2001), no. 6, 1873–1875.*

Date: November 17, 2005.

Like the previous paper, this paper is motivated by the contrast between the conjecture that free finite group actions on S^3 are conjugate to linear ones and the fact that in higher dimensions there are non-linear actions.

There are two conditions satisfied by any group G that acts freely on S^n for some n. These are the result of Zassenhaus and Smith that any abelian subgroup of G is cyclic (the p^2-condition) and the theorem of Milnor that any element of order 2 in G is central (the $2p$-condition). By deep results of Madsen, Thomas and Wall, these are the only conditions if we do not restrict dimensions.

Groups G that act freely orthogonally on S^n satisfy an additional condition - the pq-condition. This says that any subgroup of G of order pq, with p and q primes, is cyclic. For solvable groups a theorem of Wolf says that this, together with the previous conditions suffices to guarantee the existence of a free orthogonal action.

Thus geometrisation predicts that groups that act on S^3 freely must satisfy the pq-condition. I give here a simple topological proof of this.

Theorem 2.1. *Suppose the finite group G is the fundamental group of a 3-manifold M, and p and q are distinct, odd primes. Then every subgroup of G of order pq is cyclic.*

Unfortunately this does not lead to new restrictions on the class of groups that might act freely on S^3. This is because the fact that these groups have cohomology of periodicity 4, together with the result of Milnor and some deep group theory leaves two classes of possible non-orthogonal groups (one of which was eliminated by R. Lee), and both these classes satisfy the pq-condition. Nevertheless it seems worthwhile to have a conceptual proof of this fundamental property.

3. TOPOLOGICAL GEODESICS AND VIRTUAL RIGIDITY

(joint with Louis Funar), Algebr. Geom. Topol. ***1*** *(2001) 369-380.*

In this paper (joint with Louis Funar) we introduce the notion of a topological geodesic in a 3-manifold, and show that in the presence of some conditions on the fundamental group (residual finiteness and word-hyperbolicity), topological geodesics behave like geodesics in a Riemannian manifold with negative sectional curvature. These notions are applied to prove virtual rigidity results using the techniques of Gabai.

A topological geodesic in a 3-dimensional manifold M is a curve whose inverse image in the universal cover consists of unknotted components. We can look at their properties from three different points of view.

3.1. Analogy with Riemannian manifolds.

In a manifold with negative sectional curvature, each homotopy class of curves contains a unique closed geodesic. Topological geodesics are in some sense large-scale analogues of these, and satisfy analogous properties.

Theorem 3.1. *Let M be an irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic (or semi-hyperbolic). Then every conjugacy class in $\pi_1(M)$ is represented by a topological geodesic.*

Theorem 3.2. *Let M be an irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic and residually finite. Suppose c and c' are homotopic topological geodesics in M representing a primitive class in $\pi_1(M)$ (i.e., not a multiple of any other class),* *then there exists a finite cover* M' *of* M *such that* c *and* c' *lift to isotopic curves in* M'.

3.2. **Rigidity results.** Secondly, these can be viewed as the natural context for Gabai's rigidity results. One view of these proofs is that if we are given M and M' homotopy equivalent, then we can introduce boundaries by considering instead knot exteriors $M - int(N(K))$ and $M' - int(N(K'))$ (with the knots chosen so that their exteriors are irreducible). However in general we do not expect these knot exteriors to be homotopy equivalent - in some sense one needs to choose K and K' to be the same, or equivalently, make a *canonical* choice for a knot K.

Thus the key point is to find a canonical curve in a homotopy class in M. Topological geodesics play this role, if one allows passage to finite covers. In Gabai's proof what matters is not directly that K is canonical, but rather that it has a thick tube around it. We have the same technical result for topological geodesics (implicit in Gabai's paper). Not surprisingly, this technical result also plays a key role in showing the uniqueness of topological geodesics. The following result was proved earlier by Gabai.

