Quotient Manifolds by Group Actions

"Fagprojekt"

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1 Introduction

In this paper we will look at quotient constructions obtained when a Lie group acts on a manifold. It is mainly built up as the section *Geometric Interpretation* of [HKLR87] but with added details and examples. We will start by looking at group actions on topological spaces and manifolds without further structure. This part is mainly based on [tD87] and [Aud91]. We will see how a group action gives us a natural quotient construction in topological spaces. When groups act on manifolds, however, we cannot be sure that the quotient is in fact a manifold as well. By restricting ourselves to actions of compact groups, we can say something about the structure of the quotient space and in some cases we actually get a manifold. We then look at manifolds with a Riemannian metric. The basic notions of Riemannian geometry including the results about connections in Section 4.1 are from [Lee97]. In the Riemannian case we can give the the quotient a natural Riemannian structure as well. We then turn to symplectic manifolds. The basic notions described in Section 3.1 are based on [MS98]. In general the quotient manifold will not be symplectic but we will see that when a moment map exists, we can construct a quotient on which the symplectic structure from the original manifold can be transferred in a natural way. Section 3.2 which builds on [GS84] defines the moment map. The basic properties of the Lie derivative can be found in [War83] and Example 3.12 is taken from [HK98]. Finally, we look at the Kähler/hyperkähler case. The basic definitions are mainly taken from [KN69]. We will see how in this case the Riemannian quotient and symplectic quotient combine to give a quotient which is Kähler/hyperkähler. Example 4.9 is taken from [Hit92].

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2 Group Actions

2.1 Topological Spaces

We start by looking at a topological space X with no further structure.

Definition 2.1. Let G be a topological group with unit element e and X a topological space. We say that G *acts on* X *from the left* (or simply G acts on X) if there exists a continuous map $\varphi : G \times X \to X$ (where $\varphi(g, x)$ is denoted by $g \cdot x$) such that

(i) $g \cdot (h \cdot x) = (gh) \cdot x$ for $g, h \in G, x \in X$

(ii) $e \cdot x = x$ for $x \in X$

If G acts on X, the set (X, φ) (often referred to as X) is called a *G-space*.

The set $G_x = \{g \in G \mid g \cdot x = x\}$ is called the *isotropy group* of x. It is easily seen that for each $x \in X$, G_x is a subgroup of G which is closed if X is Hausdorff (since $G_x = \varphi_x^{-1}(\Delta)$ where $\varphi_x : G \to X \times X$ is the map $g \mapsto (x, g \cdot x)$ and Δ is the diagonal). An action is said to be *free* if $g \cdot x = x$ implies $g = e$. In this case the isotropy group is trivial for all x. An action is called *proper* if the map $\theta : G \times X \to X \times X$ given by $(g, x) \mapsto (x, g \cdot x)$ is proper, i.e. θ is closed and for each $y \in X \times X$, $\theta^{-1}(y)$ is compact. If V is a G-space which is also a vector space, the action is called *linear* if the map $l_g : V \to V$ given by $v \mapsto g \cdot v$ is linear for all $g \in G$.

The set $G \cdot x = \{g \cdot x \in X \mid g \in G\}$ is called the *orbit* of x. The set of orbits is denoted by X/G and is called the *orbit space*. Let $\pi : X \to X/G$ be the projection $x \mapsto G \cdot x$. We can then make the orbit space into a topological space by providing it with the quotient topology with respect to π. It is this space that we call the *quotient* of X by the action of G. Note that π is also open: If U is an open subset of X, $\pi(U)$ is open, since $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$ and hence $\pi^{-1}(\pi(U))$ is open. If X and Y are G-spaces and $f : X \to Y$ is a map, f is called *equivariant* if $f(g \cdot x) = g \cdot f(x)$ holds for all $x \in X, g \in G$. An equivariant map $f : X \to Y$ induces a map between the orbit spaces $\bar{f} : X/G \to Y/G$ where $\bar{f}(G \cdot x) = G \cdot f(x)$ for $x \in X$. $\bar{f}$ is continuous if f is continuous.

Example 2.1. Let X be the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and consider the action of $G = \mathbb{R}$ given by $t \cdot (x, y) = (x + t, y + \lambda t)$ where λ is some fixed real number. If λ is rational, an orbit will be a closed loop on the torus. If, however, λ is irrational, each orbit will be a dense subset of X. In this case, the quotient X/G

will not be a Hausdorff space: Let U_1 and U_2 be open sets in X/G containing elements $G \cdot (x_1, y_1)$ and $G \cdot (x_2, y_2)$ respectively. Then $\pi^{-1}(U_1)$ is a non-empty open set in X and since $G \cdot (x_2, y_2)$ is dense in X, $\pi^{-1}(U_1)$ must contain some element $t \cdot (x_2, y_2)$ and therefore $U_1 \cap U_2 \neq \emptyset$.

As we saw in Example 2.1, the quotient X/G of a G-space is not necessarily Hausdorff, even if X itself is Hausdorff. We do, however, have the following result.

Proposition 2.2. *Let X be a G-space where the action of G is proper. Then the quotient space X/G is Hausdorff.*

Proof. Let $\theta : G \times X \to X \times X$ be the map $(g, x) \mapsto (x, g \cdot x)$. If θ is proper, it is by definition closed. Hence $R = \theta(G \times X)$ is a closed subset of $X \times X$. Let $\pi(x_1)$ and $\pi(x_2)$ be arbitrary points in X/G such that $\pi(x_1) \neq \pi(x_2)$. Then $(x_1, x_2) \in (X \times X)\backslash R$ and since R is closed, we can find an open subset $U \subset (X \times X)\backslash R$ such that $(x_1, x_2) \in U$. We may assume that $U = U_1 \times U_2$ where $x_i \in U_i$ and U_i is an open subset of X for $i = 1, 2$. Since π is open, $\pi(U_1)$ and $\pi(U_2)$ are open sets which separate $\pi(x_1)$ and $\pi(x_2)$. □

An example of a proper action is the action of a compact group on a Hausdorff space:

Proposition 2.3. *Let X be a Hausdorff G-space where G is compact. Then the action of G on X is proper.*

Proof. Let G be a compact group acting on X and let $\theta : G \times X \to X \times X$ be the map $(g, x) \mapsto (x, g \cdot x)$. Suppose we have a closed subset C of $G \times X$ and a net $(x_j, g_j \cdot x_j)$ in $\theta(C)$ which converges to $(x, y) \in X \times X$. g_j is a net in G and so it has a subsequence which converges to some $g \in G$ such that $(g, x) \in C$. So $(x_j, g_j \cdot x_j)$ has a subsequence which converges to $(x, g \cdot x)$ and since $X \times X$ is Hausdorff, $(x, y) = (x, g \cdot x) \in \theta(C)$. Therefore $\theta(C)$ is closed. Let $(x_0, y_0) \in X \times X$.

$$\theta^{-1}(x_0, y_0) = \{(g, x_0) \in G \times X \mid g \cdot x_0 = y_0\} = \begin{cases} \emptyset & \text{if } y_0 \notin G \cdot x_0 \\ g_0 G_{x_0} & \text{if } y_0 = g_0 \cdot x_0 \end{cases}$$

Since G_{x_0} is closed, it is compact and hence $\theta^{-1}(x_0, y_0)$ is compact. □

2.2 Manifolds

Now suppose that M is a smooth manifold and that G is a Lie group. We define an action of G on M exactly as in Definition 2.1 with the only difference that we now require φ to be smooth. In this case (M, φ) (or simply M) is called a G-*manifold*. As we saw in Example 2.1 we cannot in general expect the quotient M/G to be a manifold but in some cases we can still extract information about the structure of M/G from the structure of M. We will concentrate on the special case when G is compact. In this case, we have a fundamental result called the Slice Theorem which is stated below. To understand the Slice Theorem, we will need a few more preliminary results. The proof of the following result is omitted but can be found in [tD87].

Proposition 2.4. *Let M be a G-manifold where the action of G is proper and free. Then M/G has a unique differentiable structure and the projection $\pi : M \to M/G$ is a submersion.*

By Proposition 2.3, we get the following:

Corollary 2.5. *Let M be a G-manifold where G is compact and the action of G is free. Then M/G has a unique differentiable structure and the projection $\pi : M \to M/G$ is a submersion.*

Example 2.6. If H is a subgroup of a Lie group G which is a submanifold, we can define an action of H on G by $h \cdot g = gh^{-1}$ for $h \in H, g \in G$. This is clearly a free action. It is also proper: Let $\theta : H \times G \to G \times G$ be the map $(h, g) \mapsto (g, gh^{-1})$ and let C be a closed subset of $H \times G$. Suppose $(g_j, g_j h_j^{-1})$ is a net in $\theta(C)$ which converges to $(x, y) \in G \times G$. Then g_j converges to x and $h_j = (g_j^{-1} g_j h_j^{-1})^{-1}$ converges to $(x^{-1}y)^{-1} = y^{-1}x$ and so $(y^{-1}x, x) \in C$. Therefore $(x, y) = (x, x(y^{-1}x)^{-1}) \in \theta(C)$ and so $\theta(C)$ is closed. Let $(x, y) \in G \times G$. $\theta^{-1}(x, y) = \big\{(h, g) \in H \times G \mid g = x, gh^{-1} = y\big\} = \big\{(h, g) \in H \times G \mid g = x, h = y^{-1}g\big\} = \big\{(x, y^{-1}x)\big\}$ which is clearly compact. The orbits are exactly the cosets gH and the orbit space G/H is therefore the usual group theoretic quotient. Proposition 2.4 tells us that G/H has a unique differentiable structure and that the tangent space at each $gH \in G/H$ is T_gG/T_gH.

Lemma 2.7. *Let M and N be manifolds and $f : M \to N$ an injective immersion. If M is compact, f is an embedding.*

Proof. We need to show that $f^{-1} : f(M) \to M$ is continuous. Let $f(x) \in f(M)$. Let U be any closed neighborhood of $f(x)$. Then $f^{-1}(U)$ is closed and hence compact. Since the restriction of f to $f^{-1}(U)$ is continuous and bijective, it is a homeomorphism. In particular, f^{-1} is continuous at $f(x)$. $\square$

Proposition 2.8. *Let M be a G-manifold where G is a compact Lie group. Then for each $x \in M$, $G \cdot x$ is a submanifold of M which is diffeomorphic to G/G_x.*

Proof. Let $f_x : G \to M$ be the orbit map given by $g \mapsto g \cdot x$. Since G_x is a closed subgroup of G, it is a submanifold and hence by Example 2.6, G/G_x is a compact manifold where the tangent space at gG_x is T_gG/T_gG_x. f_x induces a map $\bar{f}_x : G/G_x \to M$ with image $G \cdot x$. By Lemma 2.7, it is enough to show that this is an injective immersion for each x, i.e. that $Df_x(g)$ is non-singular and has kernel T_gG_x for all $g \in G$. Since G is a Lie group, we only need to consider $Df_x(e) : T_eG \to T_xM$. By definition, $T_eG = \mathfrak{g}$ where $\mathfrak{g}$ denotes the Lie algebra of G. Let $\xi \in \mathfrak{g}$. $t \mapsto \exp(t\xi)$, $t \in \mathbb{R}$ is a curve in G through e with tangent ξ so $\gamma_t = f_x(\exp(t\xi))$ is a curve in M through x and

$$Df_x(e)(\xi) = \tfrac{d}{dt}\Big|_{t=0}\gamma_t = \tilde{\xi}(x)$$

where $\tilde{\xi}$ denotes the vector field induced on M by A. Since

$$\gamma_{t+s} = \exp((t+s)\xi) \cdot x = \exp(t\xi)\exp(s\xi) \cdot x = f_{\gamma_s}(\gamma_t)$$

we get that

$$Df_x(\gamma_s)(\xi) = \tfrac{d}{dt}\Big|_{t=0}\gamma_{t+s} = \tfrac{d}{dt}\Big|_{t=0} f_{\gamma_s}(\gamma_t) = Df_{\gamma_s}(e)(\xi) = \tilde{\xi}(\gamma_s).$$

Suppose $\tilde{\xi}(x) = 0$. Then γ_t and the constant curve x are both integral curves of $\tilde{\xi}$ through x and uniqueness of integral curves gives us that $\exp(t\xi) \cdot x = \gamma_t = x$ for all $t \in \mathbb{R}$. Hence, $\exp(t\xi) \in G_x$ for all $t \in \mathbb{R}$ and therefore $\xi \in T_eG_x$. □

Let M be a G-manifold and let H be a closed subgroup (and thus a submanifold) of G. $G \times M$ is an H-manifold by the action $(h, (g, x)) \mapsto (gh^{-1}, h \cdot x)$. The orbit space is denoted by $G \times_H M$. This action is clearly free. If G is a compact Lie group, H is also compact. Proposition 2.3 shows that in this case the action is also proper and hence by Proposition 2.4, $G \times_H M$ is a manifold and the projection $\pi : G \times M \to G \times_H M$ is a submersion. We can define an action of G on $G \times_H M$ by $g' \cdot [g, x] = [g' \cdot g, x]$ for $g' \in G, [g, x] = H \cdot (g, x) \in G \times_H M$. We are now able to state the Slice Theorem:

Theorem 2.9 (Slice Theorem). *Let M be a G-manifold where G is a compact Lie group. For all $x \in M$ there exists a unique vector space V_x, an open neighbourhood $U \subset M$ of $G \cdot x$ and an equivariant diffeomorphism $\psi_x : G \times_{G_x} V_x \to U$ such that $\psi_x([g, 0]) = g \cdot x$ and the action of G_x on V_x is linear.*

Let us examine the statement of the Slice Theorem more closely. If M is a G-manifold where G is compact and H is a closed subgroup of G, we get a differentiable map $\tau : G \times_H M \to G/H$ by defining $\tau([g,x]) = gH$ for all $[g,v] \in G \times_H M, gH \in G/H$. τ is well-defined for if $[g,x] = [g',x']$ then $(g',x') = (gh^{-1}, h \cdot x)$ for some $h \in H$ and so $g' = gh^{-1} \in gH$. τ is equivariant since $g' \cdot \tau([g,v]) = g' \cdot gH = g'gH = \tau([g'g,x]) = \tau(g' \cdot [g,x])$. Now suppose that M is a vector space V and that the action of H on V is linear. If $g'H \in G/H$ then $\tau^{-1}(g'H) = \{[g,v] \mid g \in g'H, v \in V\} = \{[g',v] \mid v \in V\}$. If we define $[g',v_1]+[g',v_2] = [g',v_1+v_2]$ and $\lambda[g',v_1] = [g',\lambda v_1]$ for $v_1, v_2 \in V, \lambda \in \mathbb{R}$ (which is well-defined since $[g',h\cdot v_1]+[g',h\cdot v_2] = [g',h\cdot v_1+h\cdot v_2] = [g',h\cdot(v_1+v_2)] = [g',v_1+v_2]$ and $[g',\lambda(h\cdot v_1)] = [g',h\cdot(\lambda v_1)] = [g',\lambda v_1]$ for $h \in H$), then $\tau^{-1}(g'H)$ becomes a vector space isomorphic to V. So $\tau : G \times_H V \to G/H$ is a vector bundle and we can identify the zero section $\{[g,0] \mid g \in G\}$ (which we can embed as a submanifold in $G \times_H V$) with G/H. Now let $H = G_x$ for some $x \in M$. According to Proposition 2.8, G/G_x is diffeomorphic to $G \cdot x$. The Slice Theorem states that we can extend this correspondance to a neighborhood around $G \cdot x$.

$$\begin{array}{ccc} G/G_x & \subset & G \times_{G_x} V_x \\ \Big\downarrow{\scriptstyle \bar{f}_x} & & \Big\downarrow{\scriptstyle \psi_x} \\ G \cdot x & \subset & U \end{array}$$

In other words each orbit $G \cdot x'$ close to $G \cdot x$ corresponds to an orbit $\{[g,v'] \mid g \in G\}$ in $G \times_{G_x} V_x$ and by fixing a $g \in G$, the vector space V_x/G_x parametrizes the orbits close to $G \cdot x$ in M/G.

In the case where the action of G on M is free, G_x is trivial for each $x \in M$, so $G \times_{G_x} V_x$ can be identified with $G \times V_x$. Therefore the differentiable structure of V_x can be transferred to M/G. This is a special case of the situation when the action of G_x on V_x is trivial, i.e. $g \cdot v = v$ for $g \in G_x, v \in V$. In this case the map $G \times_{G_x} V_x \to G/G_x \times V_x$ given by $[g,v] \mapsto (\tau[g,v], v)$ is a well-defined diffeomorphism and we can transfer the differentiable structure of V_x to M/G. Recalling that G/G_x is a manifold diffeomorphic to $G \cdot x$, we get that the tangent space at a point $G \cdot x \in M/G$ is isomorphic to $T_xM/T_xG \cdot x$. We call the space $T_xG \cdot x$ the *vertical space*. We conclude this chapter by looking at an example where the transfer of differentiable structure to the quotient goes wrong exactly where the action is not free.

Example 2.10. Let M be the complex projective plane $\mathbb{CP}^2$ and let G be the 2-torus $\mathbb{T}^2$. $\mathbb{CP}^2$ is the space of one-dimensional subspaces of $\mathbb{C}^3$, i.e. the elements of $\mathbb{CP}^2$ can be written as $[z_0, z_1, z_2] = \{\lambda(z_0, z_1, z_2) \mid \lambda \in \mathbb{C}\}$ where $(z_0, z_1, z_2) \in$

$\mathbb{C}^3\setminus\{0\}$. $\mathbb{T}^2$ can be viewed as $S^1\times S^1=\left\{(e^{i\theta_1},e^{i\theta_2})\in\mathbb{C}^2 \mid \theta_1,\theta_2\in[0,2\pi)\right\}$. We have an action of G on M given by

$$(e^{i\theta_1},e^{i\theta_2})\cdot[z_0,z_1,z_2]=[z_0,e^{i\theta_1}z_1,e^{i\theta_2}z_2]$$

for $(e^{i\theta_1},e^{i\theta_2})\in\mathbb{T}^2,[z_0,z_1,z_2]\in\mathbb{CP}^2$.

Since G is compact, the action is proper (cf. Proposition 2.3) and since M is Hausdorff, the quotient M/G is therefore a Hausdorff space (cf. Proposition 2.2). The action is, however, not free as can be seen in e.g. the point $x=[1,0,0]$ where $G_x=G$. Therefore we cannot expect the orbit space to be a manifold, and as we will see, it is in fact a manifold with boundaries and corners. In order to see this, we define a map $\mu:\mathbb{CP}^2\to\Delta$ where $\Delta=\{(x_1,x_2)\mid x_1,x_2\in[0,\infty)\text{ and }x_1+x_2\leq 1\}$ and

$$\mu([z_0,z_1,z_2])=\left(\frac{|z_1|^2}{\|z\|^2},\frac{|z_2|^2}{\|z\|^2}\right)\text{ where }\|z\|=\sqrt{|z_0|^2+|z_1|^2+|z_2|^2}.$$

If we let G act trivially on Δ, (i.e. $g\cdot x=x$ for $g\in G,x\in\Delta$) then μ is clearly equivariant. μ is surjective: Let $(x_1,x_2)\in\Delta$. Then for $z_0=\sqrt{1-(x_1+x_2)},z_1=\sqrt{x_1},z_2=\sqrt{x_2}$, we have that $\|z\|=1$ and so $\mu([z_0,z_1,z_2])=(|z_1|^2,|z_2|^2)=(x_1,x_2)$. Since μ is also continuous, it induces a surjective continuous map $\bar\mu:\mathbb{CP}^2/\mathbb{T}^2\to\Delta$. We claim that $\bar\mu$ is actually a homeomorphism. Since $\mathbb{CP}^2$ is compact, $\mathbb{CP}^2/\mathbb{T}^2$ is compact and since $\mathbb{CP}^2/\mathbb{T}^2$ is also Hausdorff, it suffices to show that $\bar\mu$ is injective. Let $[[z_0,z_1,z_2]],[[z_0',z_1',z_2']]$ be arbitrary points in $\mathbb{CP}^2/\mathbb{T}^2$ such that $\bar\mu\left([[z_0,z_1,z_2]]\right)=\bar\mu\left([[z_0',z_1',z_2']]\right)$, i.e.

$$\frac{|z_1|^2}{\|z\|^2}=\frac{|z_1'|^2}{\|z'\|^2},\frac{|z_2|^2}{\|z\|^2}=\frac{|z_2'|^2}{\|z'\|^2}. \tag{1}$$

Since $[\lambda\xi_0,\lambda\xi_1,\lambda\xi_2]=[\xi_0,\xi_1,\xi_2]$ for $[\xi_0,\xi_1,\xi_2]\in\mathbb{CP}^2$ and $\lambda=\|\xi\|^{-1}e^{-2\pi i\theta_0}$ where $\xi_0=|\xi_0|e^{i\theta_0}$, we may assume that $\|z\|=\|z'\|=1$ and that $z_0,z_0'\in[0,\infty)$. In this case (1) implies that

$$|z_1|=|z_1'|,|z_2|=|z_2'|,z_0=\sqrt{1-(|z_1|^2+|z_2|^2)}=\sqrt{1-(|z_1'|^2+|z_2'|^2)}=z_0'$$

If $z_j=|z_j|e^{i\theta_j},z_j'=|z_j'|e^{i\theta_j'}$ for $j=1,2$, we get:

$$z_j=|z_j|e^{i\theta_j}=e^{i(\theta_j-\theta_j')}|z_j'|e^{i\theta_j'}=e^{i(\theta_j-\theta_j')}z_j'\text{ for }j=1,2$$

and hence

$$[[z_0,z_1,z_2]]=\left[(e^{i(\theta_1-\theta_1')},e^{i(\theta_2-\theta_2')})\cdot[z_0',z_1',z_2']\right]=[[z_0',z_1',z_2']].$$

So $\bar\mu$ is injective. Therefore we can identify the quotient M/G with the simplex Δ which is a manifold with boundaries and corners.

$$\begin{tikzcd}
M \arrow[d, "\pi"'] \arrow[dr, "\mu"] & \\
M/G \arrow[r, "\cong", "\bar{\mu}"'] & \Delta
\end{tikzcd}$$

We can divide Δ into three parts: The interior Δ° which is a manifold, the edges $\Delta^e = \{(x_1, x_2) \mid x_1 = 0, x_2 = 0 \text{ or } x_2 = 1 - x_1\}$ and the corners $\Delta^c = \{(0,0), (0,1), (1,0)\}$. For each $(x_1, x_2) \in \Delta$ we have that

$$\mu^{-1}(x_1, x_2) = G \cdot [z_0, z_1, z_2] = \left\{[z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2] \mid \theta_1, \theta_2 \in [0, 2\pi)\right\}$$

where $[z_0, z_1, z_2] = [\sqrt{1 - (x_1 + x_2)}, \sqrt{x_1}, \sqrt{x_2}]$. We see that if $(x_1, x_2) \in \Delta^\circ$, $z_j \neq 0$ for $j = 0, 1, 2$ and therefore $\mu^{-1}(x_1, x_2) \cong \mathbb{T}^2 = G$. Since $G \cdot z = G/G_z$ for $z \in M$, this shows that Δ° corresponds to points where the action is free. If $(x_1, x_2) \in \Delta^e \backslash \Delta^c$, exactly one of the z_j's is 0 and so $\mu^{-1}(x_1, x_2) \cong S^1$. Finally, if $(x_1, x_2) \in \Delta^c$, exactly two of the z_j's are 0 and therefore $\mu^{-1}(x_1, x_2)$ is just a single point.

2.3 Riemannian Manifolds

We now turn our attention to manifolds with a Riemannian metric:

Definition 2.2. Let M be a smooth manifold. A 2-tensor field $g \in \mathcal{T}^2(M)$ is a *Riemannian metric* on M if

(i) $g(X, Y) = g(Y, X)$ for $X, Y \in T_x M, x \in M$ (i.e. g is *symmetric*)

(ii) $g(X, X) > 0$ for $X \in T_x M \backslash \{0\}, x \in M$ (i.e. g is *positive definite*)

If g is a Riemannian metric on M we call the pair (M, g) a *Riemannian manifold*. Note that a Riemannian metric assigns an inner product to each tangent space $T_x M$ of M given by $\langle X, Y\rangle_x = g(X, Y)$ for $X, Y \in T_x M$. Thus we can define the *norm* of $X \in T_x M$ to be $|X| = \sqrt{\langle X, X\rangle_x}$ and the *angle* between $X, Y \in T_x M \backslash \{0\}$ to be the unique $\theta \in [0, \pi]$ such that $\cos\theta = \frac{\langle X, Y\rangle_x}{|X||Y|}$. If (M, g) and $(\tilde{M}, \tilde{g})$ are Riemannian manifolds and $\varphi : M \to \tilde{M}$ is a diffeomorphism, φ is called an *isometry* if $\varphi^* \tilde{g} = g$, i.e. if $\tilde{g}(\varphi_* X, \varphi_* Y) = g(X, Y)$ for all $X, Y \in TM$.

Proposition 2.11. *Let M be a smooth manifold. Then M can be given a Riemannian metric.*

Proof. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be a locally finite atlas on M. For $X, Y \in T_x M$ where $x \in U_\alpha$ define $g_\alpha(X, Y) = \langle (\varphi_\alpha)_* X, (\varphi_\alpha)_* Y\rangle$ where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. Take a partition of unity f_α subordinate to the cover $(U_\alpha)_{\alpha \in A}$

and define $g(X,Y) = \sum_{\alpha \in A} f_\alpha(x) g_\alpha(X,Y)$ for $X, Y \in T_xM, x \in M$ (where g_α can be put equal to zero outside U_α). The sum converges at each point $x \in M$ since $f_\alpha(x) = 0$ except for a finite number of $\alpha's$. It is easily seen that g is a Riemannian metric on M. $\square$

Suppose (M,g) is a Riemannian manifold and that a Lie group G acts on M. Let $l_g : M \to M$ be the map $x \mapsto g \cdot x$ for $g \in G$. l_g has inverse $l_{g^{-1}}$ and is a diffeomorphism. We say that G *acts by isometries* if l_g is an isometry for each $g \in G$, i.e.

$$\langle X, Y \rangle_x = \langle Dl_g(x)(X), Dl_g(x)(Y) \rangle_{g \cdot x} \text{ for all } x \in M, g \in G.$$

If M is a G-manifold where G is compact, we can always give M a Riemannian metric with respect to which G acts by isometries. The rough idea is the following: By Proposition 2.11, M can be given a Riemannian metric g. We can then average over G using the Haar measure μ to obtain $\bar{g}(x) = \int_{h \in G} h \cdot (g(h^{-1} \cdot x)) d\mu$ which is the desired metric.

