

Monotone preimages of dendrites

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Abstract

A counterexample to a theorem of J. J. Charatonik and K. Omiljanowski giving sufficient conditions for a dendrite to be contained in all of its monotone preimages is given, and a corrected version of the theorem is presented. An alternative proof is provided for a characterization of dendrites monotonely equivalent to a universal dendrite that was originally proved using the erroneous result. Finally, in response to a question by J. J. Charatonik, it is shown that a dendrite contained in all of its monotone preimages must have a discrete set of ramification points.

1 Introduction and Preliminaries

In 1991, J. J. Charatonik and K. Omiljanowski [2, Theorem 6.12] gave a proof of the following result:

Theorem 1.1. Let Y be a dendrite such that:

- (1) All ramification points of Y are of order three.
- (2) The set $R(Y)$ is discrete.

If a dendrite X can be mapped onto Y by a monotone mapping, then X contains a homeomorphic copy of Y .

In Section 2, we give a counterexample to this theorem, and show that by adding one additional condition, the theorem can be recovered.

The original aim of Theorem 1.1 was to show that the property of being monotonely equivalent to a universal dendrite is characterized by containment of the Omiljanowski dendrite L_0 . The original proof depends on the fact that L_0

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satisfies the conditions of Theorem 1.1, but L_0 does *not* satisfy the additional condition in the corrected version of the theorem. Therefore, in Section 4, we give an alternative proof that L_0 satisfies the *conclusion* of Theorem 1.1.

The class of dendrites that satisfy the conclusion of Theorem 1.1 (are contained in all of their monotone preimages) are also interesting in their own right. Note the resemblance of Theorem 1.1 to the following result of G. T. Whyburn about open mappings [9, (2.4), p. 188]:

Theorem 1.2. Let D be a dendrite. For each compact space X and for every light, open, surjective mapping $f : X \rightarrow Y$ with $D \subseteq Y$, there exists a homeomorphic copy D' of D in X such that the restriction $f|_{D'} : D' \rightarrow f(D') = D$ is a homeomorphism.

It is known that Theorem 1.2 can be reversed, giving a characterization of dendrites. Thus, in 2004, J. J. Charatonik asked in [3, Problem 1.3] the following question:

Question 1.3. Let MP be the class of dendrites Y such that, if a dendrite X can be mapped monotonely onto Y , then X contains a homeomorphic copy of Y . Find an intrinsic characterization for MP .

In the same paper, J. J. Charatonik proved (Theorem 4.1) that $Y \in MP$ implies that each ramification point of Y is of order three. Then, in Question 4.2, he asked whether it is also necessary that the set of ramification points of Y be discrete, seeking a characterization. In Section 3, we answer this question in the affirmative. However, because of the additional condition needed to correct Theorem 1.1, this result does not characterize MP . Therefore, Question 1.3 remains open.

1.1 Preliminaries

In this paper, all spaces are assumed to be metric, and all ordinals countable.

We use the term *continuum* to mean a compact, connected space. A *dendrite* is a locally connected continuum that contains no simple closed curves, and we will assume that all dendrites under consideration are also nondegenerate. It is known that every subcontinuum of a dendrite is a dendrite [7, §51, VI, Theorem 4, p. 301].

A *mapping* means a continuous function. A mapping $f : X \rightarrow Y$ between continua is said to be *monotone* if the preimage of each point is connected.

For a dendrite X , the *order* of a subcontinuum Y of X is the number of

components of $X \setminus Y$. We define the order of a point p in X to be the order of the singleton $\{p\}$. Points of order 1 are called *end points*, and points of order three or more are called *ramification points*. The set of end points of a dendrite X is denoted by $E(X)$, and the set of ramification points is denoted $R(X)$.

For a dendrite X , we say that $p \in X$ is a *w-point* of X if $p \notin E(X)$, and there is a sequence of ramification points lying in one component of $X \setminus \{p\}$ which converges to p . The set of *w-points* of a dendrite X will be denoted $W(X)$. We say that p is a *double w-point* of X if there is a sequence of ramification points r_n of X lying in one component of $X \setminus \{p\}$, and another sequence of ramification points q_n of X lying in a different component, such that $r_n \rightarrow p$ and $q_n \rightarrow p$.