Theorem 3.3. *Suppose* M *is a closed, irreducible* 3*-manifold with* $\pi_1(M)$ *infinite, residually finite and word-hyperbolic (or semi-hyperbolic). Then if* $f\colon M \to N$ *is a homotopy equivalence, there exist finite covers* M' *and* N' *of* M *and* N *and a lift* $f'\colon M' \to N'$ *of* f *which is homotopic to a homeomorphism.*

We can use the uniqueness of topological geodesics in an interesting way to get the following.

Theorem 3.4. *Let* M *be irreducible and with word-hyperbolic fundamental group. If* $f : M \to M$ *is a homeomorphism homotopic to the identity then there is a finite cover* M' *of* M *and a lift* $f' : M' \to M'$ *of* f *such that* f' *is isotopic to the identity.*

3.3. **Analogy with elliptic manifolds.** A third motivation comes from comparing manifolds with infinite fundamental groups with elliptic manifolds. A free action of a finite cyclic group on S^3 is linear if and only if there is a topological geodesics. Further which curves are represented by topological geodesics, together with homotopy theoretic data, suffices to classify lens spaces.

In analogy with this, the existence of topological geodesics is the first obstruction to hyperbolisation. Further their (virtual) uniqueness is the first obstruction to (virtual) topological rigidity. More significantly, these are the only known obstructions (in the first case, in fact the only obstruction).

The content of this paper may be viewed as saying that both these obstructions vanish in the case of infinite fundamental group. Roughly speaking, this says that topological rigidity cannot fail for the same reason as in the case of lens spaces, and the obstruction to hyperbolisation, if any, is independent to the obstruction to the spherical space form problem.

4. On the Geometric simple-connectivity of open manifolds
(joint with Louis Funar)

Int. Math. Res. Not. 2004, no. 24, 1193–1248.

Here we study the question of when open manifolds have a handle-decomposition without 1-handles.

In the case of compact manifolds whose dimension is at least 5, the only obstruction to the existence of such a handle-decomposition is the fundamental groups. For 3-manifolds there is a further elementary obstruction, namely simple-connectivity at infinity. This is however not an obstruction in higher dimensions - there are manifolds with handles decompositions without 1-handles that are not simply connected at infinity.

For an open manifold M, we define a series of obstructions to the existence of a handle-decomposition without 1-handles. These are defined in terms of fundamental groups corresponding to an exhaustion of M by compact sub-manifolds. The obstructions are somewhat analogous to the lower central series.

Perhaps our most interesting result is that if W is the Whitehead manifold and M a closed manifold, then $W \times M$ does not have a handle-decomposition without 1-handles.

Theorem 4.1. *If W^3 denotes the Whitehead manifold then $W^3 \times N^k$ is not ∞-compressible for any closed simply connected k-manifold N^k.*

A self-contained exposition of the construction of the obstruction and the above result is in the shorter note *One handles for open manifolds*.

We also study the 4-dimensional case. Here an argument of Casson (extended by us) shows that most compact contractible 4-manifolds do not have a handle-decomposition without 1-handles. We study the question of when the interior of such a manifold has a handle-decomposition without 1-handles.

We show that such a handle-decomposition must have a canonical form which can be specified in terms of surgeries along maximal families of disjoint curves along surfaces representing the homology of the level sets (which is a 3-manifold). If these surfaces can be chosen to be disjoint, then we show that Casson's argument extends. In general we show that while all Massey products do vanish, at least in the context of 2-complexes this does not imply the presence of disjoint surfaces.

5. EQUIVARIANT FRAMINGS, LENS SPACES AND CONTACT STRUCTURES

*Pacific Journal of Mathematics **1** (2003), 73–84*

In this paper I introduce an invariant of certain oriented 3-manifolds, namely the quotients M of S^3 by finite groups, based on the fact that the tangent bundle of any orientable 3-manifold is trivialisable. Namely, the homotopy classes of trivialisations of the tangent bundle of S^3 correspond to $\mathbb{Z}$. A trivialisation of TM pulls back to one of S^3. The homotopy class of this is well defined modulo the order of $\pi_1(M)$ if $H_1(M, \mathbb{Z}/2\mathbb{Z}) = 0$.

