

EMBEDDED SPHERES IN $S^2 \times S^1 \# \dots \# S^2 \times S^1$

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ABSTRACT. We give an algorithm to decide which elements of $\pi_2(\#_k S^2 \times S^1)$ can be represented by embedded spheres up to conjugacy. Such spheres correspond to splittings of the free group F_k on k generators. We also give an algorithm to decide when classes in $\pi_2(\#_k S^2 \times S^1)$ can be represented by disjoint embedded spheres.

Our methods may be useful in studying the splitting complex of a free group, and hence the group of outer automorphisms.

1. INTRODUCTION

We study here embedded spheres in a 3-manifold of the form $M = \#_k(S^2 \times S^1)$, i.e., the connected sum of k copies of $S^2 \times S^1$. Group theoretically such spheres correspond to splittings of the free group \mathbb{F}_k on k generators [20][29]. Understanding these is likely to be useful in studying $Out(\mathbb{F}_k)$, which is closely related to the mapping class group of the manifolds M [20], and more generally in studying the mapping class group of reducible 3-manifolds.

The first question we consider is whether a class in $\pi_2(M)$ can be represented up to conjugacy by an embedded sphere in M . Let \tilde{M} be the universal cover of M . Observe that $\pi_2(M) = \pi_2(\tilde{M}) = H_2(\tilde{M})$ by Hurewicz theorem. We shall implicitly use this identification throughout.

We first consider when $A \in H_2(\tilde{M}) = \pi_2(M)$ can be represented by an embedded sphere in \tilde{M} . We shall make use of intersection numbers (and Poincaré duality) for non-compact manifolds. Represent A by a (not necessarily connected) surface in \tilde{M} (also denoted A). Given a proper map $c : \mathbb{R} \rightarrow \tilde{M}$ which is transversal to A , we consider the algebraic intersection number $c \cdot A$ (for details see Section 2). This depends only on the homology class of A and the proper homotopy class of c . The following gives a criterion for A to be represented by an embedded sphere.

Theorem 1.1. *The class $A \in H_2(\tilde{M})$ can be represented by an embedded sphere if and only if, for each proper map $c : \mathbb{R} \rightarrow \tilde{M}$, $c \cdot A \in \{0, 1, -1\}$.*

For an embedded sphere $S \in M$ with lift $\tilde{S} \in \tilde{M}$, all the translates of \tilde{S} are disjoint from \tilde{S} . In particular, if $A = [\tilde{S}]$ is the class represented by \tilde{S} , then A and Pa can be represented by disjoint spheres for each deck transformation g . Thus, our next step is to give a criterion for when two classes A and B in $H_2(\tilde{M})$ can be represented by disjoint spheres.

Theorem 1.2. *Let A and B be classes in $H_2(\tilde{M})$ that can be represented by embedded spheres. Then A and B can be represented by disjoint embedded spheres if*

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and only if there do not exist proper maps $c, c' : \mathbb{R} \rightarrow \tilde{M}$ with $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$.

The two above theorems let us determine when, for a class $A \in \pi_2(M) = H_2(\tilde{M})$, the homology classes A and gA can be represented by disjoint spheres for each $g \in \pi_1(M)$. However to get an embedded sphere in M , we need more. Namely, such a sphere S exists if and only if there is a sphere \tilde{S} disjoint from all its translates $g\tilde{S}$.

Theorem 1.3. *Suppose $A \in \pi_2(M) = H_2(\tilde{M})$ is a class such that for each deck transformation $g \in \pi_1(M)$, A and gA can be represented by disjoint spheres in \tilde{M} . Then A can be represented up to conjugacy by an embedded sphere $S \in M$.*

Thus, we have a criterion for deciding whether a conjugacy class in $\pi_2(M)$ can be represented by an embedded sphere. However our criterion *a priori* involves checking conditions for infinitely many proper maps $c, c' : \mathbb{R} \rightarrow \tilde{M}$ and infinitely many group elements g . We shall show that it suffices to check only finitely many conditions. This gives the following result.

Theorem 1.4. *There is an algorithm that decides whether a conjugacy class $A \in \pi_2(M)$ can be represented by an embedded sphere in M .*

Our methods extend to deciding when two classes A and B can be represented by disjoint spheres in M . This is based on an analogue of Theorem 1.3.