For points p and q in a dendrite X , we denote by pq the unique arc from p to q in X .

We will also require the following simple observations about the behavior of *w-points* under monotone maps. The first result follows from the proof of Theorem 6.1 in [1], a version of which we reproduce here for convenience.

Proposition 1.4. If X, Y are dendrites and $f : X \rightarrow Y$ is a monotone surjection, then for each *w-point* $w \in Y$, there is a *w-point* $w' \in X$ such that $f(w') = w$.

Proof. Since $w \in Y$ is a *w-point*, we may find a sequence of ramification points y_n of Y lying in a single component of $Y \setminus \{w\}$ such that $y_n \rightarrow w$. Since confluent (and thus monotone) maps on dendrites have the branchpoint covering property (see [4, Theorem I.1]), we may choose, for each n , a ramification point x_n of X such that $f(x_n) = y_n$. Possibly passing to a subsequence, we may assume that the sequence $\{x_n\}$ converges to a point $w' \in X$, and by continuity of f , we have $f(w') = w$.

We claim that all points of $\{x_n\}$ lie in the same component of $X \setminus \{w'\}$. Given two such points x_i and x_j , we consider the corresponding arc $y_i y_j$ in Y . Since y_i and y_j lie in the same component of $Y \setminus \{w\}$, this arc does not contain w . Thus, $f^{-1}(y_i y_j)$ is a continuum joining x_i and x_j in X and missing $w' \in f^{-1}(w)$, and so x_i and x_j lie in the same component of $X \setminus \{w'\}$.

Furthermore, w' cannot be an end point of X . Let C be the component of $Y \setminus \{w\}$ that contains the y_n , and define K to be the closure of $f^{-1}(C)$. This is a subcontinuum of X that intersects $f^{-1}(w)$ only at w' . Therefore, either $f^{-1}(w) = \{w'\}$ and so w' has the same order as w , or $f^{-1}(w)$ is nondegenerate, and w' separates $K \cup f^{-1}(w)$. In either case, w' has order at least two, and w' is a *w-point* as required. \square

Proposition 1.5. Let X, Y be dendrites, and suppose that Y contains uncountably many w -points. If X admits a monotone map onto Y , then there exists an arc $A \subseteq X$ containing uncountably many w -points.

Proof. We can find a countable collection $\{A_n\}$ of arcs in X such that every point of X that is not an end point is contained in some element of this collection. For example, take the collection of arcs whose end points are ramification points of X , and add to this collection all arcs from an isolated end point (of which there must be countably many) to the nearest point of $\overline{R(X)}$.

By Proposition 1.4, X contains uncountably many w -points. Since w -points are by definition not end points, the set $\bigcup_n A_n \cap W(X)$ is precisely $W(X)$. Therefore, for some n , $A_n \cap W(X)$ is uncountable, and A_n is the required arc. \square

2 Counterexample and Sufficient Conditions

Figure 1 demonstrates a dendrite Y satisfying the hypotheses of Theorem 1.1, and a monotone map f from a dendrite X onto Y . Note that Y cannot be embedded in X since Y contains a double w -point and X does not, so Y provides a counterexample to Theorem 1.1.

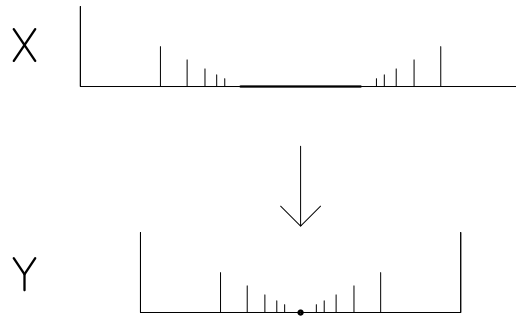


Fig. 1. Counterexample to Theorem 1.1

If we require that Y does not contain a copy of the range space of this counterexample (equivalently, does not contain a double w -point) in addition to the hypotheses of Theorem 1.1, we can show that this enlarged list of conditions on Y are sufficient for Y to be in MP .