A first application of this is to distinguish lens spaces. In the case of lens spaces with fundamental group of odd prime order, we get a proof of their classification as oriented manifolds, a bit more than from say the Reidemeister torsion (though geometric methods go give this result).

Theorem 5.1. *Suppose p is odd. Then $\mathfrak{F}(L(p,q)) = \frac{(q-1)(q^{-1}-1)}{4}$, where q^{-1} is a multiplicative inverse of q modulo p, and q and q^{-1} have been chosen to be odd representatives of their mod p equivalence class.*

Corollary 5.2. *Suppose p is a prime, then $L(p,q) = L(p,q')$ as oriented manifolds if and only if $q' = q^{\pm 1}$*

A second application is to contact geometry. A contact structure on $M = S^3/G$ with Euler class zero, in particular a contact structure on a homology sphere of this form, has canonically associated to it a trivialisation of TM. This behaves well with respect to pullbacks, and so we can use Eliashberg's classification of tight contact structures on S^3 to deduce non-existence results for universally tight contact structures.

Theorem 5.3. *The Poincaré homology sphere with one of its orientations does not admit a universally tight positive contact structure.*

Etnyre and Honda have shown that this manifold does not admit a tight contact structure, but their methods use the hypothesis much more strongly. Gompf had earlier derived the above result using methods that can be seen to be similar to the ones I use (Gompf looks at tangent plane fields rather than framings, and uses an almost complex structure on a bounding 4-manifold to define an analogous invariant).

What I believe to be the real significance of this is that it relates to the exceptional isomorphism $SO(4) = SU(2) \times SU(2)/\pm 1$. This plays a central role in elliptic 3-manifolds, and hence an invariant defined for all 3-manifolds with finite fundamental group that captures some features of the exceptional isomorphism can be very useful. The applications to contact geometry can be interpreted in this sense.

These methods can also be used to show that certain other elliptic 3-manifolds do not admit universally tight contact structures.

Remark 5.4. The equivariant framing can be generalised to an integer valued invariant associated to any finite cover. One can see that this is non-trivial. As hyperbolic manifolds are residually finite, there are plenty of invariants associated to a hyperbolic manifold. I do not know of any uses in this context.

6. Contact structures on elliptic 3-manifolds

Proc. Amer. Math. Soc. 132 (2004), no. 12, 3705–3714.

This is an extension of my paper *equivariant framings, lens spaces and contact structures*. Here I study in greater depth the existence of universally tight contact structures.

Since the work of Thurston, geometric structures on 3-manifolds have played a central role in understanding their topology. On the other hand, much of our understanding 3-dimensional manifolds has been based on co-dimension one structures – surfaces, foliations and laminations – in these manifolds. This paper explores the relation between the two kinds of structures in a particular case.

Without additional conditions co-dimension-one structures always exist, and are of not much consequence. However, the presence of *essential* co-dimension one structures – *incompressible* surfaces, *taut* foliations and *essential* laminations, leads to deep topological consequences.

By the work of Eliashberg, there is a similar dichotomy among contact structures between *tight* contact structures and *overtwisted* contact structures. Further, there are deep connections between taut foliations and tight contact structures by the work of Eliashberg and Thurston, and more recently of Honda, Kazez and Matic.

There is however one significant difference between contact structures and the other co-dimension one structures – while one of the most basic consequences of the existence of other essential co-dimension one structures in M is that the universal cover of M is $\mathbb{R}^3$ (this was in fact used to demonstrate the utility of essential laminations when they were introduced), one of the most basic examples of a tight contact structure is the *standard* contact structure on S^3 (which we recall in the next section). Several quotients of S^3 also admit tight contact structures.