Theorem 1.5. *Suppose A and B are conjugacy classes in $\pi_2(M)$ that can be represented by embedded spheres in M . Then A and B can be represented by disjoint spheres in M if and only if for each $g \in \pi_1(M)$, A and gB can be represented by disjoint spheres in \tilde{M} .*

Theorem 1.6. *There is an algorithm that decides whether conjugacy classes $A, B \in \pi_2(M)$ can be represented by disjoint embedded spheres in M .*

Note that by results of [20], homotopy classes of embedded spheres in M are the same as isotopy classes of embedded spheres in M . In group theoretic terms, isotopy classes of embedded spheres in M correspond to conjugacy classes of splittings of the free group \mathbb{F}_k . Disjoint spheres in M correspond to splittings compatible up to conjugacy. Recall that a splitting of a group G is a graph of group decomposition of G for a graph with one edge. Two splittings are said to be compatible if there is a graph of groups decomposition of G with respect to a graph with two edges e and f , so that the graph of groups decompositions obtained by collapsing the edges e and f are respectively the two given splittings. For more details, see, for instance, [26].

The analogous problem of when a conjugacy class in the fundamental group of a surface can be represented by a simple closed curve has also been solved using intersection numbers [2] [25]. The situation there is different as the universal cover is 1-ended. In [4] it was shown that in fact geodesics on a surface minimize intersection numbers and self-intersection numbers. Analogous results for least-area surfaces in 3-manifolds were shown in [5].

A related question of when a finite set in the free group can be separated, or geometrically when a finite collection of homotopy classes of simple-closed curves in M can be represented by curves disjoint from an essential sphere in M , was considered in [30] using methods of Whitehead. An algorithm was given to answer

this question, as also for the analogous question of simple closed curves on the surface of a handlebody. The space of ends of \tilde{M} was used to prove a basic result concerning free groups in [3].

The *complex of curves* of a surface has proved very useful in studying both the mapping class group of a surface and 3-manifold topology (see [1], [6], [7], [16], [17], [18], [21], [22], [23] and [24]). Analogously, the splitting complex of a free group \mathbb{F}_k [9], or equivalently the sphere complex [20] has been used to deduce results about the outer automorphisms of a free group (see [8], [10], [11], [12], [13], [14] and [19]).

Many fruitful results regarding the complex of curves have been obtained by studying the relation between distances in the complex of curves and intersection numbers. Thus one may hope that similar results regarding the splitting complex (and hence $Out(\mathbb{F}_k)$) may be obtained using our methods. A particularly interesting question is to what extent the splitting complex is δ -hyperbolicity. The analogue for the complex of curves was proved using intersection numbers by Bowditch [1].

We now give an outline of the paper. In Section 2 we recall basic facts about ends and intersection numbers, give a Serre-Bass type construction \tilde{M} and make some simple observations about its homology. In section 3 we give a proof of Theorem 1.1 and in Section 4 we give a proof of Theorem 1.2. We recall the relevant results of Scott and Swarup and use these to complete the proofs of Theorems 1.3 and 1.5 in Section 5. Finally, in section 6 we complete the proofs of Theorems 1.4 and 1.6.

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2. PRELIMINARIES

Let $M = \#_k(S^2 \times S^1)$. An alternative description of M can be given as follows. Consider the spheres S^3 and let $A_i, B_i, 1 \leq i \leq k$, be a collection of $2k$ disjoint embedded balls in S^3 . Let P be the complement of the union of the interiors of these balls and let S_i (respectively T_i) denote the boundary of A_i (respectively B_i). Then M is obtained from P by gluing together S_i and T_i with an orientation reversing diffeomorphism φ_i for each $i, 1 \leq i \leq k$.

We give a related description for \tilde{M} . Namely identify $\pi_1(M)$ with the free group G on generators $\alpha_i, 1 \leq i \leq k$, and let T be the Cayley graph of G with respect to these generators. Then T is a tree with vertices G . The space \tilde{M} is obtained by taking a copy gP of P for each vertex P , and identifying gS_i with $g\alpha_i T_i$, using φ_i , for each $g \in G$, and each generator $\alpha_i, 1 \leq i \leq k$. This construction is standard in a topological approach to Serre-Bass theory (see, for instance, [27]).