Theorem 2.1. Let Y be a dendrite. If all of the following are true:

- (1) Each $r \in R(Y)$ is of order three;
- (2) $R(Y)$ is discrete;

(3) Y contains no double w -points,

then for any dendrite X admitting a monotone map onto Y , there is an embedding of Y into X (i.e., $Y \in MP$).

Proof. Let Y be a dendrite satisfying the conditions above, and let X be a dendrite admitting a monotone map f onto Y . We will construct an embedding $h : Y \rightarrow X$.

First, we will define $h(r)$ for each $r \in R(Y)$ in such a way that the arcs from $h(r)$ to any two points in different components of $X \setminus f^{-1}(r)$ intersect only at $h(r)$. We can find such a point in general because of condition (1).

By condition (1), for each point $r \in R(Y)$, the continuum $f^{-1}(r)$ is of order three in X . Thus $\text{bd}(f^{-1}(r))$ contains at most three points.

Case 1. $\text{bd}(f^{-1}(r)) = \{a\}$. Define $h(r) = a$.

Case 2. $\text{bd}(f^{-1}(r)) = \{a, b\}$. Each component of $X \setminus f^{-1}(r)$ must contain either a or b in its closure, so either a or b is contained in two. Without loss of generality, assume that this is the case for a , and define $h(r) = a$.

Case 3. $\text{bd}(f^{-1}(r)) = \{a, b, c\}$. Since $ab \cup ac$ is the union of two arcs, it is either an arc or a simple triod. If it is an arc, then define $h(r)$ to be the element of $\{a, b, c\}$ that is not an endpoint of that arc. If $ab \cup ac$ is a triod, then define $h(r)$ to be its vertex.

Now that h is defined on each point of $R(Y)$, it is uniquely determined on $\text{cl } R(Y)$ by continuity. This extension is well-defined because of conditions (2) and (3).

For each maximal free arc ab in Y whose end points are in $\text{cl } R(Y)$, define $h|_{ab}$ to be a homeomorphism with the arc from $h(a)$ to $h(b)$.

If the dendrite Y consists of a single arc, then it trivially satisfies the conclusion of the theorem. Otherwise, for each maximal free arc pq with $q \in E(Y)$, the point p must be in $\text{cl } R(Y)$. Let e be an arbitrary point of X such that $f(e) = q$. Define $h|_{pq}$ to be a homeomorphism with the arc between $h(p)$ and e .

By our definition of $h|_{R(Y)}$, the images of these homeomorphisms intersect only at points in the image of $\text{cl } R(Y)$, where they agree. Therefore, we conclude that the complete mapping h is a homeomorphism, and thus Y can be embedded in X . \square

3 Necessary Conditions

It is known that condition (1) of Theorem 2.1 is necessary for Y to be in MP [3, Theorem 4.1, p. 121]. In this section we will show that condition (2) is also necessary, as well as a limited version of condition (3). However, in Section 4, we will see that condition (3) cannot be fully reversed, and that we therefore still lack a characterization.

Observe that for a dendrite Y and a point $p \in Y$ of finite order n , there is a dendrite X and a monotone surjection $f : X \rightarrow Y$ such that the preimage of each point in $Y \setminus \{p\}$ is a single point, and the preimage of p is a simple n -od. We may also require that each end point of the n -od be a point of its boundary in Y . We will refer to the dendrite X so constructed as the *explosion* of Y at p . In a slight abuse of notation, we will identify points of $Y \setminus \{p\}$ with their preimages.

Theorem 3.1. Let $Y \in MP$. Then $R(Y)$ is discrete.

Proof. Suppose that $R(Y)$ is not discrete. Using an inverse limit, we will construct a dendrite Y_∞ with a discrete set of ramification points (so that Y_∞ does not contain a copy of Y), and which admits a monotone map onto Y . Note that by Theorem 4.1 in [3], we may assume that each ramification point of Y is of order three.

Since the set of ramification points of a dendrites is at most countable (see [7, §51, VI, Theorem 7]), we can find an enumeration $\{r_1, r_2, \dots\}$ of $R(Y)$. Let the first factor space Y_1 be a copy of Y , and for $n \in \mathbb{Z}^+$, define Y_{n+1} to be the explosion of Y_n at r_n (strictly speaking, at the iterated preimage of r_n). We define the bonding map f_n to be the monotone map shrinking the exploded simple triod back to r_n .

