

# THE DERIVED SERIES AND VIRTUAL BETTI NUMBERS

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ABSTRACT. The virtual Betti number conjecture states that any hyperbolic three-manifold has a finite cover with positive first Betti number. We consider here when such a cover can be constructed using the derived series. Specifically, we show that the virtual Betti number conjecture would follow if it were known that the derived series of the fundamental group  $G$  of a hyperbolic three-manifold satisfies a certain stability property. The stability property is the statement that if all the quotients  $G^i/G^{i+1}$  of the derived series  $G^i$  of  $G$  are finite, then the derived series stabilises.

The proof involves basic facts regarding finite group actions on homology spheres. This is an unexpected relation between hyperbolic 3-manifolds and methods used to study elliptic 3-manifolds.

An important conjecture in 3-manifold topology and hyperbolic geometry is the so called virtual Betti number conjecture of Thurston which states that any hyperbolic 3-manifold  $M$  has a finite cover with positive first Betti number. A natural way to try to construct such a cover is to consider the derived series  $G^i$  of  $G = \pi_1(M)$  (or of some finite-index subgroup of  $G$ ). There are *a priori* three possibilities for this: the series may stabilise, some quotient  $G^i/G^{i+1}$  may be infinite or we may have finite derived quotients without the series stabilising. In the second case we obtain a finite cover with positive Betti number.

Our goal in this note is to show that to prove the virtual Betti number conjecture for a class (closed under passing to finite covers) of hyperbolic 3-manifolds, it suffices to rule out the third case for manifolds in the class. In other words, while there are *a priori* three cases, one of these in some sense does not occur in an essential way. Our proof involves basic facts regarding finite group actions on homology spheres, giving an unexpected relation between hyperbolic 3-manifolds and methods used to study elliptic 3-manifolds.

We first recall some basic definitions and elementary facts. Suppose  $G$  is a (finitely presented) group. The *derived series* of  $G$  is defined inductively as follows: let  $G^0 = G$  and, for  $i \in \mathbb{N}$ , let  $G^i = [G^{i-1}, G^{i-1}]$ . We say that the derived series stabilises if  $G^{k+1} = G^k$  for some  $k$ . In this case, for  $i \geq k$  we have  $G^i = G^k$ . The quotient  $G^i/G^{i+1}$  is just the abelianisation of  $G^i$  and hence the derived series stabilises if and only if, for some  $i$ ,  $G^i$  is perfect.

Suppose  $G = \pi_1(M)$  for a hyperbolic 3-manifold  $M$ . Then the virtual Betti number conjecture is equivalent to the statement that there is a finite-index subgroup  $H$  of  $G$  whose abelianisation is infinite. Suppose  $G^i/G^{i+1}$  is infinite for some  $i \geq 0$ , then if we let  $H = G^i$  for the *smallest* such  $i$ ,  $H$  has finite index in  $G$  and has infinite abelianisation as required.

To state our result, we need a definition.

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**Definition 0.1.** Let  $\mathcal{C}$  be a class of hyperbolic 3-manifolds that is closed under passing to finite covers. Then we say that  $\mathcal{C}$  satisfies the *stability property* if for each  $M \in \mathcal{C}$  the following holds for  $G = \pi_1(M)$ : if each quotient  $G^i/G^{i+1}$ ,  $i \in \mathbb{N}$ , of the derived series of  $G$  is finite, then the derived series stabilises.

We could take the class  $\mathcal{C}$  to be, for instance, all arithmetic manifolds or simply all hyperbolic 3-manifolds. Our main result is the following.

**Theorem 0.2.** *Suppose the class  $\mathcal{C}$  of hyperbolic 3-manifolds satisfies the stability property. Then any manifold  $M \in \mathcal{C}$  has a finite cover with positive first Betti number.*

To prove the result, we need the following lemma.

**Lemma 0.3.** *Let  $M$  be a hyperbolic 3-manifold and let  $N$  be a finite Galois cover of  $M$  with  $H$  the group of deck transformations. Suppose  $H_1(M) = H_1(N) = 0$ , then  $H$  is the binary icosahedral group (or  $M = N$ ).*

*Proof.* By Poincaré duality,  $M$  and  $N$  are homology spheres. As  $H_1(M) = 0$  and  $H$  is a quotient of  $\pi_1(M)$ , it follows that  $H$  is perfect. Thus  $H$  is a finite perfect group acting freely on the homology 3-sphere  $N$  (in particular  $H$  has periodic cohomology). It is well known (see, for instance, [3]) that  $H$  must be the binary icosahedral group (or the trivial group).  $\square$

In particular, we have the following corollary as the binary icosahedral group has order 120.

**Corollary 0.4.** *Let  $M$  be a hyperbolic 3-manifold and let  $N$  be a finite Galois cover of  $M$  with order more than 120. Then  $N$  is not a homology sphere.*

Suppose now that  $\mathcal{C}$  is a class of manifolds satisfying the hypothesis of the theorem and let  $M$  be a manifold in  $\mathcal{C}$ . Consider the covers of  $M$  corresponding to the derived subgroups of  $\pi_1(M)$ . By the stability property  $M$  either has a finite cover with positive first Betti number or  $M$  has a cover that is a homology sphere. Thus, by passing to a finite cover, we may assume that  $H_1(M) = 0$ .

By residual finiteness, we can find a finite Galois cover  $M'$  of  $M$  that has order greater than 120. Consider the derived series of  $M'$  and the corresponding covers. By the stability condition we either get a finite cover  $N$  with positive first Betti number or we have a finite cover  $N$  with  $H_1(N) = 0$ . It suffices to rule out the latter case.

Assume that we are in the latter case and that  $N$  is a finite cover of  $M'$  corresponding to a derived subgroup of  $\pi_1(M')$  with  $H_1(N) = 0$ . As the derived subgroup  $\pi_1(N)$  is a characteristic subgroup of  $\pi_1(M')$ , and  $\pi_1(M')$  is in turn a normal subgroup of  $\pi_1(M)$ ,  $\pi_1(N)$  is a normal subgroup of  $\pi_1(M)$ . Thus  $N$  is a Galois cover of  $M$ . As the degree of the covering map is greater than 120, we get a contradiction to the above corollary. This completes the proof of the theorem.  $\square$

*Remark 0.5.* Our main result also follows from the results of Shalen-Wagreich [1] and Reznikov [2], but our proof is much more elementary.

#### REFERENCES

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