Observe that if τ is a finite subtree of T , then $K_\tau = \cup_{g \in \tau} gP$ is a compact, simply-connected space homeomorphic to a space of the form $S^3 - \cup_{j=1}^n \text{int}(D_j)$ with D_j disjoint embedded balls in S^3 . Further, the boundary components of P are spheres corresponding to edges of T with exactly one vertex in τ . Let $\delta\tau$ denote the set of such edges. The sets K_τ give an exhaustion of \tilde{M} . Further, $H^1(\tilde{M})$ is the inverse limit of the groups $H^1(K_\tau)$ with respect to maps induced by inclusions (as the maps induced by inclusions are surjections and so the inverse system satisfies the Mittag-Leffler condition).

We next recall the notion of ends of a space. Let X be a topological space. For a compact set $K \subset X$, let $C(K)$ denote the set of components of $X - K$. For L

compact with $K \subset L$, we have a natural map $C(L) \rightarrow C(K)$. Thus, as compact subsets of X define a directed system under inclusion, we can define the set of ends $E(X)$ as the inverse limit of the sets $C(K)$. Further we can compute the inverse limit with respect to any exhaustion by compact sets.

It is easy to see that a proper map $f : X \rightarrow Y$ induces a map $E(f) : E(X) \rightarrow E(Y)$ and that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are proper maps, then $E(g \circ f) = E(g) \circ E(f)$. In particular, the real line \mathbb{R} has two ends which can be regarded as $-\infty$ and ∞ . Hence a proper map $c : \mathbb{R} \rightarrow X$ gives a pair of ends c_- and c_+ of X .

Now consider proper maps $c : \mathbb{R} \rightarrow \tilde{M}$. As \tilde{M} is a union of the simply-connected compact sets K_τ , the following lemma is straightforward.

Lemma 2.1. *There is a one-one correspondence between proper homotopy classes of maps $c : \mathbb{R} \rightarrow \tilde{M}$ and pairs $(c_-, c_+) \in E(\tilde{M}) \times E(\tilde{M})$.*

By a proper path in a manifold Q , possibly with boundary and possibly non-compact, we henceforth mean a map $c : I \rightarrow Q$ for a set $I \subset \mathbb{R}$ of the form $I = [a, b]$, $I = [a, \infty)$ or $I = \mathbb{R}$, such that the inverse image of ∂Q is ∂I and the inverse image of a compact set is compact.

We shall refer to a curve c as above as a proper path from c_- to c_+ or as a proper path joining c_- and c_+ . We denote such a path c by (c_-, c_+) . This is well defined up to proper homotopy. In particular, for a homology class $A \in H_2(\tilde{M})$, the intersection number $(c_-, c_+) \cdot A$ (which we define in detail below) is well defined and can be computed using any proper path joining c_- and c_+ . We shall use this implicitly throughout.

For a proper path $c : \mathbb{R} \rightarrow \tilde{M}$ and an element $A \in H_2(\tilde{M})$, we can define the algebraic intersection number between $c \cdot A$ by making c transversal to A and computing the intersection number. We formalise this using the exhaustion of \tilde{M} by the sets K_τ . Namely, if $c : \mathbb{R} \rightarrow \tilde{M}$ is a proper path, then there is an interval $[-L, L]$ such that $c^{-1}(K_\tau) \subset [-L, L]$. It follows that $c|_{[-L, L]}$ gives an element in $H_1(\tilde{M}, \tilde{M} - \text{int}(K_\tau)) = H_1(K_\tau, \partial K_\tau) = H^2(K_\tau)$, where the first isomorphism is by excision and the second by Poincaré duality. On passing to inverse limits, we see that c gives an element of $H^2(\tilde{M})$. Evaluating this element on A gives $c \cdot A$.

Note that every class $A \in H_2(\tilde{M})$ is supported in K_τ for some finite tree τ , and a proper path c gives an element of $H^2(K_\tau)$. Further, as the closures of the complementary components of K_τ in \tilde{M} are all non-compact, any proper path $\alpha : [0, 1] \rightarrow K_\tau$ can be extended to a proper path $c : \mathbb{R} \rightarrow \tilde{M}$ whose intersection with K_τ is α . In particular, the cohomology class in $H^2(K_\tau) = H_1(K_\tau, \partial K_\tau)$ corresponding to α is the image under the map induced by inclusion of the class corresponding to c . It follows that $\alpha \cdot A = c \cdot A$ for $A \in H_2(K_\tau)$.