As the inverse limit of dendrites with monotone bonding maps, $Y_\infty = \varprojlim \{Y_n, f_n\}$ is a dendrite (see [6, Theorem 10.36, p. 180]).

We claim that for $r \in R(Y_\infty)$, and for each n , the projection $\pi_n(r)$ onto the n -th factor space is a ramification point of Y_n . Let C_1, C_2, C_3 be the respective closures of three distinct components of $Y_\infty \setminus \{r\}$. Since each C_i is nondegenerate, we may choose N sufficiently large so that the projection of each component onto Y_N is nondegenerate. Since the projection mapping π_N is monotone (see [8, Corollary to Theorem 10], [5, Theorem 5]), all pairwise intersections of the three subcontinua $\pi_N(C_1)$, $\pi_N(C_2)$, and $\pi_N(C_3)$ are precisely $\{\pi_N(r)\}$. Thus, $\pi_N(r)$ is a ramification point of Y_N . Since this argument can be repeated for all $n > N$, and since the forward image of each ramification point is also a ramification point, this demonstrates the claim.

We also claim that each ramification point of Y_∞ is isolated. For each $r \in R(Y_\infty)$, the projection of r onto the first coordinate is some ramification point r_m of Y , where m is the index of r_m in the original enumeration of $R(Y)$. By construction, the preimage of r_m in Y_{m+1} is a triod T which contains exactly one ramification point of Y_{m+1} . Furthermore, for each $n \geq m+1$, the bonding map f_n restricted to the iterated preimage of T is a homeomorphism. Thus, $\pi_n^{-1}(T)$ is a triod which is a neighborhood of r and which contains no other ramification points, so r is isolated.

Since $R(Y_\infty)$ is discrete and $R(Y)$ is not, we see that Y cannot be embedded into Y_∞ . Furthermore, the projection of Y_∞ onto $Y_1 = Y$ is monotone and surjective, so we achieve the desired contradiction. \square

By a similar argument, the following partial reverse of condition (3) in Theorem 2.1 may be obtained:

Theorem 3.2. If $Y \in MP$, then Y contains either no double w -points, or uncountably many.

Proof. Seeking a contradiction, suppose that $Y \in MP$ and the set of double w -points of Y is countable and non-empty.

Let $\{w_1, w_2, \dots\}$ be an enumeration of the double w -points of Y . As above, we let Y_1 be a copy of Y , and for $n \in \mathbb{Z}^+$, we define Y_{n+1} to be the explosion of Y_n at w_n . The bonding map f_n is the monotone map shrinking the exploded arc (by Theorem 3.1, each w_n is of order two) back to w_n .

Again, since the bonding maps are monotone, $Y_\infty = \varprojlim \{Y_n, f_n\}$ is a dendrite.

We claim that if Y_∞ contains a double w -point, it must project onto double w -points in each factor space. Suppose that $w \in Y_\infty$ is a double w -point, let H and K be subcontinua of Y_∞ whose intersection is the singleton $\{w\}$, and let $h_m \in H$ and $k_m \in K$ be sequences of ramification points, each converging to w . For each n , since π_n is monotone, we have $\pi_n(H) \cap \pi_n(K) = \pi_n(H \cap K) = \pi_n(\{w\})$. As in the proof of Theorem 3.1, the projection of each h_m and each k_m onto Y_n is a ramification point. Furthermore, since the preimage of each ramification point under each bonding map is a singleton, each h_m and k_m is mapped to a *distinct* ramification point of Y_n . Therefore, $\pi_n(H)$ and $\pi_n(K)$ are nondegenerate subcontinua of Y_n which intersect only at $\pi_n(w)$, and $\pi_n(h_m) \in \pi_n(H)$, $\pi_n(k_m) \in \pi_n(K)$ are sequences of ramification points converging to $\pi_n(w)$ (by continuity), so $\pi_n(w)$ is a double w -point of Y_n , and the claim is shown.