Suppose (M,g) is a Riemannian G-manifold where G acts by isometries, G is compact and the action is free. As we saw in Section 2.2, the quotient M/G becomes a manifold where the tangent space at each point $G \cdot x$ is $T_xM/T_xG \cdot x$. We will now see how we can transfer the Riemannian structure from M to M/G. Recall that the vertical space at $x \in M$ is the subspace $T_xG \cdot x \subset T_xM$. For each $x \in M$, define the *horizontal space* at x to be $(T_xG \cdot x)^\perp = \{X \in T_xM \mid g(X,Y) = 0 \text{ for all } Y \in T_xG \cdot x\}$. The derivative of the projection at $x \in M$ denoted by $D\pi(x)$ is an isomorphism

$$D\pi(x) : (T_xG \cdot x)^\perp \xrightarrow{\cong} T_xM/T_xG \cdot x$$

for each x. So each $\hat{X} \in T_{G \cdot x}M/G$ has a unique lift $X = (D\pi(x))^{-1}(\hat{X})$ to $(T_xG \cdot x)^\perp$. Now define $\hat{g}(\hat{X}, \hat{Y}) = g(X,Y)$ where X and Y are the lifts of $\hat{X}$ and $\hat{Y}$ respectively to $(T_{g \cdot x}G \cdot x)^\perp$ for some $g \cdot x \in G \cdot x$. This is well-defined, i.e. it is independent of the point $g \cdot x$: Suppose $X', Y' \in (T_{g \cdot x}G \cdot x)^\perp$ are lifts of X, Y respectively. Since $\pi \circ l_g(x) = \pi(g \cdot x) = \pi(x)$, $D\pi(g \cdot x) \circ Dl_g(x) = D\pi(x)$ so $D\pi(g \cdot x)(Dl_g(x)(X)) = D\pi(x)(X) = \hat{X}$ and uniqueness of lifts gives us that $X' = Dl_g(x)(X)$. Similarly, $Y' = Dl_g(x)(Y)$. Since G acts by isometries, $g(X', Y') = g(Dl_g(x)(X), Dl_g(x)(Y)) = g(X,Y)$. We conclude that $(M/G, \hat{g})$ is a Riemannian manifold.

3 Symplectic Manifolds

3.1 Basic Notions

Now we will consider manifolds with a symplectic structure. We start by defining the concept of a symplectic vector space:

Definition 3.1. Let V be a finite dimensional vector space and $\omega : V \times V \to \mathbb{R}$ a bilinear form on V. (V, ω) is a *symplectic vector space* if:

(i) $\omega(v, w) = -\omega(w, v)$ for $v, w \in V$ (i.e. ω is *skew-symmetric*)

(ii) $\omega(v, w) = 0$ for all $w \in V$ implies $v = 0$ (i.e. ω is *non-degenerate*)

Proposition 3.1. *Let (V, ω) be a symplectic vector space. Then V has even dimension and there exists a basis $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ such that*

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0 \text{ and } \omega(u_j, v_k) = \delta_{jk} \text{ for } j, k \in \{1, \ldots, n\} \tag{2}$$

Proof. We prove this by induction on the dimension of V which we denote by n. For $n = 1$, $V = \text{span}\{v\}$ for some $v \neq 0$. Since

$$\omega(v, \lambda v) = \lambda\omega(v, v) = -\lambda\omega(v, v) = -\omega(v, \lambda v) \text{ for all } \lambda \in \mathbb{R},$$

$\omega(v, \lambda v) = 0$ for all $\lambda \in \mathbb{R}$ which implies that $v = 0$. This is a contradiction and hence (V, ω) cannot be a symplectic vector space. For $n = 2$, the non-degeneracy of ω implies that there exist $u_1, v_1 \in V$ such that $\omega(u_1, v_1) = 1$. Then clearly (u_1, v_1) is a basis satisfying (2). Now assume that the statement is proved for all $k < n$. Non-degeneracy of ω again gives us u_1, v_1 such that $\omega(u_1, v_1) = 1$. Let $W = \text{span}\{u_1, v_1\}$ and let $\tilde{V} = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$. Then $(\tilde{V}, \omega)$ is a symplectic vector space of dimension $n - 2$. By the induction hypothesis, $n-2$ is even and there exists a basis $u_2, \ldots, u_n, v_2, \ldots, v_n$ satisfying $\omega(u_j, u_k) = \omega(v_j, v_k) = 0$ and $\omega(u_j, v_k) = \delta_{jk}$ for $j, k \in \{2, \ldots, n\}$. Then $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ is a basis for V satisfying (2). □

We call the basis obtained in Proposition 3.1 for a *symplectic basis*. We now define symplectic structures on manifolds:

Definition 3.2. Let M be a smooth manifold and ω a 2-form on M. ω is a *symplectic structure* (or *symplectic form*) on M if:

(i) (T_xM, ω_x) is a symplectic vector space for all $x \in M$

(ii) $d\omega = 0$ (i.e. ω is *closed*).

If ω is a symplectic structure on M, the pair (M, ω) is called a *symplectic manifold*. Because of Proposition 3.1, symplectic manifolds are always of even dimension. It can be shown that locally all symplectic manifolds look alike. This is due to Darboux's Theorem:

Theorem 3.2 (Darboux's Theorem). *Let M be a smooth manifold and let ω be a symplectic structure on M. Then on a neighborhood of each $x \in M$ there exist coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that*

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

Example 3.3. The most basic example of a symplectic manifold is Euclidean space $\mathbb{R}^{2n}$ with the symplectic structure

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the usual coordinates. If $X = \sum_{j=1}^{n}(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j})$ and $X' = \sum_{j=1}^{n}(a_j' \frac{\partial}{\partial x_j} + b_j' \frac{\partial}{\partial y_j})$ then

$$\omega_0(X, X') = \sum_{j=1}^{n}(a_j b_j' - a_j' b_j)$$

so ω_0 is skew-symmetric. Since $\omega_0(X, \frac{\partial}{\partial y_j}) = a_j$ and $\omega_0(X, -\frac{\partial}{\partial x_j}) = b_j$, $\omega_0(X, Y) = 0$ for all Y implies that $a_j = b_j = 0$ for $j = 1, \ldots, n$, i.e. $X = 0$ and therefore (i) in Definition 3.2 is satisfied. Finally, $\omega_0 = d\left(\sum_{j=1}^{n} x_j dy_j\right)$ so ω_0 is in fact a symplectic structure.

Example 3.4. Let M be a manifold and consider the cotangent bundle $\pi : T^*M \to M$. The differential of the projection at $x \in M$ is $D\pi(\varphi) : T_\varphi(T^*M) \to T_xM$ where $\varphi \in T_x^*M$. Let α be the 1-form on T^*M which to each $\varphi \in T^*M$ associates $\pi^*\varphi$, i.e.

$$\alpha(\varphi)(X) = \varphi(D\pi(\varphi)(X)) \text{ for } \varphi \in T_x^*M, x \in M, X \in T_\varphi(T^*M),$$

and let $\omega = -d\alpha$. We claim that ω is a symplectic structure on T^*M.

If $(x_1, \ldots, x_n)$ are local coordinates around $x \in M$, then any element $\varphi \in T_x^*M$ can locally be written as $\varphi = \sum_{j=1}^{n} y_j dx_j$. So any $\varphi \in T^*M$ is determined by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ which we will write in short as (x, y). The projection π becomes the map $(x, y) \mapsto x$ and its differential at (x, y) denoted by $D\pi(x, y)$ becomes $(\tilde{x}, \tilde{y}) \mapsto \tilde{x}$ where $\tilde{x}, \tilde{y}$ are the coordinates with respect to the bases

$(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ and $(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})$ respectively. So we get

$$\alpha(x,y)(\tilde{x},\tilde{y}) = \sum_{j=1}^{n} y_j dx_j(\tilde{x}) = \sum_{j=1}^{n} y_j \tilde{x}_j$$

and therefore in local coordinates

$$\alpha = \sum_{j=1}^{n} y_j dx_j \quad \text{and} \quad \omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

As we saw in Example 3.3, ω satisfies conditions (i) and (ii) of Definition 3.2 and is therefore a symplectic structure on T^*M.

3.2 Moment Maps

Now let (M,ω) be a symplectic manifold and let G be a Lie group acting on M. It is clear that we cannot expect the quotient M/G to be a symplectic manifold, even if it has a differentiable structure, e.g. the quotient could have odd dimension. In the following we will see how we in some cases have a quotient construction which can be given a natural symplectic structure by the use of a so-called moment map which we will define in the end of this section. First we introduce a few more notions.

If X is any vector field, we can define maps $X_t : M \to M$ by $x' \mapsto \gamma_{x'}(t)$ where $t \mapsto \gamma_{x'}(t)$ is an integral curve of X through x'. Recall the definition of the *Lie derivative* of a differential form β with respect to a vector field X which is given by

$$(L_X\beta)_x = \lim_{t\to 0} \frac{(X_t^*\beta)_x - \beta_x}{t} \qquad \text{for } x \in M$$

where $X_t^*\beta$ denotes the pull-back of β by X_t. Also recall that the *interior product* $i(X)\beta$ of a $k+1$-form β with a vector field X is defined by $i(X)(Y_1, \ldots, Y_k) = \beta(X, Y_1, \ldots, Y_k)$ for vector fields $Y_1, \ldots, Y_k$. Some basic properties of the Lie derivative are:

$$\begin{aligned} L_X\beta &= i(X)d\beta + d(i(X)\beta) & (3)\\ L_X(\beta(Y_1,\ldots,Y_k)) &= L_X\beta(Y_1,\ldots,Y_k) \\ &+ \sum_{i=1}^{k} \beta(Y_1,\ldots,Y_{i-1},L_XY_i,Y_{i+1},\ldots,Y_k) & (4) \end{aligned}$$

where $Y_1, \ldots, Y_k$ are vector fields.

As we saw in the proof of Proposition 2.8, if G is a Lie group acting on M, each $\xi \in \mathfrak{g} = T_eG$ induces a vector field $\tilde{\xi}$ on M where $\tilde{\xi}(x) = \frac{d}{dt}\big|_{t=0} \exp(t\xi)\cdot x$

for $x \in M$. The set of vector fields on M is a Lie algebra under the Lie bracket $[X,Y] = L_Y X = -L_X Y$ and $\xi \mapsto \tilde{\xi}$ is a Lie homomorphism. Now let ω be a symplectic structure on M. We say that an action *preserves the symplectic form* if

$$\omega(Dl_g(x)(X), Dl_g(x)(Y)) = \omega(X,Y) \text{ for } X, Y \in T_x M, x \in M, g \in G$$

where l_g as before denotes the map $x \mapsto g \cdot x$. If $X = \tilde{\xi}$ is a vector field induced by a group action which preserves ω then $X_t(x) = \exp(t\xi) \cdot x = l_{\exp(t\xi)}(x)$ and

$$\begin{aligned} (X_t^* \omega)(\zeta, \eta) &= \omega(DX_t(x)(\zeta), DX_t(x)(\eta)) \\ &= \omega(Dl_{\exp(t\xi)}(x)(\zeta), Dl_{\exp(t\xi)}(x)(\eta)) \\ &= \omega(\zeta, \eta) \quad \text{for } \zeta, \eta \in T_x M, x \in M \end{aligned}$$

So in this case

$$L_X \omega = 0 \tag{5}$$

and because of (3), we get

$$d(i(X)\omega) = 0, \tag{6}$$

i.e. $i(X)\omega$ is closed. If $i(X)\omega$ is also exact, i.e. if we can find a function H such that $i(X)\omega = dH$, then X is called a *Hamiltonian vector field* and we denote X by X_H. Now we would like to examine the criteria for obtaining Hamiltonian vector fields when we have an action which preserves ω. We will see that the obstruction lies in the cohomology of the Lie algebra denoted $H^k(\mathfrak{g})$ which we will define below.

