APPENDIX:THE CHORD ALGEBRA AND FUNDAMENTAL GROUPS

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In this appendix we shall show that the chord algebra is determined by the fundamental group and peripheral structure of a knot. This allows us to generalise the construction, for instance to embeddings of S^k in S^{k+2}. We shall show that this is a non-trivial invariant for embeddings of S^2 in S^4.

Assume that we are given a knot $K \in S^3$. It is straightforward to modify the constructions here to links. Let $N(K)$ denote a regular neighbourhood of K and let $M = S^3 - int(N(K))$ denote the knot exterior. Let $G = \pi_1(M)$ and $P = \pi_1(\partial N(K)) \subset G$. Let $\mu \subset T$ denote the meridian.

Let $\mathcal{C} = \mathcal{C}(G, P, \mu) = P \backslash G / P$. We shall show that $\mathcal{C}(G, P, \mu)$ can be identified with the set $\mathcal{C}_K$ of homotopy classes of chords. Consider the algebra generated by $\mathcal{C}$. We denote the class in $\mathcal{C}$ of an element $g \in G$ by $[g]$. Let $\mathcal{A}$ be the quotient of this by relations generated by the Skein relation

$$[\alpha\mu\beta] + [\alpha\beta] + [\alpha] \cdot [\beta]$$

where α and β are elements of G. and the relation

$$[e] = -2$$

Theorem 0.1. *$\mathcal{A}$ is isomorphic to the chord algebra $\mathcal{A}_k$ of the knot K.*

Proof. Firstly, as in the proof of the Van Kampen theorem, it is easy to see that the homotopy classes of chord for K is the same as the corresponding homotopy class of chords for the knot exterior M. Here a chord in the knot exterior M is a continuous path $\alpha : [0, 1] \to M$ with $\alpha^1(\partial M) = \{0, \}$. We shall identify this chord space with $\mathcal{C}$.

Fix a base point x_0 in ∂M. Given a chord α in M we pick paths β and γ in ∂M joining x_0 to $\alpha(0)$ and $\alpha(1)$. We associate to α the equivalence class $[\beta\alpha\gamma^{-1}] \in \mathcal{C}$. This is clearly independent of the choice of β and γ.

Further, homotopic chords give the same element of $\mathcal{C}$. For, suppose α_t, $t \in [0, 1]$ is a family of chords. Pick paths β and γ joining x_0 to $\alpha_0(0)$ and $\alpha_0(1)$ and extend these to a continuous family of paths β_t and γ_t joining x_0 to $\alpha_t(0)$ and $\alpha_t(1)$. For instance, we can take β_t to be the path consisting of β_0 and $\alpha_s(0)$,$s \in [0, t]$. By construction, the corresponding loops in $\pi_1(M, x_0)$ are homotopic. Thus, we have a map from the space of chords to $\mathcal{C}$.

To construct the inverse of the above map, observe that each element in G is represented by a chord, unique up to homotopy, and hence gives an element of the fundamental group. Further, it is easy to see that if $h, k \in P$, then g and hgk give homotopic chords and that this gives the inverse of the previous construction.

To obtain the algebra isomorphisms, observe that the Skein relations on the chords translate to the given algebraic relations on the algebra generated by $\mathcal{C}$. ☐

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The above construction associates a DGA to any triple (G, P, μ), where G is a group, P a subgroup and $\mu \in P$. Such a triple is associated to any co-dimension 2 embedding $K \hookrightarrow M$ of manifolds with trivial normal bundle. More generally we can consider a family of elements μ_i and impose a Skein relation for each of them.

In particular, we have an invariant of embeddings of S^2 in S^4. We shall show that this is a non-trivial invariant.

Theorem 0.2. *The chord DGA differentiates between the unknotted S^2 in S^4 and the spun knot obtained from the trefoil.*

Proof. In the case of the unknotted S^2, $G = P = \mathbb{Z}$ and hence the chord algebra is trivial. On the other hand, in the case of the spun knot corresponding to the trefoil, G is the fundamental group of the complement of the trefoil, μ is the element corresponding to the meridian of the trefoil, and P is a free abelian group generated by μ. It follows that the chord algebra surjects onto the chord algerbra corresponding to the trefoil knot, and hence is non-trivial. □

Viterbo has shown that the Floer homology of the tangent bundle of a manifold is isomorphic to the homology of its loop space. Here we have shown how the zero-dimensional contact homology can similarly be determined in terms of the algebraic topology of the space of chords. It seems possible that higher contact homology may have a description analogous to our description, except that one takes into account not just the homotopy classes of chords, but the full homotopy type of the space of chords.

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