We use the above observations and the fact that K_τ is a compact, simply-connected space homeomorphic to S^3 with finitely many balls deleted, with the boundary components corresponding to the edges in $\delta\tau$ to deduce some elementary results concerning the homology of \tilde{M} .

First note that each edge of T has corresponding to it a sphere of the form gS_i for some i , and such spheres are the boundary components of K_τ . As $H_2(K_\tau)$ is generated by its boundary components it follows that these spheres generate $H_2(\tilde{M})$.

Next, note that if A and B are two homology classes, then for some finite subtree $\tau \subset T$ they are both supported by K_τ . If A is not homologous to B , then as

$H_1(K_\tau) = 0$, by Poincaré duality there exists a proper path α in K_τ such that $\alpha \cdot A \neq \alpha \cdot B$. By extending α to a proper path $c : \mathbb{R} \rightarrow \tilde{M}$, we deduce that there is a proper path $c : \mathbb{R} \rightarrow \tilde{M}$ with $c \cdot A \neq c \cdot B$. Thus an element $A \in H_2(M)$ is determined by the intersection numbers $c \cdot A$ for proper paths $c : \mathbb{R} \rightarrow \tilde{M}$.

Finally, if S is an embedded sphere in M , then S separates \tilde{M} into two components. If the closure of one of these is compact, then S is homologically trivial. Otherwise we can find a proper path $c : \mathbb{R} \rightarrow \tilde{M}$ with $c \cdot S = 1$, from which it follows that S is primitive.

3. EMBEDDED SPHERES IN \tilde{M}

We now characterise which homology classes in \tilde{M} can be represented by embedded spheres.

Proof of Theorem 1.1. Suppose A can be represented by an embedded sphere S . Then the complement of S consists of two components with closures X_1 and X_2 . As S is compact, the space of ends of \tilde{M} is also partitioned into sets $E_i = E(X_i)$. For a pair of ends (c_-, c_+) , if both c_- and c_+ are contained in the same E_i , we have a corresponding proper path c disjoint from S . Otherwise we can choose c intersecting S in one point. In either case, $c \cdot A$ is 0, 1 or -1 . Computing intersection numbers $(c_-, c_+) \cdot A$ using these paths, it follows that $c \cdot A$ is always 0, 1 or -1 .

Conversely, assume that for each $c = (c_-, c_+)$, $c \cdot A$ is one of 0, 1 or -1 . Let A be represented by a (not necessarily connected) smooth, closed surface, which we also denote A . Let $K = K_\tau \supset A$. Then K is a compact, 3-dimensional, connected manifold contained in \tilde{M} such that the closure W_i of each complementary component of K is non-compact. As \tilde{M} is simply-connected and K is connected, $N_i = \partial W_i$ is connected for each W_i . Note that there are finitely many sets W_i and $E(\tilde{M})$ is partitioned into the sets $E(W_i)$.

We define a relation on the space of ends $E(\tilde{M})$ as follows. For a pair of ends e_0 and e_1 , let c be a proper path joining e_0 to e_1 . We define $e_0 \sim e_1$ if $c \cdot A = 0$. We shall show that the relation \sim is an equivalence relation. When A is a non-trivial homology class we show that there are exactly two equivalence classes.

We first need a lemma.

Lemma 3.1. *For ends e, f and g of \tilde{M} .*

- $(e, f) \cdot A = -(f, e) \cdot A$
- $(e, g) \cdot A = (e, f) \cdot A + (f, g) \cdot A$

Proof. The first part is immediate from the definitions. Suppose now e, f and g are ends and let c and c' be proper paths from e to f and from f to g respectively. Let k be such that $f \in E(W_k)$. Then there exist $T \in \mathbb{R}$ such that $c([T, \infty)) \subset W_k$ and $c'((-\infty, -T]) \subset W_k$. Let γ be a path in W_k joining $c(T)$ and $c'(-T)$. Consider the path $c'' = c|_{(-\infty, T]} * \gamma * c'|_{[-T, \infty)} : \mathbb{R} \rightarrow \tilde{M}$. This is a proper path from e to g and its intersection points with A are the union of those of c with A and c' with A , with the signs associated to the points of $c'' \cap A$ agreeing with the signs for $c \cap A$ and $c' \cap A$. Computing $(e, g) \cdot A$ using c'' , we see $(e, g) \cdot A = (e, f) \cdot A + (f, g) \cdot A$ as claimed. \square

By the above, \sim is an equivalence relation. We next show that there are at most two equivalence classes. This follows from the next lemma.

