LIMITS OF FUNCTIONS AND ELLIPTIC OPERATORS

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Abstract. We show that a subspace S of the space of real analytical functions on a manifold that satisfies certain regularity properties is contained in the set of solutions of a linear elliptic differential equation. The regularity properties are that S is closed in $L^2(M)$ and that if a sequence of functions f_n in S converges in $L^2(M)$, then so do the partial derivatives of the functions f_n.

The limit f of a sequence f_n of complex analytical functions (under uniform convergence on compact sets) is complex analytical. Furthermore all partial derivatives of f_n converge to the corresponding partial derivatives of f. This is in contrast to the case of real analytical functions. In fact, by the Weierstrass approximation theorem, every continuous real function on a compact domain is the uniform limit of real analytical functions on this domain.

The reason for the contrast between the complex and the real analytical cases is of course that complex analytical functions satisfy an elliptic differential equation, namely the Cauchy-Riemann equation (or alternatively because they satisfy the Laplace equation), while no such equation is satisfied in the analytical case.

Here we show that this phenomenon is universal, namely, whenever we have a class S of (real analytical) functions on a closed manifold M that have regularity properties similar to those of holomorphic functions, all functions in $f \in S$ satisfy an elliptic differential equation $Pf = 0$.

Our motivation is that in many geometric situations rigidity phenomena are associated with elliptic operators which are often *hidden*, i.e., not *a priori* related to the geometry. Two striking instances of this are the Seiberg-Witten equations for smooth four-dimensional manifolds and J-holomorphic curves in Symplectic topology. Hence it is of interest to show that there are situations where there must be elliptic operators, even though they are not *a priori* present.

First we recall the definition of the Sobolev spaces $W^{2,k}$ where $k \geq 0$ is an integer. We will not need the general case when $k \in R$.

Definition 0.1. Let $k \geq 0$ be an integer. Suppose f and g are smooth, real valued functions on $\mathbb{R}^n$ with compact support, we define the Sobolev inner product $\langle f, g\rangle_{2,k}$ by

$$\langle f, g\rangle_{2,k} = \Sigma_{j=0}^{k} \Sigma_{|I|=j} \int_{\mathbb{R}^n} \partial^I f(x) \partial^I g(x) dx$$

where I is a multi-index and ∂^I denotes the partial derivative with respect to I.

Definition 0.2. Suppose M is a manifold, let $\{U_i\}$ be a locally finite cover of M by subsets homeomorphic to $\mathbb{R}^n$ and let $\{\pi_i\}$ be a partition of unity subordinate to

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this cover with $supp(\pi_i) \subset U_i$ compact. For smooth compactly supported functions f and g on M, define $\langle f, g\rangle_{2,k}$ by

$$\langle f, g\rangle_{2,k} = \Sigma_i \langle \pi_i f, \pi_i g\rangle_{2,k}$$

where $\langle \pi_i f, \pi_i g\rangle_{2,k}$ denotes the Sobolev inner product on $U_i = \mathbb{R}^n$

The above definition depends on the choice of the cover U, but different covers give equivalent inner products.

Definition 0.3. The Sobolev space $W^{2,k}(M)$ is the Hilbert space completion of the space $C_c^\infty(M)$ of smooth functions on M with compact support with respect to the Sobolev inner product $\langle\ ,\ \rangle_{2,k}$.

When $k = 0$ we get the Hilbert space $L^2(M)$ with its usual inner product. The definitions above coincide with the definitions using Fourier transforms.

We can now state our main result.

Theorem 0.4. *Let S be a subspace of real analytical functions on a compact real analytical manifold M that is closed under the L^2-norm on M. Assume further that if $f_n \in S$ is a sequence of functions such that $f_n \to f$ in $L^2(M)$, then $f_n \to f$ in all Sobolev spaces $W^{2,k}(M)$, $k \in \mathbb{N}$. Then there is an analytical elliptic differential operator P on M such that $\forall f \in S$, $Pf = 0$.*

Remark 0.5. The analogous result for sections of a bundle on M holds and can be proved in exactly the same way.