Consider the action of a Lie Group G on itself given by left multiplication $g \cdot h = l_g(h) = gh$ for $g, h \in G$. We say that a vector field X on G is a *left-invariant vector field* if $X(h) = (Dl_{g^{-1}}(gh)X)(h)$ for all $h, g \in G$. A differential form β on G is called a *left-invariant form* if $l_g^*\beta = \beta$ for all $g \in G$. The set of left-invariant vector fields can be identified with the Lie algebra $\mathfrak{g}$ with the Lie bracket $[X,Y] = L_X Y$ and it can be shown that any left-invariant differential form is completely determined by its values on left-invariant vector fields. Thus we may identify the set of left-invariant k-forms with $\Lambda^k(\mathfrak{g}^*)$. Let $H^k(\mathfrak{g}) = \frac{\ker \delta_k}{\mathrm{im} \delta_{k-1}}$ be the kth cohomology group of $\mathfrak{g}$ where $\delta_k : \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+1}(\mathfrak{g}^*)$ is the map induced by $\beta \mapsto d\beta$. This makes sense since if β is a left-invariant form, then $d\beta$ is also left-invariant. Using (3) and (4), we can find explicit expressions for δ: If $\beta \in \Lambda^1(\mathfrak{g}^*)$, $i(X)\beta = \beta(X)$ is constant for any left-invariant vector field X and so if Y is also a left-invariant vector field,

$$\begin{aligned} 0 = L_X(\beta(Y)) &= L_X\beta(Y) + \beta(L_X Y) \\ &= (i(X)d\beta)(Y) + \beta([X,Y]) \\ &= d\beta(X,Y) + \beta([X,Y]). \end{aligned}$$

So

$$d\beta(X,Y) = -\beta([X,Y]). \tag{7}$$

Inductively, if β is a $k-1$-form we get:

$$d\beta(Y_1,\dots,Y_k) = \sum_{i<j}(-1)^{i+j}\beta([Y_i,Y_j],Y_1,\dots,\hat{Y}_i,\dots,\hat{Y}_j,\dots,Y_k). \tag{8}$$

Let $\psi : \Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ be the map $X \wedge Y \mapsto -[X,Y]$. Then , identifying $(\Lambda^2(\mathfrak{g}))^*$ with $\Lambda^2(\mathfrak{g}^*)$ (where $\varphi : \Lambda^2(\mathfrak{g}) \to \mathbb{R}$ is identified with $\tilde{\varphi} : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ where $\tilde{\varphi}(X,Y) = \varphi(X\wedge Y)$),(7) shows that δ_1 is the dual of ψ which we denote by ψ^*:

$$\psi^*(\beta)(X\wedge Y) = \beta(\psi(X\wedge Y)) = \beta(-[X,Y]) = d\beta(X,Y)$$

This shows that if $H^1(\mathfrak{g}) = 0$ then ψ^* is injective and hence ψ is surjective. Therefore any left-invariant vector field can be written as a linear combination $\sum_j [X_j,Y_j]$ where X_j, Y_j are left-invariant vector fields.

Now suppose X is a vector field induced by an action of G which preserves the symplectic form ω on a symplectic manifold (M,ω) and suppose X is a vector field of the form $\widetilde{[\zeta,\eta]}$ where $\zeta,\eta \in \mathfrak{g}$. Since $\xi \mapsto \tilde{\xi}$ is a Lie homomorphism from $\mathfrak{g}$ to the set of vector fields on M, $X = \widetilde{[\zeta,\eta]} = [\tilde{\zeta},\tilde{\eta}]$. By (5), $L_{\tilde{\zeta}}\omega = 0$ and by (6), $di(\tilde{\eta})\omega = 0$. Now using (3) and (4) we get that

$$\begin{aligned} i(X)\omega(Y) &= i([\tilde{\zeta},\tilde{\eta}])\omega(Y) \\ &= -i(L_{\tilde{\zeta}}\tilde{\eta})\omega(Y) \\ &= -\omega(L_{\tilde{\zeta}}\tilde{\eta},Y) - L_{\tilde{\zeta}}\omega(\tilde{\eta},Y) \\ &= -L_{\tilde{\zeta}}(\omega(\tilde{\eta},Y)) \\ &= -L_{\tilde{\zeta}}(i(\tilde{\eta})\omega)(Y) \\ &= -i(\tilde{\zeta})(di(\tilde{\eta})\omega)(Y) - d(i(\tilde{\zeta})i(\tilde{\eta})\omega)(Y) \\ &= -d(i(\tilde{\zeta})i(\tilde{\eta})\omega)(Y) \end{aligned}$$

So $i(X)\omega = dH$ where $H = -i(\zeta)i(\eta)\omega$ and therefore X is a Hamiltonian vector field.

If $H^1(\mathfrak{g}) = 0$, then by (7), all vector fields $\tilde{\xi}$ are of the a linear combination of vector fields of the form $\widetilde{[\zeta,\eta]}$ and therefore Hamiltonian. Let $\mathrm{Ham}(M)$ denote the set of Hamiltonian vector fields. $\mathrm{Ham}(M)$ is a Lie subalgebra of the set of vector fields on M. So in this case the map $\xi \mapsto \tilde{\xi}$ is a Lie homomorphism from $\mathfrak{g}$ to $\mathrm{Ham}(M)$. Suppose M is connected. Since for each smooth function H on M, the equation $i(X)\omega = dH$ uniquely determines a vector field $X = X_H$ on M, we have a map $\rho : C^\infty(M) \to H(M)$ where $\ker\rho$ is the constant functions. So we get an exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\rho} H(M) \longrightarrow 0$$

The set of smooth functions on M denoted $C^\infty(M)$ is also a Lie algebra under the Poisson bracket $[H_1, H_2] = \{H_1, H_2\} = -i(X_{H_1})i(X_{H_2})\omega$. Since $H \in \ker\rho$ implies that H is constant, we see that

$$[H_1, H_2] = 0 \text{ for } H_1 \in \ker\rho, H_2 \in C^\infty(M). \tag{9}$$

Furthermore, ρ is a Lie homomorphism: $\rho(\{H_1, H_2\})$ is the unique vector field X satisfying $i(X)\omega = d\{H_1, H_2\}$. Since $i(X_{H_2})\omega = dH_2$, (3) implies that

$$\begin{aligned} L_{X_{H_1}}(i(X_{H_2})\omega) &= L_{X_{H_1}}(dH_2) = d(i(X_{H_1})(dH_2)) \\ &= d(i(X_{H_1})i(X_{H_2})\omega) = -d\{H_1, H_2\} \end{aligned}$$

and so since by (5) $L_{X_{H_1}}\omega = 0$,

$$\begin{aligned} i(L_{X_{H_1}}X_{H_2})\omega &= i(L_{X_{H_1}}X_{H_2})\omega + i(X_{H_2})L_{X_{H_1}}\omega \\ &= L_{X_{H_1}}(i(X_{H_2})\omega) = -d\{H_1, H_2\} \end{aligned}$$

Therefore

$$\rho(\{H_1, H_2\}) = -L_{X_{H_1}}X_{H_2} = [\rho(H_1), \rho(H_2)]$$

So we get the following diagram where the first row is exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r, "\rho"] & H(M) \arrow[r] & 0 \\ & & & \mathfrak{g} \arrow[u, "\xi \mapsto \tilde{\xi}"'] & \end{tikzcd}$$

The following proposition is essential for the definition of the moment map:

Proposition 3.5. *Let (M, ω) be a connected symplectic G-manifold such that the action of G preserves ω where G is a connected Lie group. Suppose $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$. Then there exists a unique Lie homomorphism λ such that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r, "\rho"] & H(M) \arrow[r] & 0 \\
 & & & \mathfrak{g} \arrow[u, "\xi \mapsto \tilde{\xi}"'] \arrow[ul, dashed, "\lambda"] &
\end{tikzcd}$$

Proof. Let κ be the map $\xi \mapsto \tilde{\xi}$. Considering $\mathfrak{g}$ and $C^\infty(M)$ as vector spaces, we can find a linear map $\tau : \mathfrak{g} \to C^\infty(M)$ such that $\rho \circ \tau = \kappa$ (since $\mathfrak{g}$ has finite dimension). Now if $\xi, \eta \in \mathfrak{g}$, the element $c(\xi, \eta) = \tau[\xi, \eta] - [\tau(\xi), \tau(\eta)]$ lies in $\ker\rho = \mathbb{R}$ since

$$\rho(\tau[\xi,\eta] - [\tau(\xi),\tau(\eta)]) = \rho \circ \tau[\xi,\eta] - \rho[\tau(\xi),\tau(\eta)] = \kappa[\xi,\eta] - [\kappa(\xi),\kappa(\eta)] = 0.$$

So c defines an element of $\Lambda^2(\mathfrak{g}^*)$. Now (9) implies

$$[\tau([\xi,\eta]),\tau(\zeta)] - [[\tau(\xi),\tau(\eta)],\tau(\zeta)] = [c(\xi,\eta),\tau(\zeta)] = 0$$

so using (8) and the Jacobi identity we get:

$$\begin{aligned}
dc(\xi,\eta,\zeta) &= -\left(c([\xi,\eta],\zeta) + c([\zeta,\xi],\eta) + c([\eta,\zeta],\xi)\right) \\
&= -\big(\tau([[\xi,\eta],\zeta]) - [[\tau(\xi),\tau(\eta)],\tau(\zeta)] \\
&\quad +\tau([[\zeta,\xi],\eta]) - [[\tau(\zeta),\tau(\xi)],\tau(\eta)] \\
&\quad +\tau([[\eta,\zeta],\xi]) - [[\tau(\eta),\tau(\zeta)],\tau(\xi)]\big) = 0
\end{aligned}$$

Hence c represents an element $[c]$ of $H^2(\mathfrak{g})$. This element is independent of the choice of linear function τ: Suppose τ' is another linear function such that $\rho \circ \tau' = \kappa$. Then $\tau' = \tau + b$ for some linear function $b : \mathfrak{g} \to \ker\rho$. By (9), $[b(\xi), H] = 0$ for $\xi \in \mathfrak{g}, H \in C^\infty(M)$ and therefore

$$\begin{aligned}
c'(\xi,\eta) &= (\tau + b)[\xi,\eta] - [(\tau+b)(\xi),(\tau+b)(\eta)] \\
&= c(\xi,\eta) + b([\xi,\eta]) = (c - \delta b)(\xi,\eta).
\end{aligned}$$

So $[c'] = [c]$. If $H^2(\mathfrak{g}) = 0$, $[c] = 0$, i.e. $c = \delta b$ for some $b : \mathfrak{g} \to \ker\rho$ and $\lambda = \tau + b$ is a Lie homomorphism. Since $H^1(\mathfrak{g}) = 0$, this b is unique. □

Now we are finally able to define the moment map. Suppose that the Lie homomorphism $\lambda : \mathfrak{g} \to C^\infty(M)$ described in Proposition 3.5 exists, i.e. for each $\xi \in \mathfrak{g}$, $H_\xi = \lambda(\xi)$ is the unique function satisfying $i(\tilde{\xi})\omega = dH_\xi$. Then we can define a map $\mu : M \to \mathfrak{g}^*$ by

$$\mu(x)(\xi) = H_\xi(x) \text{ for } \xi \in \mathfrak{g}, x \in M. \tag{10}$$

This map is called the *moment map.*

There is a natural action of G on $\mathfrak{g}^*$: Let $\varphi_g : G \to G$ be the homomorphism $h \mapsto ghg^{-1}$. Then $\mathrm{Ad}g = D\varphi_g(e) : \mathfrak{g} \to \mathfrak{g}$ defines a map Ad: $G \to \mathrm{Aut}(\mathfrak{g})$ where

$\mathrm{Aut}(\mathfrak{g})$ is the automorphism group of $\mathfrak{g}$. This defines an action on $\mathfrak{g}$ given by $g\cdot\xi = \mathrm{Ad}g(\xi)$ for $\xi \in \mathfrak{g}, g \in G$ which in turn gives the action $(g\cdot\psi)(\xi) = \psi(g^{-1}\cdot\xi)$ for $\psi \in \mathfrak{g}^*$.

Proposition 3.6. *Let (M, ω) be a symplectic G-manifold on which a unique moment map μ exists. Then μ is equivariant.*

Proof. First we show that the map $\xi \mapsto \tilde{\xi}$ from $\mathfrak{g}$ into $\mathrm{Ham}(M)$ is equivariant where the action on $\mathrm{Ham}(M)$ is given by $(g \cdot X)(x) = Dl_{g^{-1}}(g\cdot x)(X(g\cdot x))$ for $g \in G, X \in \mathrm{Ham}(M), x \in M$:

$$\begin{aligned}
(g\cdot\tilde{\xi})(x) &= Dl_{g^{-1}}(g\cdot x)\left(\tilde{\xi}(g\cdot x)\right) \\
&= Dl_{g^{-1}}(g\cdot x)\left(\frac{d}{dt}\Big|_{t=0}\exp(t\xi)\cdot(g\cdot x)\right) \\
&= Dl_{g^{-1}}(g\cdot x)\left(\frac{d}{dt}\Big|_{t=0} g(g^{-1}\exp(t\xi)g)\cdot x\right) \\
&= Dl_{g^{-1}}(g\cdot x)\left(\frac{d}{dt}\Big|_{t=0}(g\exp(\mathrm{Ad}g^{-1}(t\xi))\cdot x\right) \\
&= \left(\widetilde{\mathrm{Ad}g^{-1}(\xi)}\right)(x) = (\widetilde{g\cdot\xi})(x)
\end{aligned}$$

Now we can show that $g^{-1}\cdot\mu(g\cdot x) = \mu(x)$ for $g \in G, x \in M$. Since

$$g^{-1}\cdot\mu(g\cdot x)(\xi) = \mu(g\cdot x)(\mathrm{Ad}g(\xi))$$

for all $\xi \in \mathfrak{g}$, (10) gives us that

$$\begin{aligned}
d(g^{-1}\cdot\mu(g\cdot x)(\xi)) &= d(\mu(g\cdot x)(\mathrm{Ad}g(\xi))) \\
&= i(\widetilde{\mathrm{Ad}g(\xi)}(g\cdot x))\omega \\
&= i(\widetilde{g^{-1}\cdot\xi}(g\cdot x))\omega \\
&= i(\tilde{\xi})\omega = d(\mu(x)\xi)
\end{aligned}$$

