

Math 566 Problem Sheet 3: Due Nov. 20

Please do four of these problems and try to think about how you might solve the problems you don't choose.

1. "loose ends" = two questions

(a) "Generalization of the Inverse Function Theorem:" Let $f: M \rightarrow N$ be a smooth map that is 1 – 1 on a compact submanifold L of M . Suppose that for all $p \in L$,

$$T_p f: T_p M \rightarrow T_{f(p)} N$$

is an isomorphism. Prove that f maps L diffeomorphically onto $f(L)$.

Prove that f , in fact, maps an open neighborhood of L diffeomorphically onto an open neighborhood of $f(L)$.

Hint: Prove that a local diffeomorphism $g: X \rightarrow Y$ is actually a diffeomorphism of X onto an open subset of Y if g is 1 – 1. So, to prove the question above it suffices to show that f is 1 – 1 on some neighborhood of L .

(b) Use a partition-of-unity argument to prove the following noncompact version the previous result. Suppose that the differential of $f: M \rightarrow N$ is an isomorphism for all p in a submanifold L of M , and assume f maps L diffeomorphically onto $f(L)$. Prove that f maps an open neighborhood of L diffeomorphically onto an open neighborhood of $f(L)$.

Hint: Let $\{U_i\}$ be a locally finite open covering of $f(L)$ on which $g_i: U_i \rightarrow M$ is a local inverse of f . Set

$$W = \{q \in N \mid g_i(q) = g_j(q) \text{ whenever } q \in U_i \cap U_j\}.$$

Show that the g_i "patch together" to define a smooth inverse $g: W \rightarrow M$. Then prove that W contains an open neighborhood of $f(L)$.

2. Let $f: M \rightarrow N \times \mathbb{R}^k$ be a smooth map. Show that there exists an arbitrarily small vector $v \in \mathbb{R}^k$ such that the map

$$g: M \rightarrow N \times \mathbb{R}^k, \quad p \mapsto f(p) + v$$

is transverse to the submanifold $N \times \{0\} \subset N \times \mathbb{R}^k$.

3. (a) Let $M^m \subset \mathbb{R}^p$ be a smooth submanifold embedded in \mathbb{R}^p . Show that there exists a hyperplane in \mathbb{R}^p which intersects M^m transversely.

(b) Let M^m be a compact manifold and $f: M^m \rightarrow \mathbb{R}^{m+1}$ a smooth map such the image $f(M^m)$ does not contain the origin. Show that there exists a line through the origin which meets $f(M^m)$ in only a finite number of points.

4. Show that if M is a compact k -dimensional manifold, then there exists a map $M \rightarrow \mathbb{R}^{2k-1}$ that is an immersion except at finitely many points of M .

To do this let $f: M \rightarrow \mathbb{R}^{2k}$ be an immersion and consider the map

$$F: TM \rightarrow \mathbb{R}^{2k} \quad (p, v) \in T_p M \mapsto T_p f(v).$$

Let a be a regular value of F and show that $\pi_a \circ f$ is an immersion except at $f^{-1}(a)$. The exceptional points $f^{-1}(a)$ are called *cross caps*.

Hint: Fix a metric g on TM and show that there are only finitely many preimages of a in the compact set $\{(p, v) \in TM \mid \|v\|_g^2 = g(v, v) \leq 1\}$. If there are infinitely many points (p_i, v_i) then there is a subsequence with $p_i \rightarrow p$ and $v_i/\|v_i\|_g \rightarrow w$. Show $T_p f(w) = 0$.

(Whitney showed that for maps of two-dimensional manifolds into \mathbb{R}^3 , a typical cross cap looks like the map $(x, y) \mapsto (x, xy, y^2)$. Take a look at this map.)

5. Show that there is no surjective smooth map $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$.
6. (a) If $f: M \rightarrow N$ is homotopic to a constant map, show that $I_2(f, L) = 0$ for all closed submanifolds $L \subset N$ of complementary dimension, except perhaps if $\dim M = 0$.
- (b) Prove that intersection theory is vacuous in contractible manifolds: if N is contractible and $\dim N > 0$, then $I_2(f, L) = 0$ for every $f: M \rightarrow N$, compact M , and closed L with $\dim M + \dim L = \dim N$.
7. Suppose $f: M \rightarrow S^k$ is smooth, where M is compact and $0 < \dim M < k$. Then for all closed $L \subset N$ of dimension complementary to M , $I_2(f, L) = 0$. (So the degrees ($\dim M = k$) are the only interesting intersection numbers on spheres.)
8. Prove that there is a complex number z such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

Find an upper bound for $\|z\|$.

(You should definitely not try to compute a root!)