A differential operator P that satisfies elliptic regularity on every open set U (i.e., if u is a distribution on U with $Pu = f$, f smooth, then u is smooth) is called *hypoelliptic*. Such operators have been characterised among operators with constant coefficients by Hörmander [2]. What we consider here is a different situation where our class of functions may not be given by a differential equation. What we can conclude is also weaker - we only know that S is contained in the set of solutions to an elliptic differential equation.

We now outline the proof. By using the hypothesis, we show that on the space S, the L^2 norm is equivalent to the $W^{2,2}$ norm. From this we deduce that the space S is finite dimensional. Next, for each $x \in M$, the partial derivatives at x give linear functionals on S. By using the finite-dimensionality of S, we show that at x we can find an elliptic differential equation satisfied by S. The same method yields elliptic differential equations on certain semi- analytical sub-varieties. Finally, we use the local Noetherian property of real analytic varieties to deduce that we can globally construct an elliptic differential operator P with $Pf = 0\ \forall f \in S$.

Only the final step in the above outline uses analyticity. We shall show, however, that the hypothesis of analyticity is essential for our result.

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1. Finite dimensionality of S

In this section we show that S is finite dimensional. First we make an elementary observation about subspaces of Hilbert spaces.

Let H_1 and H_2 be Hilbert spaces with norms $\|\ \|_1$ and $\|\ \|_2$ respectively. Assume that as sets $H_2 \subset H_1$ with $\|x\|_1 \leq \|x\|_2$, $\forall x \in H_2$. The following result will be applied to the case when $H_1 = L^2(M)$ and $H_2 = W^{2,2}(M)$.

Proposition 1.1. *Let S be a subspace of H_2 that is closed in H_1 and H_2 so that the subspace topologies induced by H_1 and H_2 coincide. Then there is a constant $C > 0$ such that $\|x\|_2 \leq C\|x\|_1$, $\forall x \in S$.*

Proof. By hypothesis, the identity map from S with its topology as a subspace of H_1 to S with its topology as a subspace of H_2 is continuous. Hence it must be bounded from which the conclusion follows. □

Note that by the hypothesis in our main theorem, the above result applies to S with $H_1 = L^2(M)$ and $H_2 = W^{2,2}(M)$. We next show that a space S satisfying the hypothesis of the main theorem is finite dimensional.

Lemma 1.2. *Let M be a closed manifold and let S be a subspace of $W^{2,2}(M) \subset L^2(M)$ such that there exists $C > 0$ such that for $f \in S$, $\|f\|_{2,2} < C\|f\|_2$. Then S is finite dimensional.*

Proof. Suppose S is infinite dimensional, then let $\{x_n\}$ denote an L^2-orthonormal sequence in S. By hypothesis, for all $j \in \mathbb{N}$, $x_n \in W^{2,2}$ and $\|x_n\|_{2,2} < C$. By the Rellich lemma it follows that the sequence x_n has a convergent sequence in L^2. But, as the vectors x_n are L^2-orthonormal, this is impossible. Thus, S must be finite dimensional. □

2. Pointwise differential equations

We can now construct elliptic differential equations satisfied by the functions in S at a single point $x \in M$. Choose a system of local co-ordinates. Observe that partial derivatives at x give linear functionals on S, i.e., elements of the dual S' of S. These generate a subspace V_x of S'. As S' is finite dimensional, V_x is generated by finitely many partial derivatives, and hence those of order at most k for some k. We denote these differential operators by $P_1,\dots, P_m$.

Now let E be an elliptic operator of order greater than k, for instance a power of the Laplacian. Then $f \mapsto Ef(x)$ is an element of S', hence is spanned by the P_i. Thus, at x each $f \in S$ satisfies a relation $(E - \Sigma c_i P_i)f(x) = 0$. As this has the same leading term as E, this is an elliptic differential equation.

Note that the relations $P_1,\dots P_m$ are independent as elements of S' on an open set (as independence is an open condition). Let $r(x) = dim V_x$. Let $f_1,\dots,f_N$ be a basis for S. In the special case where $r(x)$ is a constant function (for instance $r(x) = dim(S)$, the maximum possible value, at all points), we shall see that we have a global elliptic operator even in the absence of analyticity.