Hence $\mu(g\cdot x) = g\cdot\mu(x)$ for $g \in G, x \in M$. □

Example 3.7. Let $M = S^2 \subset \mathbb{R}^3$. At each vector $\underline{x} \in S^2$, the tangent space consists of the vectors in $\mathbb{R}^3$ perpendicular to $\underline{x}$. So if $\underline{v}, \underline{w} \in T_{\underline{x}}M$, $\underline{v}\times\underline{w} = s(\underline{v},\underline{w})\underline{x}$ for $s(\underline{v},\underline{w}) \in \mathbb{R}$ where $\times$ denotes the cross product on $\mathbb{R}^3$. $\omega(\underline{v},\underline{w}) = s(\underline{v},\underline{w})$ defines a symplectic structure on M. Now consider the action of G =SO(3) on M given by the usual action of SO(3) on $\mathbb{R}^3$. The Lie algebra of SO(3) denoted $\mathfrak{so}(3)$ is the set of 3 by 3 skew-symmetric matrices and can therefore be identified with $\mathbb{R}^3$: If $(a_{ij})_{1\le i,j\le 3} \in \mathfrak{so}(3)$ it can be identified with $\underline{a} = (a_{12}, a_{13}, a_{23})$. Under this identification, the Lie bracket becomes $[\underline{a},\underline{b}] = \underline{b}\times\underline{a}$. So according to (7), $\delta_1(\beta)(\xi,\eta) = -\beta(\eta\times\xi)$ for

$\xi, \eta \in H^1(\mathfrak{so}(3)^*) = \Lambda^1(\mathfrak{so}(3)^*) = \mathbb{R}^3$. Since any vector in $\mathbb{R}^3$ can be written in the form $\eta \times \xi$, $\delta_1(\beta) = 0$ implies $\beta = 0$, i.e. $\delta_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is injective and therefore bijective. Hence $H^1(\mathfrak{so}(3)) = \frac{\ker\delta_1}{\mathrm{im}\delta_0} = 0$ and $H^2(\mathfrak{so}(3)) = \frac{\ker\delta_2}{\mathrm{im}\delta_1} = \frac{\ker\delta_2}{\mathbb{R}^3} = 0$ and so by Proposition 3.5 a moment map exists.

Now let $\mu : S^2 \to \mathbb{R}^3$ be the map $\underline{x} \mapsto -\underline{x}$, i.e. as an element of $(\mathbb{R}^3)^*$, $\mu(\underline{x})(\underline{\xi}) = -\underline{\xi} \cdot \underline{x}$ for $\underline{\xi} \in \mathbb{R}^3$ where $\cdot$ is the usual inner product on $\mathbb{R}^3$. Therefore if $\underline{v} = \sum_{j=1}^3 v_j \frac{\partial}{\partial x_j} \in T_{\underline{x}}M$,

$$d\mu(\underline{x})(\underline{\xi})(\underline{v}) = \sum_{j=1}^{3} \tfrac{\partial \mu}{\partial x_j}\big|_{\underline{x}} dx_j|_{\underline{x}}(\underline{v}) = -\sum_{j=1}^{3} \xi_j dx_j(\underline{v}) = -\underline{\xi} \cdot \underline{v}$$

Each $\underline{\xi} \in \mathbb{R}^3$ induces the vector field $\widetilde{\underline{\xi}}$ given by

$$\begin{aligned} \widetilde{\underline{\xi}}(\underline{x}) &= \tfrac{d}{dt}\big|_{t=0} t[\underline{\xi}, \underline{x}] \\ &= \tfrac{d}{dt}\big|_{t=0} t(\underline{x} \times \underline{\xi}) = \underline{x} \times \underline{\xi}. \end{aligned}$$

and therefore

$$i(\widetilde{\underline{\xi}})\omega(\underline{v}) = \omega(\underline{x} \times \underline{\xi}, \underline{v}).$$

Since $\underline{x} \perp \underline{v}$, $(\underline{x} \times \underline{\xi}) \times \underline{v} = \underline{\xi}(\underline{x} \cdot \underline{v}) - \underline{x}(\underline{\xi} \cdot \underline{v}) = -(\underline{\xi} \cdot \underline{v})\underline{x}$ and so

$$i(\widetilde{\underline{\xi}})\omega(\underline{v}) = -\underline{\xi} \cdot \underline{v} = d\mu(\underline{x})(\underline{\xi})(\underline{v}).$$

This shows that μ is the moment map.

If $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$, Proposition 3.5 ensures us that the moment map exists and is unique. When, however, $H^1(\mathfrak{g}) \neq 0$, we cannot be sure that vector fields induced by the action are Hamiltonian and if $H^2(\mathfrak{g}) \neq 0$ we do not know whether the homomorphism λ exists. The following examples look at actions of $G = S^1$. S^1 has Lie algebra $\mathbb{R}$ and therefore $H^k(\mathfrak{g}) = \Lambda^k(\mathbb{R})$. In particular $H^2(\mathfrak{g}) = 0$ and $H^1(\mathfrak{g}) = \mathbb{R} \neq 0$.

Example 3.8. Let M be $\mathbb{C}^n$ which, identified with $\mathbb{R}^{2n}$, has the symplectic form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$. Consider the action of $G = S^1$ given by $e^{i\theta} \cdot z = e^{i\theta} z$. Let $\mu : \mathbb{C}^n \to \mathbb{R}$ be the map $z = (x_1, y_1 \ldots, x_n, y_n) \mapsto -\frac{1}{2}\|z\|^2 = -\frac{1}{2}\sum_{j=1}^n x_j^2 + y_j^2$ (so under the identification $\mathbb{R} = \mathfrak{g}^*$, $\mu(z)(\xi) = -\frac{1}{2}\xi\|z\|^2$ for $\xi \in \mathfrak{g}$). Hence

$$\begin{aligned} d\mu(z)(\xi) &= -\tfrac{1}{2}\xi \left(\sum_{j=1}^{n} \frac{\partial \mu}{\partial x_j}\bigg|_z dx_j\bigg|_z + \frac{\partial \mu}{\partial y_j}\bigg|_z dy_j\bigg|_z \right) \\ &= -\tfrac{1}{2}\xi \left(\sum_{j=1}^{n} 2x_j dx_j + 2y_j dy_j \right) \\ &= -\xi \sum_{j=1}^{n} x_j dx_j + y_j dy_j. \end{aligned}$$

Each $\xi \in \mathfrak{g}$ induces the vector field $\tilde{\xi}$ given by:

$$\begin{aligned}\tilde{\xi}(z) &= \tfrac{d}{dt}\big|_{t=0} e^{i\xi t} z = i\xi z \\ &= (-y_1\xi, x_1\xi, \ldots, -y_n\xi, x_n\xi)\end{aligned}$$

So if $Y = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j}$ then

$$\begin{aligned}i(\tilde{\xi})\omega_0(Y) &= \left(\sum_{j=1}^n dx_j \wedge dy_j\right)(\tilde{\xi}, Y) = \sum_{j=1}^n -\xi y_j b_j - (a_j \xi x_j) \\ &= -\xi\left(\sum_{j=1}^n x_j dx_j + y_j dy_j\right)(Y) \\ &= d\mu(z)(\xi)(Y).\end{aligned}$$

So in this case μ is a moment map.

Example 3.9. Now let M be the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which has the symplectic form $\omega = dx \wedge dy$ induced from $\mathbb{R}^2$. Identifying $G = S^1$ with $\mathbb{R}/\mathbb{Z}$ we can define an action of G on M by $a \cdot (x, y) = (x + a, y)$. If $\xi = 1 \in \mathfrak{g}$, it induces the vector field $\tilde{\xi} = \frac{\partial}{\partial x}$ and if $Y = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$

$$i(\tilde{\xi})\omega(Y) = (dx \wedge dy)(\tfrac{\partial}{\partial x}, Y) = b = dy(Y)$$

But the function y does not exist on $\mathbb{T}^2$. Therefore there cannot be a moment map for this action.

3.3 Symplectic Quotients

Suppose (M, ω) is a symplectic G-manifold and that a moment map $\mu : M \to \mathfrak{g}^*$ exists. In the this section we will see how we can use the moment map to create a quotient manifold which can be endowed with a natural symplectic structure. We will restrict ourselves to the case where G is compact and the action of G is free. Let $N = \mu^{-1}(\mu(x_0))$ for some $x_0 \in M$. We claim that N has dimension $2n - k$ where n and k are the dimensions of M and G respectively. Consider the differential of μ at the point $x \in M$ denoted by $D\mu(x) : T_xM \to \mathfrak{g}^*$ (where we have identified $T_{\mu(x)}\mathfrak{g}^*$ with $\mathfrak{g}^*$). By (10), this is given by:

$$D\mu(x)(Y)(\xi) = \omega(\tilde{\xi}(x), Y) \text{ for } Y \in T_xM, \xi \in \mathfrak{g}^* \tag{11}$$

So $Y \in \ker(D\mu(x))$ if and only if $\omega(\tilde{\xi}(x), Y) = 0$ for all $\xi \in \mathfrak{g}$. As we saw in the proof of Proposition 2.8, since the action of G is free, the vector field $\tilde{\xi}$ is given by:

$$\tilde{\xi}(x) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot x = Df_x(e)(\xi)$$

where $f_x : G \to M$ is the orbit map $g \mapsto g{\cdot}x$. Furthermore, $Df_x(e)$ is an injective map with image $T_xG{\cdot}x$. So $\ker D\mu(x) = \{Y \in T_xM \mid \omega(X,Y) = 0 \text{ for all } X \in T_xG \cdot x\}$. Since ω is non-degenerate, $\ker D\mu$ has dimension $2n - k$. Now it follows that $D\mu : TM \to \mathfrak{g}^*$ must be surjective. The Implicit Function Theorem therefore gives us that N is a manifold and that

$$T_xN = \ker D\mu(x) = \{Y \in T_xM \mid \omega(X,Y) = 0 \text{ for all } X \in T_xG \cdot x\}\,. \qquad (12)$$

In particular, N is of dimension $2n - k$.

Remark 3.10. Note that to obtain the result that N is a submanifold of dimension $2n - k$, we do not need the action to be free on all of M. As long as it is free on N, $D\mu(x)$ will be surjective for all $x \in N$ which gives the desired result.

Now suppose that $\mu(x_0) = \varphi_0$ where φ_0 is fixed under the action of G on $\mathfrak{g}^*$. Then since μ is equivariant, the action of G on M can be restricted to an action on N. Since the action is free and G is compact, we get a quotient N/G which is a manifold of dimension $2(n - k)$. In particular, N/G has even dimension. We now want to define a symplectic structure on N/G. Let $\pi : N \to N/G$ be the projection $x \mapsto G \cdot x$. Consider the 2-form on N/G defined by

$$\hat{\omega}(\hat{X}, \hat{Y}) = \omega(X, Y) \text{ for } \hat{X}, \hat{Y} \in T(N/G)$$

for $X \in (D\pi(x))^{-1}(\hat{X}), Y \in (D\pi(x))^{-1}(\hat{Y})$. $\hat{\omega}$ is well-defined: If X' and Y' are other lifts of $\hat{X}$ and $\hat{Y}$ then since $T_{G\cdot x}N/G \cong T_xN/T_xG \cdot x$, $X' = X + A$ and $Y' = Y + B$ for $A, B \in T_xG \cdot x$. Then (12) implies

$$\omega(X', Y') = \omega(X, Y) + \omega(A, Y) + \omega(X, B) + \omega(A, B) = \omega(X, Y).$$

$\hat{\omega}$ is non-degenerate: Suppose $\hat{\omega}(\hat{X}, \hat{Y}) = 0$ for all $\hat{Y} \in T_xN/T_xG \cdot x$. Then $\omega(X, Y) = 0$ for all $Y \in T_xN = \ker D\mu(x)$. So X must be of the form $X = \hat{\xi}(x)$ for some $\xi \in \mathfrak{g}$. Therefore $D\pi(x)(X) = 0$ and so $\hat{\omega}$ is non-degenerate. $\hat{\omega}$ is closed: Let $\iota : N \to M$ be the inclusion map. Observe that

$$\pi^*\hat{\omega}(X, Y) = \iota^*\omega(X, Y) \text{ for } X, Y \in T_xN.$$

Therefore

$$\pi^*(d\hat{\omega}) = d(\pi^*\hat{\omega}) = d(\iota^*\omega) = \iota^*(d\omega) = 0$$

and since $D\pi(x)$ is surjective for all $x \in M$, π^* is injective and therefore $d\hat{\omega} = 0$. So $\hat{\omega}$ is a symplectic form on N/G.