Thus, tight contact structures clearly reveal a different aspect of 3-manifolds, at least in some cases. Our main goal in this paper is to relate the existence of contact structures on elliptic manifolds (i.e., quotients of S^3 by a group of isometries) with tight universal covers to the isomorphism $SO(4) = (SU(2) \times SU(2))/\pm 1$, and more generally to *spherical structures*.

Our first main result is the following.

Theorem 6.1. *Suppose $M = S^3/G$ where G is a group of isometries of S^3. Then the oriented manifold M has a positive contact structure with tight universal cover if and only if G (after possibly conjugating by an isometry) leaves invariant the standard contact structure on S^3.*

This statement can be strengthened as the obstructions depend only on the fundamental group.

7. Limits of functions and Elliptic operators

Proc. Indian Acad. Sci. Math. Sci. 114 (2004), no. 2, 153–158.

The motivation for this paper is that in many geometric situations rigidity phenomena are associated with elliptic operators which are often *hidden*, i.e., not *a priori* related to the geometry. Two striking instances of this are the Seiberg-Witten equations for smooth four-dimensional manifolds and J-holomorphic curves in Symplectic topology. Hence it is of interest to show that there are situations where there must be elliptic operators, even though they are not *a priori* present. My aim was to show how, at least in the simplest case, regularity properties lead to elliptic operators not a priori present in the problem.

Below I include the MathSciNet review of this paper (as it is as good a summary as I can give):

‘As is well known, complex analytic functions satisfy an elliptic differential equation, namely the Cauchy-Riemann equation, while no such equation is satisfied in the real analytic case.

‘In this article the author shows that this phenomenon is universal; namely, a class S of (real analytic) functions on closed manifold M that have regularity properties similar to those of holomorphic functions in $f \in S$ satisfies an elliptic differential equation $Pf = 0$.

‘The main theorem of this paper affirms that a subspace S of the space of real analytic functions on a manifold that satisfies certain regularity properties is contained in the set of solutions of a linear elliptic differential equation. The regularity properties are that S is closed in $L^2(M)$ and that if the sequence of functions f_n in S converges in $L^2(M)$, then so do the partial derivatives of the functions f_n.

‘Some words about the proof of this theorem. By using the hypothesis, the author shows that on the space S, the L^2 norm is equivalent to the $W^{2,2}$ norm. From this he deduces that the space S is finite-dimensional. Next, for each $x \in M$, the partial derivatives at x give linear functionals on S. By using the finite-dimensionality of S, the author shows that at x we can find an elliptic differential equation satisfied by

S. The same method yields elliptic differential equations on certain semi-analytic sub-varieties. Finally, the author uses the local Noetherian property of real analytic varieties to deduce that it is possible to globally construct an elliptic differential operator P with $Pf = 0$ for all $f \in S$.'

8. EMBEDDED SPHERES IS $\#_k S^1 \times S^1$

to appear in *Topology and its applications.*

In this paper, we address the question of which elements in $\pi_2(\#_k S^2 \times S^1)$ are represented up to conjugacy by homotopy spheres. Group theoretically such spheres correspond to splittings of the free group $\mathfrak{F}_k$ on k generators. Understanding these is likely to be useful in studying $Out(\mathfrak{F}_k)$, which is closely related to the mapping class group of the manifolds M, and more generally in studying the mapping class group of reducible 3-manifolds.

We solve this in terms of intersection numbers in the universal cover $\tilde{M}$ of $M = \#_k(S^2 \times S^1)$ in a series of steps.

Theorem 8.1. *The class* $A \in H_2(\tilde{M}) = \pi_2(\tilde{M})$ *can be represented by an embedded sphere if and only if, for each proper map* $c : \mathbb{R} \to \tilde{M}$, $c \cdot A \in \{0, 1, -1\}$.

Theorem 8.2. *Let* A *and* B *be classes in* $H_2(\tilde{M})$ *that can be represented by embedded spheres. Then* A *and* B *can be represented by disjoint embedded spheres if and only if there do not exist proper maps* $c, c' : \mathbb{R} \to \tilde{M}$ *with* $c \cdot A = 1 = c \cdot B$ *and* $c' \cdot A = 1 = -c' \cdot B$.