Proposition 2.1. *Suppose $r(x) = m$ is a constant. Then there is an elliptic differential operator E such that $Ef = 0$, for all $f \in S$.*

Proof. We first show that there is a uniform degree k so that the operators of degree at most k span V_x for all $x \in M$. For $x \in M$, let $P_1,\dots P_m$ be operators independent at x and let U_x be the set where these operators are independent. This is an open set as linear independence is an open condition (for instance, by considering determinants). By hypothesis, M is the union of the sets U_x. By compactness we can find finitely many such sets U_j whose union is M. Let k be the maximal degree of the differential operators associated to these sets.

Now, let E be an elliptic operator of order greater than k. For each U_j, we have differential operators $P_1,\dots,P_m$ which are independent at each $x \in U_j$ and hence span V_x. Hence we have a relation $Ef(x) = \Sigma c_i(x) P_i f(x)$ for $f \in S$.

We next show that each $c_i(x)$ is smooth as a function of $x \in U_j$. Let $x_0 \in U_j$ be an arbitrary point. We shall show that $c_i(x)$ is smooth at x_0.

As the operators P_j, $1 \leq j \leq m$, are independent at x_0 as functionals on S, there exist $g_i \in S$, $1 \leq i \leq m$, such that $P_j g_i(x_0) = \delta_{ij}$. It follows that for $x \in U_j$, $P_j g_i(x) = \delta_{ij} + a_{ij}(x)$ with $a_{ij}(x)$ smooth functions and $a_{ij}(x_0) = 0$.

Let $A(x)$ denote the matrix with entries $a_{ij}(x)$ and let $V(x)$ (respectively $C(x)$) denote the (column) vector with entries $Eg_i(x)$ (respectively $c_i(x)$). Note that $V(x)$ is smooth as a function of x. As $Eg_i(x) = \Sigma_j P_j g_i(x) c_j(x)$, i.e., $V(x) = (I + A(x))C(x)$, we have $C(x) = (I + A(x))^{-1} V(x)$.

Now, $A(x_0) = 0$ and it is well known that $M \mapsto (I + M)^{-1}$ is smooth as a function of M at $M = 0$ (by using the implicit function theorem or Taylor expansions). Hence $C(x)$ is smooth at x_0, i.e., each c_i is smooth at x_0, as required. Let $E'_j = E - \Sigma_i c_i P_i$. This is an elliptic operator annihilating S.

To construct an elliptic operator globally, we take a partition of unity $\{\pi_i\}$ subordinate to the cover $\{U_j\}$ and let $E_0 = \Sigma_i \pi_i E_i$. Then each $f \in S$ is in the kernel of E_0 and E_0 is elliptic as, by construction, the leading term of E_0 is the same as that of E. □

Without assuming analyticity, however, our main result fails in general. To see this, we let $M = S^1 = \mathbb{R}/\mathbb{Z}$ and construct a function f on S^1 so that, for $n > 1$, $f^{(n)}(1/n) \neq 0$ but $f^{(k)}(1/n) = 0$ for $k < n$. Let S be the (one-dimensional) span of f.

An elliptic differential operator E on a 1-dimensional manifold is a differential operator $P(D)$ whose leading coefficient is non-zero at all points. The function f does not satisfy $Ef = 0$ for any such operator as if d is the order of E, by construction $Ef(1/d) \neq 0$.

3. Globalisation in the analytical case

In the analytical case, we shall construct sets similar to the U_j above. These are now open in the real analytic topology, i.e., one whose sub-basis is generated by sets of the form $f(x) \neq 0$ where f is an analytical function.

We shall need two basic facts regarding the real analytical topology (see, for instance, [4]). Firstly, any closed set is defined by a single equation, as given a closed set $F = Z(g_1, \dots, g_p) = \{x : g_i(x) = 0, 1 \leq i \leq p\}$, we have $F = Z(g_1^2 + \cdots + g_p^2)$. Secondly, as the ring of power series is Noetherian, the real analytical topology is locally Noetherian. As M is compact, the real analytical topology on M is Noetherian.