Example 3.11. As in Example 3.8 let $M = \mathbb{C}^n = \mathbb{R}^{2n}$ with symplectic structure ω_0 and let $G = S^1$ act on M by $e^{i\theta} \cdot z = e^{i\theta} z$. We saw that this gave us a moment map $\mu : \mathbb{C}^n \to \mathbb{R}$ where $\mu(z) = -\frac{1}{2}\|z\|^2$. Now for each $c > 0$, $\mu^{-1}(c) = \{z \in \mathbb{C}^n \mid \|z\| = \sqrt{c}\} = S^{2n-1}(\sqrt{c})$. So the quotient becomes $S^{2n-1}(\sqrt{c})/S^1 = \mathbb{CP}^{n-1}$

Example 3.12. Now we will see how we can use Example 3.7 to construct the so-called polygon space $\text{Pol}(\alpha)$ for $\alpha \in \mathbb{R}^n_+$. $\text{Pol}(\alpha)$ is the space of configurations (modulo rotation) in $\mathbb{R}^3$ of an n-polygon with edges of length $\alpha_1, \dots \alpha_n$ where $\alpha = (\alpha_1, \dots \alpha_n)$. Let $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{R}^n_+$ and for each $i \in \{1, \dots, n\}$, let $S^2_{\alpha_i}$ denote the sphere with radius α_i. Let $M = \prod_{i=1}^n S^2_{\alpha_i}$. We can think of each point $x = (x_1, \dots, x_n) \in M$ as a path which starts from the origin and has n successive steps x_i, each of length α_i. M has a natural symplectic structure given by $\omega((X_1, \dots, X_n), (Y_1, \dots, Y_n)) = \sum_{i=1}^n \omega_i(X_i, Y_i)$ where ω_i is the symplectic structure described in Example 3.7 on $S^2_{\alpha_i}$ for $i \in \{1, \dots, n\}$. Now let $G = \text{SO}(3)$ act on M by acting component-wise as described in Example 3.7. Clearly, the map $\mu(x) = -\sum_{i=1}^n x_i$ is a moment map for this action. $\mu(x)$ is minus the endpoint of the path x so $\mu^{-1}(0)$ are exactly the paths which form loops. Now we would like to obtain $\text{Pol}(\alpha)$ as the symplectic quotient $\mu^{-1}(0)/G$. We do, however, not know whether the action is free on $\mu^{-1}(0)$. The fixed points under the action of an element of $SO(3)$ are exactly the points on the axis of rotation. An object of $\mu^{-1}(0)$ can therefore only remain fixed if it lies in some one-dimensional subspace. We may assume that this is span{(1,0,0)}, i.e. $x_i \in \{(\pm\alpha_i, 0, 0)\}$ for $i \in \{1, \dots, n\}$. Now suppose that $\sum_{i=1}^n s_i\alpha_i = 0$ has no solutions for $s_i \in \{\pm 1\}$. Then the action of $SO(3)$ is free on $\mu^{-1}(0)$. For such α we therefore get $\text{Pol}(\alpha) = \mu^{-1}(0)/G$.

4 Kähler Manifolds

4.1 Basic Notions

Now we would like to look at Kähler manifolds. Kähler manifolds are both Riemannian and symplectic and we will see how we can use the quotient constructions described above to get a Kähler quotient. Before we can give the definition of a Kähler manifold, we will need to look at some further aspects of Riemannian geometry and we will also need to examine complex structures on manifolds.

Definition 4.1. Let M be a smooth manifold and let $\pi : E \to M$ be a vector bundle over M with smooth sections $\mathcal{E}(M)$. Let $(X,Y) \mapsto \nabla_X Y$ be a map $\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$. ∇ is called a *connection* if it satisfies:

(i) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for $f, g \in C^\infty(M)$.

(ii) $\nabla_X(sY_1 + tY_2) = s\nabla_X Y_1 + t\nabla_X Y_2$ for $s, t \in \mathbb{R}$

(iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)(Y)$ for $f \in C^\infty(M)$.

If $\mathcal{E}(M) = \mathcal{T}(M)$, ∇ is called a *linear connection.*

A connection is a local operator since it can be shown that if $x \in M$ and $X = X', Y = Y'$ on a neighborhood of x then $\nabla_X Y = \nabla_{X'} Y'$ at x.

Lemma 4.1. *Let M be a smooth manifold with a linear connection ∇. Let U be an open subset of M and let $E_1, \ldots, E_n$ be a frame on U. If $X = \sum_i X_i E_i$ and $Y = \sum_i Y_i E_i$ then*

$$\nabla_X Y = \sum_{ijk} (XY_k + X_i Y_j \Gamma_{ij}^k) E_k \tag{13}$$

where Γ_{ij}^k are the smooth functions on U defined by $\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$.

Proof. We have that:

$$\begin{aligned} \nabla_X Y &= \sum_j \nabla_X (Y_j E_j) \\ &= \sum_j (XY_j) E_j + Y_j \nabla_X E_j \\ &= \sum_j (XY_j E_j + \sum_i X_i Y_j \nabla_{E_i} E_j) \\ &= \sum_{ijk} (XY_k + X_i Y_j \Gamma_{ij}^k) E_k \end{aligned}$$

□

The n^3 functions Γ_{ij}^k are called the *Christoffel symbols* of ∇ and Lemma 4.1 shows that these completely determine ∇ on U.

Proposition 4.2. *Let M be a smooth manifold. Then there exists a linear connection ∇ on M.*

Proof. Let $(U_\alpha)_{\alpha \in A}$ be a locally finite atlas on M with coordinates $x_1^\alpha, \ldots x_n^\alpha$ on U_α. For each α, define the operator ∇^α on U_α by inserting $E_k = \frac{\partial}{\partial x_k}$ into (13) for some smooth functions Γ_{ij}^k on M:

$$\nabla_X^\alpha Y = \sum_{ijk} (XY_k + X_i Y_j \Gamma_{ij}^k) \tfrac{\partial}{\partial x_k}$$

Direct computation shows that ∇^α satisfies (i)-(iii) in Definition 4.1 on U_α. Now let $(f_\alpha)_{\alpha\in A}$ be a partition of unity subordinate to $(U_\alpha)_{\alpha\in A}$. Define

$$\nabla_X Y = \sum_\alpha f_\alpha \nabla^\alpha_X Y.$$

Direct computation again shows that this is a connection on M. ☐

A simple example of a linear connection is the Euclidean connection on $\mathbb{R}^n$ defined by

$$\nabla_X Y = \sum_{j=1}^{n}(XY_j)\tfrac{\partial}{\partial x_j} = \sum_{i,j} X_i \tfrac{\partial Y_j}{\partial x_i}\tfrac{\partial}{\partial x_j} \text{ for } X = \sum_i X_i \tfrac{\partial}{\partial x_j}, Y = \sum_j Y_j \tfrac{\partial}{\partial x_j} \quad (14)$$

This corresponds to $\Gamma^k_{ij} = 0$ for all i, j, k.

So far we have only defined linear connections on vector fields. The following result enables us to extend any linear connection to all tensor fields $\mathcal{T}_l^k M$.

Proposition 4.3. *Let M be a smooth manifold and $\overline{\nabla}$ be a linear connection on M. There exists a unique connection $\nabla : \mathcal{T}(M) \times \mathcal{T}_l^k(M) \to \mathcal{T}_l^k(M)$ for $k, l \in \mathbb{Z}$ such that*

(i) $\nabla_X Y = \overline{\nabla}_X Y$ *for* $Y \in \mathcal{T}(M)$

(ii) $\nabla_X f = Xf$ *for* $f \in \mathcal{T}^0(M)$

(iii) $\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$

(iv) $\nabla_X(\mathrm{tr}Y) = \mathrm{tr}(\nabla_X Y)$

where tr *denotes the trace operator.*

The connection defined in Proposition 4.3 satisfies the following properties:

$$\nabla_X(\beta(Y)) = \nabla_X\beta(Y) + \beta(\nabla_X Y) \text{ for } \beta \in \mathcal{T}^1(M), Y \in \mathcal{T}(M)$$

$$\begin{aligned}(\nabla_X F)(\beta_1, \dots, \beta_l, Y_1, \dots, Y_k) &= X(F(\beta_1, \dots, \beta_l, Y_1, \dots, Y_k))\\ &- \sum_{j=1}^{l} F(\beta_1 \dots, \nabla_X\beta_j, \dots, \beta_l, Y_1, \dots, Y_k)\\ &- \sum_{i=1}^{k} F(\beta_1 \dots, \beta_l, Y_1, \dots, \nabla_X Y_i, \dots, Y_k) \text{ for } F \in \mathcal{T}_l^k(M) \qquad (15)\end{aligned}$$

Now suppose we have a Riemannian manifold (M, g) and suppose that ∇ is a linear connection on M. We say that ∇ is *compatible with g* if

$$\nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all vector fields X, Y, Z, i.e. $\nabla_X g = 0$ for all X. The *torsion* of a linear connection ∇ is defined as

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

A linear connection is called *symmetric* if it has no torsion, i.e. $\tau \equiv 0$. The following theorem is a fundamental result of Riemannian Geometry:

Theorem 4.4. *Let (M, g) be a Riemannian manifold. Then there exists a unique linear connection ∇ on M which is*

(i) *compatible with g*

(ii) *symmetric.*

The connection defined in this way is called the *Riemannian connection* or the *Levi-Civita connection* of g.

Now we turn our attention to complex manifolds. A complex manifold is defined in a similar way as a smooth manifold, except the coordinate charts map into $\mathbb{C}^n$ and we require the coordinate changes to be holomorphic. It will, however, turn out to be useful to introduce the concept of almost complex structures which we will define below. First we look at complex structures on vector spaces: Let V be a real vector space and let $I : V \to V$ be an endomorphism of V. I is called a *complex structure* on V if $I^2 = -1$ where 1 denotes the identity map. We can then think of V as a complex vector space by defining

$$(a + ib)v = av + bIv \text{ for } a + ib \in \mathbb{C}, v \in V.$$

If $X_1, \ldots, X_n$ is a basis for V as a complex vector space then

$$X_1, \ldots, X_n, IX_1, \ldots, IX_n$$

is easily seen to be a basis for the underlying real vector space. In particular, V has even dimension. If V is a complex vector space, multiplication by i clearly defines a complex structure on the underlying real vector space.

Example 4.5. Let $M = \mathbb{C}^n$. By the identification of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$, we see that multiplication by i corresponds to $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (-y_1, \ldots, -y_n, x_1, \ldots, x_n)$ which is a complex structure I_0 on $\mathbb{R}^{2n}$. In the usual basis, I_0 can be written as

$$I_0 = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

where 1_n is the identity matrix of degree n.

Now let M be a real smooth manifold and let $I \in \mathcal{T}_1^1(M)$ be a $\binom{1}{1}$ tensor field on M. There is a one to one correspondance between $\binom{1}{1}$ tensors on a vector space and its endomorphisms: If $\varphi : V \to V$ is an endomorphism it defines a unique $\binom{1}{1}$ tensor, also denoted φ, by $\varphi(\beta, X) = \beta(\varphi X)$. So we consider I to be an endomorphism of T_xM for all $x \in M$. If I is in this way a complex structure on each T_xM, I is called an *almost complex structure* on M and (M, I) is called an *almost complex manifold*. Note that in this case M has even dimension. If I is an almost complex structure on M, we define the *torsion* of I to be the $\binom{1}{2}$ tensor field σ where

$$\sigma(X,Y) = 2([IX,IY] - [X,Y] - I[X,IY] - I[IX,Y])$$

for vector fields X, Y. It can be shown that

Theorem 4.6. *An almost complex manifold (M, I) is a complex manifold if and only if I has no torsion.*

In the following we will therefore consider complex manifolds as almost complex manifolds satisfying the integrability condition described above. In this case we call I the complex structure on M.

Using the notions of Riemannian connection and complex structure, we are now able to define a Kähler manifold:

Definition 4.2. Let (M, I, g) be a complex Riemannian manifold. Let ∇ denote the Riemannian connection of g. If

(i) $\nabla_X I = 0$ for all vector fields X

(ii) $g(IX, IY) = g(X,Y)$ for all vector fields X, Y

then (M, I, g) is a *Kähler manifold*

Let F be an endomorphism on each tangent space. Then

$$F(Y) = F(\sum_k Y_k \tfrac{\partial}{\partial x_k}) = \sum_k Y_k F \tfrac{\partial}{\partial x_k} = \sum_j (\sum_k Y_k dx_j(F \tfrac{\partial}{\partial x_k}) \tfrac{\partial}{\partial x_j}).$$

So $F(Y)$ has j'th coordinate $F(Y)_j = \sum_k Y_k dx_j(F\frac{\partial}{\partial x_k})$. The correspondence between $\binom{1}{1}$ tensors and endomorphisms mentioned above gives us that

$$\begin{aligned}(\mathrm{tr}(F \otimes Y))_j &= \sum_k F \otimes Y(dx_j, \tfrac{\partial}{\partial x_k}, dx_k) \\ &= \sum_k F(dx_j, \tfrac{\partial}{\partial x_k}) Y(dx_k) \\ &= \sum_k dx_j(F\tfrac{\partial}{\partial x_k}) Y_k = F(Y)_j\end{aligned}$$

So $F(Y) = \mathrm{tr}(F \otimes Y)$. Hence if I is the complex structure on M, we can use Proposition 4.3 to get:

$$\begin{aligned}\nabla_X(I(Y)) &= \nabla_X(\mathrm{tr}(I \otimes Y)) \\ &= \mathrm{tr}((\nabla_X I) \otimes Y + I \otimes (\nabla_X Y)) \\ &= (\nabla_X I)(Y) + I(\nabla_X Y).\end{aligned}$$

So the condition $\nabla_X I = 0$ for all X is equivalent to the condition that I commutes with ∇_X for all X.