Theorem 8.3. *Suppose* $A \in \pi_2(M) = H_2(\tilde{M})$ *is a class such that for each deck transformation* $g \in \pi_1(M)$, A *and* gA *can be represented by disjoint spheres in* $\tilde{M}$. *Then* A *can be represented up to conjugacy by an embedded sphere* $S \in M$.

Theorem 8.4. *There is an algorithm that decides whether a conjugacy class* $A \in \pi_2(M)$ *can be represented by an embedded sphere in* M.

For the first two results, the main subtlety is guessing the correct statements. One then uses a variant of standard methods of Stallings and Whitehead in the setting of proper maps. The third result uses recent work of Scott and Swarup, and is probably the first such application outside group theory. Given our criterion, it is straightforward to obtain the algorithm.

9. HOMOLOGY AND HOMEOMORPHISMS OF NON-ORIENTABLE SURFACES

Proc. Indian Acad. Sci. Math. Sci. 115 (2005), no. 3, 251–257.

We show that, for a closed non-orientable surface F, an automorphism of $H_1(F, \mathbb{Z})$ is induced by a homeomorphism of F if and only if it preserves the (mod 2) intersection pairing. We also prove the corresponding result for punctured surfaces.

The proof has two main steps. In the first, we show that any element in the Kernel K of the natural map $Aut(H_1(F, \mathbb{Z})) \to Aut(H_1(F, \mathbb{Z}/2\mathbb{Z}))$ is induced by a homeomorphism. This involves an algebraic (Euclid's algorithm type) lemma where we determine the generators of K. These can then be geometrically shown to be realised by homeomorphisms of the surface.

In the second step, we prove inductively the stronger statement that isomorphisms between $H_1(\cdot, \mathbb{Z}/2\mathbb{Z})$ of non-orientable surfaces that preserve the intersection pairing are induced by homeomorphisms. The proof consists of showing that the image of a *generator*, which corresponds to the central curve in a Möbius band, can be represented by the central curve in a Möbius band and then proceeding inductively.

An algebraic corollary (presumably well-known) to our second step is a collection of involutions that generate the orthogonal group over $\mathbb{Z}/2\mathbb{Z}$.

10. The Chord algebra and fundamental groups

(joint with Lenny Ng), Appendix to *Knot and braid invariants from contact homology II* by Lenny Ng, Geom. Topol. 4 (2005), 1603-1637.

Using contact homology, Lenny Ng has introduced invariants of knots in the 3-sphere. In this appendix to his paper, we show how the *cord algebra* defined by Lenny can be expressed in terms of the fundamental group and peripheral structure of the knot exterior.

11. Extremes of the Indian summer monsoon rainfall, ENSO and equatorial Indian Ocean oscillation

(joint with Sulochana Gadgil, PN Vinayachandran, PA Francis) Geophysical Research letters 2004.

This paper concerns prediction of the extreme events (droughts and surpluses) of the Indian monsoon based on two indices - the well known El Nino and the so called *Equitorial Indian oscillation* studied by my co-authors in a previous paper. My contribution was to use *order statistics* to show some striking patterns. One can see a very strong relation of the indices to extreme years and *no* relation within the years of rainfall close to normal.

12. Degree-one maps, surgery and 4-manifolds

submitted for publication.

Degree-one maps between closed, oriented 3-manifolds have been extensively studied, from the viewpoint that if there is a degree-one map from M to N, then N is in some sense simpler than M. This is motivated by various considerations - for instance there is a degree-one map from any closed, oriented 3-manifold to the 3-sphere and degree-one maps induce surjections in terms of fundamental groups.

In this paper, we first give an interpretation of the existence of degree-one maps in terms of surgery.