Since the iterated preimage of each double w -point of Y is eventually a free arc (containing no double w -points), there is no point of Y_∞ projecting onto

double w -points in each factor space, and we conclude that Y_∞ has no double w -points. Therefore Y cannot be embedded into Y_∞ , but since π_1 maps Y_∞ monotonely onto Y , this gives the desired contradiction. \square

4 The Omiljanowski Dendrite

We refer to the dendrite L_0 defined in [2, Example 6.9, p. 182] as the *Omiljanowski dendrite* (see Figure 2). As we will require the intermediate steps of the construction, we recall the definition here.

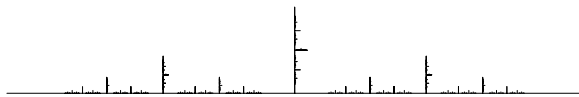


Fig. 2. Omiljanowski Dendrite

Definition 4.1. We begin with a unit straight line segment in the plane, which we denote by L_1 . Let a_1, a_4 be the end points of L_1 , and assume that L_1 is equipped with the natural ordering from a_1 to a_4 . Divide L_1 into three equal subsegments, and denote the middle subsegment by M . Let a_2, a_3 be the end points of M such that $a_1 < a_2 < a_3 < a_4$. In M , locate a thrice-diminished copy of the Cantor ternary set C . At the midpoint of each component K of $M \setminus C$, erect a straight line segment perpendicular to L_1 and with length equal to the length of K . We denote by L_2 the union of L_1 and the collection of all raised segments. We perform the same construction on each of the raised arcs: divide each such segment into three equal subsegments, embed in the middle segment M an appropriately diminished copy of the Cantor set C , and at the midpoint of each component K of $M \setminus C$ raise an arc perpendicular to K with length equal to that of K . We denote the union of L_2 and all attached arcs by L_3 . Continuing the process, we produce a nested sequence of dendrites $L_1 \subset L_2 \subset L_3 \subset \dots$ in the plane. We define $L_0 = \text{cl}(\bigcup_i L_i)$.

Note that by the construction of L_2 , the arc $a_1a_4 \subseteq L_2$ contains a Cantor set of w -points. In particular, the “irrational” points of the Cantor set which are not endpoints of deleted intervals correspond to double w -points of L_2 , so L_2 (and therefore L_0) does not satisfy condition (3) of Theorem 2.1. Our aim in this section is to show that L_0 is, regardless, a member of MP . This will allow us to recover a theorem characterizing dendrites which are monotonely equivalent to a universal dendrite, but it also shows that Theorem 2.1 cannot be reversed, and that we therefore do not have a characterization for MP .

As an intermediate step in showing that $L_0 \in MP$, we will show that $L_2 \in MP$. In the construction of a copy of L_2 in some monotone preimage, the

following technical lemma will roughly play the role of “deleting intervals” in the construction of the Cantor set:

Lemma 4.2. Let pq be an arc equipped with a convex metric d and with the natural order from p to q . If R is a countable subset of pq such that the sets $\text{cl}(B_\epsilon(p)) \cap \text{cl}(R)$ and $\text{cl}(B_\epsilon(q)) \cap \text{cl}(R)$ are uncountable for every $\epsilon > 0$, then there exist $a, b, c \in pq$ such that

- (1) $p < a < b < c < q$
- (2) $b \in R$
- (3) $d(a, c) \geq \frac{1}{3}d(p, q)$
- (4) For every $\epsilon > 0$, the sets $\text{cl}(B_\epsilon(a)) \cap pa \cap \text{cl}(R)$ and $\text{cl}(B_\epsilon(b)) \cap cq \cap \text{cl}(R)$ are uncountable

Proof. Let b be any element of R that is not an end point of the arc. Then there is some subarc $a'c' \subseteq pq$ such that $p < a' < b < c' < q$ and $d(a', c') \geq \frac{1}{3}d(p, q)$. The set $\{x \in pa' \mid aa' \cap \text{cl}(R) \text{ is countable}\}$ must be closed, so we may take a to be the least element. Similarly, we may take c to be the greatest element of $c'q$ such that $c'c \cap \text{cl}(R)$ is countable. Then a, b , and c are the required points. \square

We will also need the following lemma concerning subsets of an arc that are of the same form as $R(L_2)$.