If (M, I, g) is a Kähler manifold, we can define a 2-form ω by

$$\omega(X, Y) = g(IX, Y).$$

ω is clearly non-degenerate. ω is also skew-symmetric:

$$\omega(X, Y) = g(IX, Y) = g(IX, -I^2 Y) = g(X, -IY) = -g(IY, X) = -\omega(Y, X)$$

Now we claim that ω is closed. A calculation similar to the one used to obtain (8) shows that if β is any k-form,

$$\begin{aligned}d\beta(Y_1, \ldots, Y_k) &= \sum_{i=1}^{k} (-1)^{i+1} Y_i \beta(Y_1, \ldots, \hat{Y_i}, \ldots, Y_k) \\ &+ \sum_{i<j} (-1)^{i+j} \beta([Y_i, Y_j], Y_1, \ldots, \hat{Y_i}, \ldots, \hat{Y_j}, \ldots, Y_k)\end{aligned}$$

Therefore

$$\begin{aligned}d\omega(X, Y, Z) &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &- \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)\end{aligned} \tag{16}$$

We want to show that this is 0. Observe that since ∇ is compatible with g and since I commutes with ∇_X for all X:

$$\begin{aligned}X\omega(Y, Z) &= \nabla_X(\omega(Y, Z)) = \nabla_X(g(IY, Z)) \\ &= g(\nabla_X(IY), Z) + g(IY, \nabla_X Z) \\ &= g(I\nabla_X Y, Z) + g(IY, \nabla_X Z) \\ &= \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)\end{aligned}$$

Since ∇ is symmetric, i.e. $[X, Y] = \nabla_X Y - \nabla_Y X$ for all X, Y we get:

$$\begin{aligned}&-\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &\quad = -\omega(\nabla_X Y - \nabla_Y X, Z) + \omega(\nabla_X Z - \nabla_Z X, Y) - \omega(\nabla_Y Z - \nabla_Z Y, X) \\ &\quad = -(\omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)) + (\omega(\nabla_Y X, Z) + \omega(X, \nabla_Y Z)) \\ &\qquad - (\omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y)) \\ &\quad = -X\omega(Y, Z) + Y\omega(X, Z) - Z\omega(X, Y).\end{aligned}$$

Inserting this into (16) gives us $d\omega(X, Y, Z) = 0$. Hence ω is a symplectic form. We call ω the *Kähler form* on M.

4.2 Kähler Quotients

Let (M, I, g) be a Kähler manifold with Kähler form ω. Let G be a compact Lie group acting freely on M preserving both ω and g. Then as we saw in sections 2.3 and 3.3, when a moment map μ exists, we can transfer the Riemannian and symplectic structure to the quotient N/G where $N = \mu^{-1}(0)$. We denote these by $\hat{g}$ and $\hat{\omega}$ respectively. We can define a complex structure on N/G by

$$\hat{I}(\hat{X}) = \widehat{IX} \tag{17}$$

where $X \in T_x M$ is the unique horizontal lift of $\hat{X} = D\pi(x)(X)$ and where $\widehat{IX} = D\pi(x)(IX)$ (cf. Section 2.3). We have to check that this is well-defined. First we ensure that if $X \in (T_x G \cdot x)^{\perp}$ then $IX \in TN$: Recall from (12) that $T_x N = \{X' \mid \omega(X', Y) = 0 \text{ for all } Y \in T_x G \cdot x\}$. If $Y \in T_x G \cdot x$, then $\omega(IX, Y) = -g(X, Y) = 0$ so $IX \in TN$. Secondly we have to see that (17) is independent of the lift. Let X' be another lift of $\hat{X}$. Then as we saw in Section 2.3, $X' = Dl_h(x)(X)$ for some $h \in G$. Since for all Y

$$\begin{aligned}
&g(I(Dl_h(x)(X)) - Dl_h(x)(IX), Dl_h(x)(Y)) \\
&\quad = \omega(Dl_h(x)(X), Dl_h(x)(Y)) - g(Dl_h(x)(IX), Dl_h(x)(Y)) \\
&\quad = \omega(X, Y) - g(IX, Y) = 0,
\end{aligned}$$

we get that $I(X') = I(Dl_h(x)(X)) = Dl_h(x)(IX)$. Therefore

$$\widehat{IX'} = D\pi(h \cdot x)(IX') = D\pi(h \cdot x)(Dl_h(x)(IX)) = D\pi(x)(IX) = \widehat{IX}.$$

Clearly, $\hat{I}^2 = 1$ so all that is left to show is that $\hat{I}$ is torsion-free. Let $\hat{X}$ be a vector field on N/G and X be the unique horizontal lift of $\hat{X}$, i.e. $D\pi(x)(X(x)) = \hat{X}(\pi(x))$ for $x \in M$. If $f \in C^{\infty}(N/G)$ then

$$\hat{X} f(\pi(x)) = \hat{X}(\pi(x))(f) = D\pi(x)(X(x))(f) = X(f \circ \pi)(x)$$

i.e. $(\hat{X} f) \circ \pi = X(f \circ \pi)$. Taking the Lie bracket with another vector field $\hat{Y}$ therefore yields

$$\begin{aligned}
([\hat{X}, \hat{Y}]f) \circ \pi &= (\hat{X}(\hat{Y} f)) \circ \pi - (\hat{Y}(\hat{X} f)) \circ \pi \\
&= X((\hat{Y} f) \circ \pi) - Y((\hat{X} f) \circ \pi) \\
&= X(Y(f \circ \pi)) - Y(X(f \circ \pi)) \\
&= [X, Y](f \circ \pi).
\end{aligned}$$

Therefore

$$D\pi(x)([X,Y](x)) = [\hat{X},\hat{Y}](\pi(x)) \tag{18}$$

which we can write in short as $\widehat{[X,Y]} = [\hat{X},\hat{Y}]$. So the torsion of $\hat{I}$ becomes

$$\begin{aligned}\hat{\sigma}(\hat{X},\hat{Y}) &= 2([\hat{I}\hat{X},\hat{I}\hat{Y}] - [\hat{X},\hat{Y}] - \hat{I}[\hat{X},\hat{I}\hat{Y}] - \hat{I}[\hat{I}\hat{X},\hat{Y}]) \\ &= \widehat{\sigma(X,Y)} = \hat{0} = 0.\end{aligned}$$

Hence $\hat{I}$ is a complex structure on N/G. We would like to see that $(N/G,\hat{I},\hat{g})$ is a Kähler manifold with Kähler form $\hat{\omega}$. From the definition of $\hat{g}$ and $\hat{I}$, it is clear that condition (ii) in Definition 4.2 is satisfied. We need to check that $\widehat{\nabla}_{\hat{X}}$ commutes with $\hat{I}$ for all vector fields $\hat{X}$ on N/G where $\widehat{\nabla}$ is the Riemannian connection with respect to $\hat{g}$.

First we examine what the Riemannian connection looks like on N. We can split each tangent space of the points $x \in N$ into a direct sum $T_xM = T_xN \oplus (T_xN)^{\perp}$. The spaces $(T_xN)^{\perp}$ constitute a vector bundle over N which we call the *normal bundle*. The orthogonal projections at each point x give us the *tangential projection* denoted by $\pi^{\top} : TM\big|_N \to TN$ and the *normal projection* denoted by $\pi^{\perp} : TM\big|_N \to TN^{\perp}$. Let $\widetilde{\nabla}$ be the Riemannian connection on M. If X, Y are vector fields on N we can extend them to vector fields on M which we also denote by X, Y. Decomposing $\widetilde{\nabla}_X Y$ at each point $x \in N$ we get

$$\widetilde{\nabla}_X Y = \pi^{\top}(\widetilde{\nabla}_X Y) + \pi^{\perp}(\widetilde{\nabla}_X Y).$$

Define $\nabla : \mathcal{T}(N) \times \mathcal{T}(N) \to \mathcal{T}(N)$ by $\nabla_X Y = \pi^{\top}(\widetilde{\nabla}_X Y)$. It can be shown that ∇ is well-defined and in fact a connection on N. ∇ is symmetric:

$$\begin{aligned}\nabla_X Y - \nabla_Y X &= \pi^{\top}(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X) \\ &= \pi^{\top}[X,Y] = [X,Y].\end{aligned}$$

∇ is compatible with g: Let X, Y, Z be tangent fields on N. Then

$$\begin{aligned}\nabla_X(g(Y,Z)) &= Xg(Y,Z) \\ &= g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z) \\ &= g(\pi^{\top}\widetilde{\nabla}_X Y, Z) + g(Y, \pi^{\top}\widetilde{\nabla}_X Z) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z).\end{aligned}$$

Thus uniqueness of the Riemannian connection gives us that ∇ is the Riemannian connection on N.

Now we would like to look at the Riemannian connection on N/G with respect to $\hat{g}$. As we saw in Section 2.3, for each $x \in N$, $D\pi(x)$ is an isomorphism from the horizontal space $H_x = (T_xG \cdot x)^\perp$ to the tangent space $T_{G\cdot x}N/G = T_xN/T_xG \cdot x$. The horizontal spaces constitute a subbundle $H \subset TN$. We can now pull back the Riemannian connection $\widehat{\nabla}$ on N/G to a connection $\pi^*\nabla$ on H:

$$(\pi^*\nabla)_X Y = \widehat{\nabla}_{\hat{X}}\hat{Y}.$$

We claim that this is the connection ∇^H obtained by orthogonal projection of the Riemannian connection ∇ on TN to H. If π_H denotes the projection $TN \to H$ then

$$\nabla^H_{X'}Y' = \pi_H(\nabla_X Y)$$

for $X' = \pi_H(X), Y' = \pi_H(Y)$. When we identify H with $T(N/G)$, we can think of ∇^H as a connection on N/G:

$$\nabla^H_{\hat{X}}\hat{Y} = \widehat{\nabla^H_{X'}Y'} = \pi_H\widehat{(\nabla_X Y)}$$

where X', Y' are horizontal lifts of $\hat{X}, \hat{Y}$ respectively and $X \in \pi_H^{-1}(X'), Y \in \pi_H^{-1}(Y')$. Our claim is that in this way, ∇^H is symmetric and compatible with $\hat{g}$. Because of (18) we get

$$\begin{aligned}
\nabla^H_{\hat{X}}\hat{Y} - \nabla^H_{\hat{Y}}\hat{X} &= \pi_H\widehat{(\nabla_X Y)} - \pi_H\widehat{(\nabla_X Y)} \\
&= \nabla_X\widehat{Y - \nabla_Y}X = \widehat{[X,Y]} = [\hat{X}, \hat{Y}].
\end{aligned}$$

So ∇^H is symmetric. Since

$$\begin{aligned}
\nabla^H_{\hat{X}}(\hat{g}(\hat{Y}, \hat{Z}) &= \nabla^H_{X'}(g(Y', Z')) = X'g(Y', Z') \\
&= g(\nabla_{X'}Y', Z') + g(Y', \nabla_{X'}Z') \\
&= g(\nabla^H_{X'}Y', Z') + g(Y', \nabla^H_{X'}Z') \\
&= \hat{g}(\nabla^H_{\hat{X}}\hat{Y}, \hat{Z}) + \hat{g}(\hat{Y}, \nabla^H_{\hat{X}}\hat{Z})
\end{aligned}$$

∇^H is compatible with $\hat{g}$. We conclude that $\widehat{\nabla} = \nabla^H$.

Now we would like to see that $\hat{\nabla}_{\hat{X}}$ commutes with $\hat{I}$ for all vector fields $\hat{X}$ on N/G. Observe that

$$\begin{aligned}
T_xN &= \{X \mid \omega(X, Y) = 0 \text{ for all } Y \in T_xG \cdot x\} \\
&= \{X \mid g(IY, X) = 0 \text{ for all } Y \in T_xG \cdot x\} = I(T_xG \cdot x)^\perp
\end{aligned}$$

Therefore $(T_xG \cdot x \oplus I(T_xG \cdot x))^\perp = (T_xG \cdot x)^\perp = H_x^\perp$. So the complement of H and therefore H itself is a complex vector bundle and hence I commutes with

π_H (where we think of π_H as a mapping from TM). Let $\hat{X}, \hat{Y}$ be vector fields on N/G and let X, Y be the horizontal lifts of $\hat{X}, \hat{Y}$ respectively. Then

$$\begin{aligned}\widehat{\nabla}_{\hat{X}} \hat{I}(\hat{Y}) &= \pi_H(\widehat{\nabla_X(IY)}) = \pi_H(\widehat{\tilde{\nabla}_X(IY)}) \\ &= \pi_H(\widehat{I\tilde{\nabla}_X Y}) = I(\pi_H(\widehat{\nabla_X Y})) \\ &= \hat{I}(\widehat{\nabla_{\hat{X}} \hat{Y}}).\end{aligned}$$

Therefore $(N/G, \hat{I}, \hat{g})$ is a Kähler manifold and the quotient symplectic structure $\hat{\omega}$ is the Kähler form:

$$\hat{\omega}(\hat{X}, \hat{Y}) = \omega(X, Y) = g(IX, Y) = \hat{g}(\widehat{IX}, \hat{Y}) = \hat{g}(\hat{I}\hat{X}, \hat{Y}).$$