Theorem 12.1. *For closed oriented 3-manifolds M and N, there is a degree-one map from M to N if and only if M can be obtained from N by surgery about a link in N each of whose components is an unknot in N.*

We then show that the existence of a degree-one map is naturally related to the topology of smooth 4-manifolds. This is motivated by the relation to *topological field theories*

Theorem 12.2. *For closed orientable 3-manifolds M and N, there is a degree-one map from M to N if and only if there is a smooth embedding of M in $int(W)$, $W = (N \times I)\#_n\overline{\mathbb{C}P^2}\#_m\mathbb{C}P^2$ for some $m > 0$, $n > 0$, which separates the boundary components of W, with the embedding having the appropriate orientation.*

Finally, we show that, even in the topological category, the above result cannot be sharpened to having an embedding of M in $N \times [0, 1]$.

13. Open manifolds, Ozsvath-Szabo invariants and Exotic $\mathbb{R}^4$'s

submitted for publication

Exotic $\mathbb{R}^4$'s, i.e., manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^4$ are one of the dramatic consequences of the work of Freedman and Donaldson, with further results coming from Seiberg-Witten theory. Previous constructions of exotic $\mathbb{R}^4$'s relied on the failure of surgery or the h-cobordism theorem in the smooth category.

In this paper, we construct invariants of smooth, open 4-manifolds and show that these can detect exotic $\mathbb{R}^4$'s, giving the first invariants that can do this. These are based on taking direct limits of Ozsvath-Szabo invariants with respect to an exhaustion.

One has to get around the subtlety that Ozsvath-Szabo invariants are not a genuine field theory but instead satisfy a more complicated product formula. This is done by restricting to so-called *admissible exhaustions* (after proving some basic algebraic topology lemmas) and by using *twisted coefficients.*

The non-vanishing of the invariants is proved by embedding in symplectic manifolds, using the work of Gabai, Eliashberg and Kronheimer-Mrowka. To construct the exotic $\mathbb{R}^4$'s, one adds a 2-handle along a slice knot in $S^3 = \partial B^4$, followed by a Casson handle (using the work of Gompf) and then surgers out a 2-sphere (which exists as the knot chosen is slice).

14. The Derived series and virtual Betti numbers

This note was an observation that in studying virtual Betti numbers, one can use the classification of perfect groups with periodic cohomology. As I was not aware of such an observation in the literature, I submitted this to ‘Topology proceedings’. Though the paper cannot be published, I include some of the referee's comments.

‘Gadgil's paper does implicitly contain an interesting observation that is not in the paper of Shalen with Wagreich: if M is a homology 3-sphere and $\tilde{M}$ is a finite-sheeted covering space of M with covering group different from A_5 or the binary icosahedral group, then either the derived series of $\tilde{M}$ does not terminate, or some term of the derived series is of infinite index (in which case a finite solvable cover of $\tilde{M}$ has positive first betti number). However, this observation doesn't seem to be new. I think it is implicit in the proof of Corollary 3.2 of the paper "Regular coverings of homology 3-spheres by homology 3-spheres" by E. Luft and D. Sjerve, Trans. Amer. Math. Soc., vol. 311 (1989), pp. 467–481.

If I'm reading things correctly, and Luft and Sjerve really do give almost the same argument, then I guess I have to recommend turning down the paper. This is a little hard for me to do, because I have to say that I've learned something interesting (even if apparently not new) from Gadgil's paper, and it's even something that may be useful in my research. I would certainly encourage the author to keep looking at questions of this kind.’

15. On the Andrews-Curtis conjecture and Algorithms form Topology

preprint, arXiv:math.GR/0108053

The Andrews-Curtis conjecture, one of the basic problems in combinatorial group theory, asserts that any balanced presentation (i.e., one with the same number of relations and generators) is related to a standard presentation by the so-called

Andrews-Curtis moves. These are conjugating a relation by a generator, inverting a relation and multiplying one relation by another.

In this paper we relate the Andrews-Curtis conjecture to another basic problem in group theory, namely the existence of an algorithm to decide whether a group given by a *balanced* presentation is trivial. For not necessarily balanced presentations, it is well known that such an algorithm does not exist. The main result of this paper is the following.