As M is analytical, by a theorem of Morrey [3] and Grauert [1] there is an analytical Riemannian metric on M. Hence the Laplacian (with respect to an analytical metric) is an analytical elliptic operator on M and so are it powers. It follows that there are analytical elliptic differential operators on M of arbitrarily high orders.

In the rest of this section, we make the convention that all differential operators we consider are analytical ones globally defined on M. In particular we shall use the notation of the previous sections, but with V_x now being the subspace of S' generated by global analytical operators acting on S at x and $r(x)$ its dimension.

We shall inductively construct sequences of sets F_i and V_i, with F_i a decreasing sequence of closed sets and V_i open, and finitely many elliptic differential operators that span V_x for $x \in F_i \cap V_i$.

Let $F_1 = M$ and note that this is a closed subset of M. On $F_1 = M$, let $m_1 = r(z)$ be the maximum value of $r(x)$ (which is attained as $r(x) \in \mathbb{Z}$, $0 \leq r(x) \leq dim(S)$ for all $x \in M$) and let $P_1^1, \dots, P_m^1$, $m = m_1$, be (analytic) differential operators with $f \mapsto P_i^1 f(z)$ independent in S'. Then the set V_1 where the P_j^1's are independent is an open set in the analytical topology. It follows that for all $x \in V_1$, the functionals $P_j^1(x)$ span V_x. Let q_1 be the maximum order of differential operators P_j^1.

Next, let F_2 be the complement of V_1. Let $r(z) = m_2 \leq m_1$, $z \in F_2$, be the maximum value of $r(x)$ on F_2 and let $P_1^2, \dots, P_m^2$, $m = m_2$, be different operators with $f \mapsto P_j^2 f(z)$ independent functionals in S'. Let V_2 be the open set (in the real analytical topology) where P_j^2's are independent as functionals in S'. Then these span V_x for all $x \in F_2 \cap V_2$. Let q_2 be the maximum order of these differential operators.

Inductively, given F_k and V_k, we define $F_{k+1} = F_k \setminus V_k$ and let m_{k+1} be the maximum rank of V_x on F_{k+1}. As above, we construct differential operators P_j^{k+1} and let V_{k+1} be the set on which these are independent. These span V_x for all $x \in F_{k+1} \cap V_{k+1}$. Let q_k be the maximal order of the differential operators constructed above.

Now, by the local Noetherian property, the above process must stabilise. It follows that for some n, $F_n \subset V_n$. Let q be the maximum of the numbers q_j, $1 \leq j \leq n$ and note that on each set $F_j \cap V_j$, we have differential operators of degree at most q that span at each point the subspace V_x. Let g_n be analytical functions such that $F_n = Z(g_n)$.

Let E be an analytic elliptic operator of order greater than q. We construct inductively analytic elliptic operators $E_n, \dots, E_1$ with E_1 being the required operator. First, note that on $F_n \cap V_n$, we can find an operator G_n, with analytical coefficients, of order at most q so that $Ef(x) = G_n f(x)$ for all $x \in F_n \cap V_n$, $f \in S$. Let $E_n = E - G_n$.

Next, observe that on $F_{n-1} \cap V_{n-1}$, the function g_n does not vanish, and hence the operator E_n/g_n is well defined. Hence there is an operator G_{n-1}, with analytical coefficients, of order at most k so that $(E_n/g_n)f(x) = G_{n-1}f(x)$ for all $x \in F_{n-1} \cap V_{n-1}$, $f \in S$. Let $E_{n-1} = E_n - g_n G_{n-1}$. This annihilates S on $F_{n-1} \cap V_{n-1}$ by construction and also on F_n as it coincides with E_n on F_n. Thus E_{n-1} annihilates S on F_{n-1}

Similarly, given an elliptic operator E_k that annihilates S on F_k, we can construct an elliptic operator E_{k-1} that annihilates S on F_{k-1}. Proceeding inductively, we obtain an operator that annihilates S on $F_1 = M$. □

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