Example 4.7. As in example (3.11) we look at the action of $G = S^1$ on $M = \mathbb{C}^n$. $\mathbb{C}^n$ has the complex structure I_0 (cf. Example 4.5) and the Euclidean metric $g(X, X') = \sum_{j=1}^n A_j A'_j + B_j B'_j$ where $X = \sum_{j=1}^n A_j + iB_j, X' = \sum_{j=1}^n A'_j + iB'_j$. The Euclidean connection is (cf. (14))

$$\nabla_X X' = \sum_{i,j} (A_i \tfrac{\partial A'_j}{\partial x_i} + B_i \tfrac{\partial A'_j}{\partial y_i}) \tfrac{\partial}{\partial x_j} + (A_i \tfrac{\partial B'_j}{\partial x_i} + B_i \tfrac{\partial B'_j}{\partial y_i}) \tfrac{\partial}{\partial y_j}.$$

Therefore

$$\begin{aligned}\nabla_X(I_0 X') &= \sum_{i,j} (A_i \tfrac{\partial(-B'_j)}{\partial x_i} + B_i \tfrac{\partial(-B'_j)}{\partial y_i}) \tfrac{\partial}{\partial x_j} + (A_i \tfrac{\partial A'_j}{\partial x_i} + B_i \tfrac{\partial A'_j}{\partial y_i}) \tfrac{\partial}{\partial y_j} \\ &= \sum_{i,j} -(A_i \tfrac{\partial B'_j}{\partial x_i} + B_i \tfrac{\partial B'_j}{\partial y_i}) \tfrac{\partial}{\partial x_j} + (A_i \tfrac{\partial A'_j}{\partial x_i} + B_i \tfrac{\partial A'_j}{\partial y_i}) \tfrac{\partial}{\partial y_j} \\ &= I_0 \nabla_X X'.\end{aligned}$$

Furthermore

$$g(I_0 X, I_0 X') = \sum_j (-B_j)(-B'_j) + A_j A'_j = g(X, X').$$

So $(\mathbb{C}^n, I_0, g)$ is a Kähler manifold. The Kähler structure is ω_0:

$$\omega(X, X') = g(I_0 X, X') = \sum_j -B_j A'_j + A_j B'_j = \omega_0(X, X')$$

We conclude that the quotient $N/G = \mathbb{CP}^{n-1}$ is a Kähler manifold with the induced symplectic structure $\widehat{I_0}$.

Example 4.8. Now look at the polygon spaces described in Example 3.12, i.e. let $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{R}^n_+$ and $M = \prod_{i=1}^n S^2_{\alpha_i}$ where $S^2_{\alpha_i}$ is the sphere with radius α_i. S^2 can be given an almost complex structure by defining $I\underline{v} = \underline{v} \times \underline{x}$ for $\underline{v} \in T_{\underline{x}} S^2$. That this is an integrable structure is a special case of the more general result that all almost complex structure on manifolds of real dimension 2 are integrable. This result is proved in the following way: Let X be a manifold of real dimension $2n$ with complex structure J. Extend J to a complex linear operator also denoted by J on the complexification of the tangent bundle denoted by $TX \otimes \mathbb{C}$ such that $J^2 = -1$. For each $x \in X$, $T_xX \otimes \mathbb{C}$ splits into the direct sum $T^{1,0} \oplus T^{0,1}$ where $T^{1,0} = \{v \mid Jv = iv\}$ and $T^{0,1} = \{v \mid Jv = -iv\}$ are the eigenspaces of J. Through the mapping $v \mapsto v - iJv$ for $v \in T_xX$, we may identify T_xX with $T^{1,0}$. Now since

$$\begin{aligned} J[v - iJv, w - iJw] &- i[v - iJv, w - iJw] \\ &= J([v,w] - [Jv, Jw] - i[Jv, w] - i[v, Jw]) \\ &\quad -i[v,w] - i[Jv, Jw] + [Jv, w] + [v, Jw] \\ &= J[v,w] - J[Jv, Jw] - [Jv, w] - [v, Jw] \\ &\quad +i([Jv, Jw] - [v,w] - J[Jv, w] - J[v, Jw]) \\ &= \tfrac{1}{2}(J\sigma(v,w) + i\sigma(v,w)), \end{aligned}$$

the integrability condition on J is equivalent to the condition that $J[v', w'] = i[v', w']$ for all $v', w' \in T^{1,0}$. Now consider the case $n = 1$, i.e. X has complex dimension 1. If v', w' are vector fields on X with $Jv' = iv'$, $Jw' = iw'$, then $v' = fw'$ for some scalar function f on X. So $J[v', w'] = J[v', fv'] = J(v'(f)v') = v'(f)iv' = i[v', w']$. Therefore J is integrable. (S^2, g) is also a Riemannian manifold where g denotes the Euclidean metric. We see that

$$g(I\underline{v}, I\underline{w}) = (\underline{v} \times \underline{x}) \cdot (\underline{w} \times \underline{x}) = (\underline{v} \cdot \underline{w})(\underline{x} \cdot \underline{x}) - (\underline{v} \cdot \underline{x})(\underline{x} \cdot \underline{w}) = (\underline{v} \cdot \underline{w}) = g(\underline{v}, \underline{w}).$$

Now consider the Riemannian connection on S^2 with respect to g. This is given by

$$\nabla_{\underline{v}}\underline{w} = (D\underline{v}(\underline{w}))^\top$$

where $^\top$ denotes the tangential projection and $D\underline{v}(\underline{w}) = \frac{d}{dt}\big|_{t=0}\underline{w} \circ \gamma$ where γ is a curve with $\gamma'(0) = \underline{v}$. Hence

$$\begin{aligned} \nabla_{\underline{v}} I\underline{w}\big|_{\underline{x}} &= \nabla_{\underline{v}}(\gamma \times \underline{w})\big|_{\underline{x}} = (D\underline{v}(\gamma \times \underline{w}))^\top\big|_{\underline{x}} \\ &= (\tfrac{d}{dt}\big|_{t=0}(\gamma \times \underline{w}) \circ \gamma(t))^\top \\ &= (D\underline{v}(\gamma \times \underline{w}))^\top\big|_{\underline{x}} + (\gamma \times D\underline{v}(\underline{w}))^\top\big|_{\underline{x}}. \end{aligned}$$

Now since $\gamma \cdot \gamma = 1$, $0 = D\underline{v}(\gamma \cdot \gamma) = 2(\gamma \cdot D\underline{v}(\gamma))$ so $D\underline{v}(\gamma)$ is a tangent vector and therefore $(D\underline{v}(\gamma) \times \underline{w})^\top = 0$. Hence

$$\nabla_{\underline{v}} I\underline{w}\big|_{\underline{x}} = \underline{x} \times D\underline{v}(\underline{w}) = \underline{x} \times \nabla_{\underline{v}}\underline{w} = I\nabla_{\underline{v}}\underline{w}\big|_{\underline{x}}.$$

Therefore (S^2, I, g) is a Kähler manifold. We can now give M the complex structure $I_1 \otimes \ldots \otimes I_n$ where I_i is the complex structure on $S^2_{\alpha_i}$ for $i \in \{1, \ldots, n\}$ and the Riemannian structure $g_1 \otimes \ldots \otimes g_n$ where g_i is the Euclidean metric on $S^2_{\alpha_i}$ for $i \in \{1, \ldots, n\}$. This makes M into a Kähler manifold.

4.3 Hyperkähler Manifolds

We now look at hyperkähler manifolds which are defined as follows:

Definition 4.3. Let (M, g) be a Riemannian manifold which is Kähler with respect to the three complex structures I, J, K. Suppose that I, J, K satisfy $IJK = -1$. Then (M, I, J, K, g) is called a *hyperkähler manifold.*

Suppose (M, I, J, K, g) is a hyperkähler manifold. From each complex structure I, J and K we get Kähler structures ω_I, ω_J and ω_K respectively. Suppose G is a compact Lie group acting freely on M preserving $g, \omega_I, \omega_J, \omega_K$ and that for each Kähler form a moment map μ_I, μ_J, μ_K respectively exists. We can write this as a single map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$. We claim that the quotient $\mu^{-1}(0)/G = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)/G$ is a hyperkähler manifold.

We start by seeing that $\mu^{-1}(0)/G$ is a Kähler manifold with respect the quotient Riemannian structure denoted by $\tilde{g}$ and the complex structure induced by I . Consider the map $\mu_+ = \mu_J + i\mu_K : M \to \mathfrak{g}^* \otimes \mathbb{C}$. Let Y be a vector field on M and $\xi \in \mathfrak{g}$. As usual $\tilde{\xi}$ denotes the vector field generated by ξ. Then

$$\begin{aligned} D\mu_+(\xi)(Y) &= \omega_J(\tilde{\xi}, Y) + i\omega_K(\tilde{\xi}, Y) \\ &= g(J\tilde{\xi}, Y) + ig(K\tilde{\xi}, Y) \end{aligned}$$

so

$$\begin{aligned} D\mu_+(\xi)(IY) &= g(J\tilde{\xi}, IY) + ig(K\tilde{\xi}, IY) \\ &= -g(K\tilde{\xi}, Y) + ig(J\tilde{\xi}, Y) \\ &= i(g(J\tilde{\xi}, Y) + ig(K\tilde{\xi}, Y)) = iD\mu_+(\xi)(Y), \end{aligned}$$

i.e. μ_+ is holomorphic with respect to I. Therefore if $X \in T_xN$ for $x \in N$ where $N = \mu_+^{-1}(0)$ then $D\mu_+(x)(X) = 0$ and therefore

$$D\mu_+(x)(IX) = iD\mu_+(x)(X) = 0$$

so $IX \in T_xN$. Hence T_xN is I-invariant for all $x \in N$ and therefore $N = \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ is a complex submanifold with complex structure $I\big|_N$. Hence, $(N, I\big|_N, g\big|_N)$ is a Kähler manifold. The action of G on N has moment map

$\mu_I\big|_N$. So as we saw in Section 4.2, $\mu^{-1}(0)/G = \mu_I^{-1}(0) \cap N/G = \mu_I\big|_N^{-1}(0)/G$ is a Kähler manifold with respect to the complex structure $\tilde{I} = \widehat{I\big|_N}$ and $\tilde{g}$.

Now repeating the procedure for J and K we get complex structures $\tilde{J}, \tilde{K}$ with respect to which $(\mu^{-1}(0)/G, \tilde{g})$ is a Kähler manifold. Clearly $\tilde{I}\tilde{J}\tilde{K} = -1$, so $(\mu^{-1}(0)/G, \tilde{I}, \tilde{J}, \tilde{K}, \tilde{g})$ is a hyperkähler manifold.

Example 4.9. Let $M = \mathbb{H}^n = \mathbb{C}^n \oplus \mathbb{C}^n j$ which we can identify with $\mathbb{C} \oplus (\mathbb{C}^n)^*$ through the mapping $z + wj \mapsto z + \tilde{w}$ where $\tilde{w}$ is defined by $\tilde{w}(\zeta) = \sum_l \zeta_l \overline{w_l}$. On $\mathbb{C}^n \oplus \mathbb{C}^n j$ right multiplication by i, j, k gives us three complex structures. The corresponding complex structures I, J, K on M make (M, I, J, K, g), where g is the Euclidean metric, into a hyperkähler manifold. Let $G = S^1$ act by

$$e^{i\theta} \cdot (z_1, \dots, z_n, w_1, \dots, w_n) = (e^{i\theta} z_1, \dots, e^{i\theta} z_n, e^{-i\theta} w_1, \dots, e^{-i\theta} w_n).$$

The relevant moment maps can be seen to be defined by:

$$\mu_I(z, w) = i(\|z\|^2 - \|w\|^2)$$
$$\mu_+(z, w) = w(z)$$

Let $\eta = (i, 0, 0) \in \mathfrak{g}^* \otimes \mathbb{R}$. Then

$$\mu^{-1}(\eta) = \left\{(z, w) \mid w(z) = 0, i(\|z\|^2 - \|w\|^2) = 1\right\}.$$

The hyperkähler quotient $\mu^{-1}(\eta)/G$ can be shown to be the cotangent bundle of $\mathbb{CP}^n$.

5 Summary

We started out by looking at group actions on topological spaces and how this gave us a natural quotient construction. We generalized to manifolds but in this case the quotient will not necessarily be a manifold itself. We saw how a quotient manifold is obtained when the action of the group is free and the group is compact. More generally, The Slice Theorem gives a local model of the quotient in the case when the group acting on the manifold is compact. We then looked at manifolds with a Riemannian structure. We saw how the Riemannian structure descends to the quotient in a natural way when the action is free, the group is compact and the group acts by isometries. After a short introduction to symplectic geometry, we saw how the construction of a quotient manifold from the group action on a symplectic manifold does not in general yield a symplectic quotient even if the action preserves the symplectic form. When a moment map μ exists, we could, however, restrict ourselves to $\mu^{-1}(0)$ and this will give a symplectic quotient. The existence and uniqueness of the moment map was seen to depend on the first and second cohomology groups of the Lie algebra of the group acting on the manifold. We then looked at connections and complex structures to be able to define Kähler manifolds. Kähler manifolds have both a Riemannian and a symplectic structure and the quotient constructions we looked at earlier combine to give a quotient which is a Kähler manifold. Finally, we looked at hyperkähler manifolds which are roughly Kähler manifolds with respect to three different complex structures. Here we can use the Kähler quotient construction (three times) to obtain a quotient which is hyperkähler.

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