Lemma 3.2. *Suppose $e \not\sim f$ and $e \not\sim g$. Then $f \sim g$ and $(e, f) \cdot A = (e, g) \cdot A$.*

Proof. By Lemma 3.1, we have

$$(f, g) \cdot A = (e, g) \cdot A - (e, f) \cdot A$$

By hypothesis, each of $(e, g) \cdot A$ and $(e, f) \cdot A$ is ± 1 and their difference $(f, g) \cdot A$ is 0, 1 or -1 . It follows that $(e, f) \cdot A = (e, g) \cdot A$ and $(f, g) \cdot A = 0$, i.e., $f \sim g$. \square

Now as $A \neq 0$ in homology, as elements of $H_2(M)$ are determined by intersection numbers with proper paths, there are ends e and f such that $(e, f) \cdot A \neq 0$, i.e., $e \not\sim f$. Thus there are exactly two equivalence classes of ends which we denote E_1 and E_2 .

Next, observe that given two points in $E(W_i)$, for some i , there is a path joining these in the complement of K , hence of A . It follows that these are equivalent. Hence each $E(W_i)$ is contained in E_1 or E_2 . We now construct a proper function $f : \tilde{M} \rightarrow \mathbb{R}$. Namely, for each i , if $E(W_i) \subset E_1$ (respectively $E(W_i) \subset E_2$), we construct a proper function $f : W_i \rightarrow [-1, -\infty)$ (respectively $f : W_i \rightarrow [1, \infty)$). We extend this across K to get a proper function $f : \tilde{M} \rightarrow \mathbb{R}$.

As \tilde{M} is simply-connected, we can use standard techniques due to Whitehead and Stallings [28][29] to show that, after a proper homotopy of f , we can assume that $S = f^{-1}(0)$ is a sphere. The details are analogous to the proof of Kneser's conjecture in [15]. Namely, by Lemma 6.5 of [15], we can assume that each component of $f^{-1}(0)$ is incompressible, hence a sphere. Suppose $f^{-1}(0)$ has more than one component, we take a path α joining two components. As in the proof of Theorem 7.1 of [15], we can replace α by a path joining two components whose interior is disjoint from $f^{-1}(0)$. As \mathbb{R} is simply-connected, $f \circ \alpha$ is a homotopically trivial loop in \mathbb{R} . Modifying f in a neighbourhood of α reduces the number of components of $f^{-1}(0)$. After finitely many iterations of this process, $f^{-1}(0)$ is connected.

The sphere S now separates \tilde{M} into subsets X_1 and X_2 . By construction, $E(X_i) = E_i$. Hence, as a homology class in \tilde{M} is determined by intersection numbers $c \cdot A$ for proper paths $c : \mathbb{R} \rightarrow \tilde{M}$, $A = [S]$ after possibly changing the orientation of S . \square

Remark 3.3. By construction $S \subset K$.

4. DISJOINT SPHERES IN \tilde{M}

Suppose now that A and B are classes in $H_2(\tilde{M}) = \pi_2(M)$ which can be represented by embedded spheres S and T . We shall deduce when S and T can be chosen to be disjoint. Denote the closures of the components of the complement of S (respectively T) by X_1 and X_2 (respectively Y_1 and Y_2) so that $(e, f) \cdot A = 1$ if and only if $e \in X_1$ and $f \in X_2$ and $(e, f) \cdot B = 1$ if and only if $e \in Y_1$ and $f \in Y_2$. Recall that $(f, e) \cdot A = -(e, f) \cdot A$ and $(f, e) \cdot B = -(e, f) \cdot B$.