Lemma 4.3. Let $\mathbb{C}_1, \mathbb{C}_2$ be subsets of an arc I that are each homeomorphic to the Cantor set, and each of which contain both end points of the arc. Let R_1 be a subset of I formed by choosing exactly one point from each component of $I \setminus \mathbb{C}_1$, and let R_2 be a set formed similarly from $I \setminus \mathbb{C}_2$. Then there exists a homeomorphism h of I onto itself such that $h(R_1) = R_2$.

Proof. Let h_1 be a homeomorphism of I onto itself such that $h_1(\mathbb{C}_1) = \mathbb{C}_2$.

We may construct a second homeomorphism h_2 from I onto itself as follows. For each component C of $I \setminus \mathbb{C}_2$, there must be exactly one point $c_1 \in C$ that is in the image of R_1 , and exactly one point $c_2 \in C$ that is in R_2 . Define $h_2|_{\text{cl}(C)}$ to be a homeomorphism of the arc $\text{cl}(C)$ onto itself that fixes the endpoints and takes c_1 to c_2 . For each point $x \in \mathbb{C}_2$, define $h_2(x) = x$.

Since h_2 takes $h_1(R_1)$ onto R_2 , the composition $h = h_2 \circ h_1$ is the required homeomorphism. \square

With these tools in hand, we are prepared to prove the first step of the main result:

Theorem 4.4. Let X be a dendrite that contains an arc pq such that $pq \cap W(X)$ is uncountable. Then X contains a copy of L_2 .

Proof. It is well known that every locally connected continuum has an equivalent convex metric, so we may assume that X is equipped with a convex metric d . We will also equip the arc pq with the natural order from p to q . Possibly taking a subarc, we may assume without loss of generality that p and q are not end points of X and that $\text{cl}(B_\epsilon(p)) \cap pq \cap W(X)$ and $\text{cl}(B_\epsilon(q)) \cap pq \cap W(X)$ are uncountable for every $\epsilon > 0$.

We will construct, by induction, a collection I_m of subarcs of pq indexed by all finite (possibly empty) binary sequences. We begin by setting I (the subarc indexed by the empty sequence) to be the entire arc pq .

Suppose that for some finite binary sequence $m = \langle m_1, m_2, \dots, m_n \rangle$, we have defined an arc $I_m \subseteq pq$ with end points $L_m < R_m$. Suppose furthermore that $\text{cl}(B_\epsilon(L_m)) \cap I_m \cap W(X)$ and $\text{cl}(B_\epsilon(R_m)) \cap I_m \cap W(X)$ are uncountable for every $\epsilon > 0$. By Lemma 4.2, we may choose $a_m, b_m, c_m \in I_m$ such that

- $L_m < a_m < b_m < c_m < R_m$,
- $b_m \in R(X)$,
- $d(a_m, c_m) \geq \frac{1}{3}d(L_m, R_m)$, and
- $\text{cl}(B_\epsilon(a_m)) \cap L_m a_m \cap W(X)$ and $\text{cl}(B_\epsilon(c_m)) \cap c_m R_m \cap W(X)$ are uncountable for every $\epsilon > 0$.

Define $I_{m,0} = L_m a_m$ and $I_{m,1} = c_m R_m$.

By induction, this defines a collection I_m of subarcs of pq and points $b_m \in I_m$, indexed by the set of all finite binary sequence. Let B be the collection of all b_m , and let \mathbb{C}_1 be the closure of the collection of endpoints of the intervals I_m . By construction, \mathbb{C}_1 is homeomorphic to the Cantor set, and B contains exactly one point from each component of $pq \setminus \mathbb{C}_1$.

However, by construction, $W(L_2)$ is also homeomorphic to the Cantor set, and $R(L_2)$ consists of exactly one point from each component of $a_2 a_4 \setminus W(L_2)$. Therefore, by Lemma 4.3, there is a homeomorphism h from $a_2 a_4$ onto pq that takes $R(L_2)$ onto B . Since $B \subseteq R(X)$ and $p, q \notin E(X)$, this h may be extended to an embedding of L_2 into X . \square

Finally, we can use this result to show that $L_0 \in MP$:

Theorem 4.5. If X is a dendrite admitting a monotone surjection f onto L_0 , then X contains a copy of L_0 .

Proof. We may assume without loss of generality that X is irreducible with

respect to mapping *onto* L_0 . In particular, we may take the subcontinuum $\bigcup_{x \in L_0} \text{irr}(\text{bd}(f^{-1}(x)))$.