Theorem 15.1. *At least one of the following holds*

- *There is an algorithm to recognise balanced presentations of the trivial group, or*
- *The (balanced) Andrews-Curtis conjecture is false.*

This is based on generalising recent algorithmic methods in 3-manifold topology, in particular the Rubinstein-Thompson algorithm to recognise S^3, to the context of groups. The goal is to exploit the fact that while algorithmic problems in 3-manifold topology often have solutions, those in group theory do not.

This is done by associating to a group presentation a pair (M, S) of topological spaces, called a *handle-structures*. The space M is a compact 3-manifold and S is a subsurface of ∂M. The model for (M, S) is the collection of 1-handles and 2-handles in a 3-manifold, with M corresponding to the union of this handles and S corresponding to the surface along which these are attached to the 0-handle.

Such a structure can be associated, canonically up to finitely many choices, to any finite presentation. Further, the Andrews-Curtis transformations for a presentation correspond to handle-slides, and cancellations also have a natural meaning.

We can construct an algorithm, based on the Rubinstein-Thompson algorithm, to recognise a certain class of balanced presentations of the trivial group. Using this, one can relate the Andrews-Curtis conjecture to the triviality problem for balanced presentations.

16. Conformal structures and harmonic functions

preprint (joint with Harish Seshadri)

A conformal structure on a smooth manifold M is an equivalence class of Riemannian metrics on M, where two metrics g and g' are equivalent if for some positive smooth function $f : M \to \mathbb{R}^+$, $g' = f^2 g$. Our goal here is to give an alternative characterisation of a conformal structure, in terms of the kernel of the conformal Laplacian, and use this to study various notions of conformal geometry in a manner that applies also to the case of Carnot-Caratheodary metrics.

We first outline our characterisation. Given a Riemannian metric $g(.,.)$ on a manifold M, we consider for every open set U of M the kernel of the conformal Laplacian on U, which we call the $\Box$-harmonic functions. The projective class of this sheaf of $\Box$-harmonic functions depends only on the conformal class of g. We show conversely in section that the sheaf of functions in turn determines the conformal class of g. Thus, we can characterise conformal structures in terms of these functions.

Perhaps the most interesting result of this paper is a characterisation of conformal flatness in terms of a sheaf of functions satisfying certain axioms (which we formulate), in particular sheafs associated to Riemannian metrics. The basic idea is that (conformally) flat structures admit (conformal) homotheties, and hence sheafs coming from metrics that are conformally flat up to a certain order admit homotheties that are conformal up to that order. The obstruction to existence of such homotheties gives curvatures.

As all Riemannian metrics are flat up to first order, we obtain *first-order curvatures* that are the obstructions to a sheaf coming from a Riemannian metric. If these as well as an invariant which we call the *torsion* vanish then the sheaf is associated to a Riemannian metric. We then obtain *conformal curvatures* that determine whether the metric is conformally flat. Analogous curvatures can also be constructed in the case of Carnot-Caratheodary metrics.

The other advantage of our approach is in cases where we can construct a sheaf of $\Box$-harmonic functions even though there is no underlying Riemannian metric (at least *a priori*). In this paper we consider two such applications. The first is to study the conformal geometry of piecewise-linear metrics with respect to a triangulation. A PL-metric gives rise to a piecewise conformal structure in our sense, though there is no classical analogue for this.

Our second application is to consider convergence and degeneration of conformal structures. In Riemannian geometry, generally the difficulty in studying this lies in *choosing* metrics in each conformal class in the sequence appropriately, typically with some control on the curvature. From our point of view, however, one can simply take the weak limit of the sheaf of $\Box$-harmonic functions. The resulting sheaf may not correspond to a Riemannian metric. In this case we get a *degenerate* conformal structure. In this paper we provide a framework to understand such degenerations, which we hope will have applications in the future.

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