Suppose S and T are disjoint. We first consider the case $T \subset X_2$. Then X_1 is contained in one of Y_1 and Y_2 . If $X_1 \subset Y_1$, then for $c' = (e, f)$, if $c' \cdot A = 1$ then $e \in X_1 \subset Y_1$ hence $(f, e) \cdot B \neq 1$, i.e., $c' \cdot B \neq -1$. Thus, there does not exist c' with $c' \cdot A = 1 = -c' \cdot B$.

By considering other cases similarly, we see that there do not exist proper maps $c, c' : \mathbb{R} \rightarrow \tilde{M}$ with $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$.

Conversely, suppose there do not exist proper maps $c, c' : \mathbb{R} \rightarrow \tilde{M}$ with $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$. We define three equivalence relations \sim_A, \sim_B and

\sim on $E(\tilde{M})$. Namely, $e \sim_A f$ (respectively $e \sim_B f$) if $(e, f) \cdot A = 0$ (respectively $(e, f) \cdot B = 0$) and $e \sim f$ if $e \sim_A f$ and $e \sim_B f$. We shall see that \sim partitions $E(\tilde{M})$ into at most three equivalence classes.

Let $e \in E(\tilde{M})$ be an end. We shall assume that e is fixed. By Lemma 3.2, for ends f , $(e, f) \cdot A$ has only two possible values, 0 and one of 1 and -1 . By replacing A by $-A$, we assume that for every end f , $(e, f) \cdot A \in \{0, 1\}$. Similarly, we assume that for every end f , $(e, f) \cdot B \in \{0, 1\}$. Thus, for ends f , the pair $((e, f) \cdot A, (e, f) \cdot B)$ has four possible values. By Lemma 3.1, if $((e, f) \cdot A, (e, f) \cdot B) = ((e, g) \cdot A, (e, g) \cdot B)$, then $f \sim g$. Hence there are at most four equivalence classes under the relation \sim .

We now show that at least one of these classes is empty. If all the four classes are non-empty, we can find f, g and h with $(e, f) \cdot A = 1$, $(e, f) \cdot B = 0$, $(e, g) \cdot A = 0$, $(e, g) \cdot B = 1$, $(e, h) \cdot A = 1$ and $(e, h) \cdot B = 1$. Taking $c = (e, h)$ and $c' = (g, f)$, by Lemma 3.1 we see that $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$, a contradiction.

Remark 4.1. As the four equivalence classes under \sim are the four intersections $E(X_i) \cap E(Y_j)$, we see that one of these sets must be empty, i.e. one of the sets $X_i \cap Y_j$ is compact. This is important in the sequel.

If A and B are not independent, then either $A = B$ or $A = -B$ as both A and B are represented by embedded spheres and are hence primitive. In this case they can be represented by disjoint embedded spheres. Hence we may assume that they are independent. As elements of $H_2(M)$ are determined by intersection numbers with proper paths, it follows that there must be three equivalence classes. Let e, f and g represent the equivalence classes. By changing signs and permuting if necessary, we can assume that $(e, f) \cdot A = 1$, $(e, f) \cdot B = 0$, $(e, g) \cdot A = 0$ and $(e, g) \cdot B = 1$.

We now proceed as in the previous section. Choose surfaces representing A and B and a compact submanifold K containing these as in the previous section. Let \mathcal{T} (a tripod) denote the union of three half lines R_e, R_f and R_g , each homeomorphic to $[0, \infty)$, with the points 0 in all of them identified. We construct a proper map $f: \tilde{M} \rightarrow \mathcal{T}$ by mapping the components W_i of $\tilde{M} - \text{int}(K)$ equivalent to e properly onto R_e and analogously for components equivalent to f and g . We extend the map f over K . Let $1_f \in R_f$ and $1_g \in R_g$ denote points in \mathbb{R}_f and \mathbb{R}_g corresponding to 1 in the identifications of \mathbb{R}_f and \mathbb{R}_g with $[0, \infty)$.

As in the previous section, we use techniques of Whitehead and Stallings to see that, after a proper homotopy of f , $S = f^{-1}(1_e)$ and $S' = f^{-1}(1_h)$ are disjoint spheres representing A and B . To do this, first note that as before we can assume that S and S' are incompressible. Suppose one of them, say S , has two components and α is a path joining these components. Then as \mathcal{T} is simply-connected, we can pass to a subinterval to obtain an arc whose endpoints are both in S or both in S' , and are moreover contained in different components of S or S' so that the interior of the arc is disjoint from S and S' . We can modify f in a neighbourhood of this arc to reduce the number of components. Iterating this procedure we obtain f so that S and S' are connected. As in the previous section they represent A and B .