Since the arc $a_1a_4 \subseteq L_0$ contains uncountably many w -points, by Proposition 1.5, X contains an arc with uncountably many w -points. Therefore, by Theorem 4.4, there is an embedding $e_2 : L_2 \rightarrow X$. Consider the image of an arbitrary ramification point $r \in L_2$. The point $e_2(r)$ is a ramification point, so we can find a component C_r of $X \setminus e_2(r)$ whose closure intersects $e_1(a_1a_4)$ only at $e_2(r)$. By our irreducibility assumption on X , the image of $\text{cl}(C_r)$ under f must be the closure of some component of $X \setminus \{r'\}$ for some $r' \in R(L_0)$. By construction of L_0 , then, the image $f(\text{cl}(C_r))$ is homeomorphic to L_0 . Therefore, applying Theorem 4.4 again, $\text{cl}(C_r)$ must contain a copy L_2^r of L_2 . Let $e_r : L_2 \rightarrow L_2^r$ be the corresponding embedding. Let p_r be the nearest point of $e_r(a_1a_4)$ to $e_2(r)$. By the construction of L_2 , at least one component of $\text{cl}(C_r) \setminus \{p_r\}$ contains a copy of L_2 such that the corresponding embedding \hat{e}_r sends the arc a_1a_4 into $e_r(a_1a_4)$. By symmetry, we may assume without loss of generality that the arc from $\hat{e}_r(a_1)$ to p_r intersects the arc $\hat{e}_r(a_1)\hat{e}_r(a_4)$ only at $\hat{e}_r(a_1)$. Since the end points $a_1, a_4 \in L_2$ are isolated, we see that the set $\hat{e}_r(L_2) \cup \hat{e}_r(a_1)p_r \cup p_re_2(r)$ is also homeomorphic to L_2 , and attaches to $e_2(L_2)$ in the same way that components of $L_3 \setminus L_2$ attach to L_2 .

Since r was arbitrary, we may repeat this process for each $r \in L_2$, and we see that in this way, e_2 may be extended to an embedding e_3 of L_3 into X . Again, for each ramification point $r' \in L_3 \setminus L_2$, a component of $X \setminus e_3(r')$ can be found whose closure maps onto a homeomorphic copy of L_0 . It is clear that this process can be repeated to find an increasing sequence of dendrites $e_n(L_n)$ in X such that $e_n(L_n)$ is homeomorphic to L_n for each n . The closure of the union of this collection is a copy of L_0 in X . \square

Observe that by Theorem 4.5, we can easily recover the result [2, Theorem 6.14, (64), p. 185] that a dendrite X is monotonely equivalent to a universal dendrite (a dendrite having a dense set of ramification points) if and only if X contains a copy of L_0 . This result was initially proved using Theorem 1.1, but since L_0 does not satisfy the hypotheses of the corrected Theorem 2.1, the additional Theorem 4.5 is required.

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References

- [1] D. Arévalo, W. J. Charatonik, P. Pellicer-Covarrubias, L. Simón, Dendrites with a closed set of end points, *Topology Appl.* 115 (2001), 1–17.
- [2] J. J. Charatonik, Monotone mappings of universal dendrites, *Topology Appl.* 38 (1991), 163–187.
- [3] J. J. Charatonik, Dendrites and monotone mappings, *Math. Pann.* 15 (2004), 115–125.
- [4] C. A. Eberhart, J. B. Fugate, G. R. Gordh Jr., Branchpoint covering theorems for confluent and weakly confluent maps, *Proc. Amer. Math. Soc.* 55 (1976), 409–415.
- [5] K. R. Gentry, Some properties of the induced map, *Fund. Math.* 66 (1970), 55–59.
- [6] S. B. Nadler Jr., *Continuum Theory*, M. Dekker, New York, Basel and Hong Kong, 1992.
- [7] K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, 1968.
- [8] E. Puzio, Limit mappings and projections of inverse systems, *Fund. Math.* 80 (1973), 57–73.
- [9] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ., Vol. 28, Amer. Math. Soc., Providence, RI, 1942; reprinted with corrections 1971.