5. INTERSECTION NUMBERS AND EMBEDDED SPHERES

Suppose now that the class $A \in \pi_2(M) = H_2(\tilde{M})$ can be represented by an embedded sphere S in \tilde{M} . Further assume that for all $g \in \pi_1(M)$, A and gA can be represented by disjoint embedded spheres. We show that the class A is represented by a splitting of the free group $G = \pi_1(M)$ and hence an embedded sphere.

This follows from the work of Scott and Swarup [26] using Remark 4.1. We begin by recalling the relevant notions and results in the special case that is relevant to us.

For a set $E \subset G$, we denote the complement of E by E^* and by $E^{(*)}$ we mean one of the sets E and E^* . Two subsets E and E' of the group G are said to be almost equal if their symmetric difference is finite, and a set E is said to be non-trivial if both E and E^* are infinite. The set E is said to be almost invariant if E is almost equal to Eg for all $g \in G$. An equivalent condition in terms of the Cayley graph $\Gamma(G)$ is that the set δE of edges of $\Gamma(G)$ with one vertex in E and the other in E^* is finite.

By the construction of \tilde{M} in Section 2, there is a natural embedding of $\Gamma(G)$ in \tilde{M} , and in particular G is identified with a subset of \tilde{M} . Suppose that $S \subset \tilde{M}$ is an embedded sphere. Let X_1 and X_2 be the closures of the complementary components of S . Let $E_i = X_i \cap G$. Then E_1 and E_2 form complementary almost-invariant sets as only finitely many edges of the Cayley graph $\Gamma(G)$ intersect S , and hence the set δE_1 of edges of $\Gamma(G)$ with one vertex in E_1 and the other in E_1^* is finite. The sets $E_1^{(*)}$ are called the almost invariant sets corresponding to S . Note that if embedded spheres S and S' are isotopic (equivalently homologous) then the corresponding almost invariant sets are almost equal.

Note that the sets $E(X_i)$ are determined by the sets E_i (as can be seen by considering the exhaustion by sets of the form K_τ). Hence, for two embedded spheres S and S' , if the corresponding almost invariant sets are almost equal, then S and S' are homologous (as homology classes in \tilde{M} are determined by intersection numbers $c \cdot A$).

By the Kneser conjecture (proved by Stallings), splittings of G correspond to embedded spheres in M . An embedded sphere in M lifts to a collection of embedded spheres in \tilde{M} . The corresponding almost invariant sets in G are called the almost invariant sets corresponding to the splitting of G .

Theorem 1.12 of [26] gives conditions under which (in our situation) an almost invariant set E gives rise to a splitting, one of whose almost invariant sets is almost equal to E . We shall verify that these conditions are satisfied. As a consequence we get a sphere $\Sigma \subset \tilde{M}$ that, up to conjugacy, represents $A \in \pi_2(\tilde{M}) = H_2(\tilde{M})$.

Theorem 1.12 of [26] applies for A if for the sets E_1 and E_2 defined as above, for each $g \neq 1$, $g \in G$, exactly three of the sets $gE_i \cap E_j$ are infinite. By Remark 4.1, at most three of the sets are compact, and exactly three are compact unless A and gA are homologous. Thus it suffices to show that for $g \neq 1$, A and gA are not homologous. The theorem is easy in the case when $k = 1$ and when A is homologically trivial, so we may assume that neither of these happens.

Suppose now $gA = A$ as elements in homology for $g \neq 1$. Then $g^m A = A$ for all $m \in \mathbb{Z}$. Let τ be a finite subtree with the support of A contained in K_τ . Then for m large enough, the distance between τ and $g\tau$ is at least two. The following lemma gives a contradiction, showing that we cannot have $A = gA$.

Lemma 5.1. *Assume $k \geq 2$. Let τ and τ' be finite subtrees in T so that the distance between them is at least two. Suppose A and B are non-trivial classes in $H_2(\tilde{M})$ supported respectively in K_τ and $K_{\tau'}$. Then A and B are not homologous.*

Proof. Let W be the closure of the component of $\tilde{M} - K_\tau$ containing $K_{\tau'}$. Then as $k \geq 2$, it is easy to see that the closure of $W - K_{\tau'}$ is not compact., hence there is

a proper path $c' : [1, \infty) \rightarrow W - K_{\tau'}$ with $c'(1) \in \partial W$. Using this path, it follows that any proper path $\alpha : [0, 1] \rightarrow K_{\tau}$ extends to a proper path $c : \mathbb{R} \rightarrow \tilde{M}$ which is disjoint from $K_{\tau'}$.

Now as A is not homologically trivial, there is a proper path $\alpha : [0, 1] \rightarrow K_{\tau}$ with $\alpha \cdot A \neq 0$. We extend this to a path c as above with $c \cdot A \neq 0$. As c is disjoint from $K_{\tau'}$, $c \cdot B = 0$. Hence A is not homologous to B . \square

Thus exactly three of the sets $gE_i \cap E_j$ are infinite. Hence we can apply Theorem 1.12 of [26] to complete the proof of Theorem 1.3.

The proof of Theorem 1.5 is very similar. We may assume that A and gB are not homologous for all $g \in G$, as the case when they are homologous is easy. Then we define sets E_i as above and analogous sets E'_j corresponding to B . The final part of Theorem 1.12 of [26] shows that we get compatible splittings corresponding to A and B provided for each $g \neq 1$, $g \in G$, exactly three of the sets $gE_i \cap E'_j$ is infinite. The proof that this is the case is as above.

6. THE ALGORITHMS

We now have necessary and sufficient conditions for deciding whether a class $A \in \pi_2(M)$ can be represented by an embedded sphere in M . However there are *a priori* infinitely many conditions. To make this into an algorithm, we reduce these to finitely many conditions.

We use the construction of \tilde{M} given in Section 2. Recall that there is a natural embedding of the Cayley graph T of the free group $G = \pi_1(M)$ in \tilde{M} . Further we have a canonical sphere in \tilde{M} associated to each edge of the Cayley graph, and these spheres generate $H_2(\tilde{M})$. Note that each of these generating spheres intersects exactly one edge e of T and with $S \cap e$ is a single point with transversal intersection.

Observe that any proper path c is properly homotopic to an edge path in T . By the above, elements of $\pi_2(M)$ correspond to finite linear combinations of edges of T . Let A be such an element, and let $\tau \subset T$ be a finite subtree such that K_{τ} contains the support of A . Then for an edge-path c , $c \cdot A$ depends only on the finite edge path $\alpha = c \cap \tau$ contained in τ with endpoints on $\partial\tau$. Further, as T is a tree without any terminal vertices, any finite edge path α in τ with endpoints on $\partial\tau$ is of the form $\alpha = c \cap \tau$ for a proper path c . Hence A is represented by an embedded sphere in \tilde{M} if and only if for every finite edge path α in τ with endpoints on $\partial\tau$, $\alpha \cdot A$ is 0, 1 or -1 . As this is a finite condition, it can be verified algorithmically.

Similarly, given two homology classes A and B in $H_2(\tilde{M})$, we have an algorithm to decide whether A and B can be represented by disjoint embedded spheres by taking τ containing the supports of both A and B .

Finally, if A is a homology class with τ a tree supporting A , we first verify whether A can be embedded in \tilde{M} . Next there are at most finitely many elements g_1, \dots, g_n in G such that $\tau \cap g_i \tau$ is non-empty. For each of these g_i we check whether A and $g_i A$ can be represented by disjoint spheres. Assume henceforth that A has this property.

By Remark 3.3, A can be represented by an embedded sphere S in $K = K_{\tau}$. If $\tau \cap g\tau$ is empty, so is $K \cap gK$ and hence $S \cap gS$, i.e. A and gA can be represented

by disjoint embedded spheres. Thus we need to check only finitely many conditions for finitely many g_i , which can be done algorithmically.

Similar considerations, using Theorem 1.5 gives an algorithm to decide whether two classes in $\pi_2(M)$ (more generally finitely many classes in $\pi_2(M)$) can be represented by disjoint spheres. \square

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