

Math 566 Problem Sheet 2: Due Oct. 9

Please do five of these problems and try to think about how you might solve the problems you don't choose.

- (a) Prove that if $E_1 \oplus E_2$ and E_1 are orientable then so is E_2 . Similarly, show that if E_1 and E_2 are orientable then so is $E_1 \oplus E_2$.
(b) If E is an orientable vector bundle and $F \subset E$ is a subbundle, show that E/F is orientable iff F is orientable.
- Prove that a vector bundle is orientable iff it possesses a bundle atlas, all of whose transition functions ($g_{\beta\alpha}$'s) map into

$$GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det A > 0\}.$$

- (a) Let (E, π, X) be a vector bundle and $\tilde{\pi}: \tilde{X}(E) \rightarrow X$ its orientation cover. Show that $\tilde{\pi}^*E$ possesses a (canonical) orientation.
(b) By the orientation cover $\tilde{M} \rightarrow M$ of a manifold M , one means the orientation cover of TM . Show that \tilde{M} is orientable regardless of whether or not M is.
- A vector bundle is *stably trivial* if the Whitney sum with a suitable trivial bundle is trivial. Show that TS^n is stably trivial.
- Let M be a manifold and Δ_M the diagonal in $M \times M$:

$$\Delta_M = \{(x, x) \in M \times M \mid x \in M\}.$$

Show that Δ_M is a submanifold of $M \times M$ for which the tangent bundle and the normal bundle are isomorphic: $T\Delta_M \simeq \perp M$.

- Let E be a vector bundle over X with fibres of dimension k . Let \mathcal{U} be a bundle atlas for E all of whose transition functions are maps into $O(n) \subset GL(n, \mathbb{R})$. Show there is precisely one Riemannian metric $\langle \cdot, \cdot \rangle$ on E , such that all charts of \mathcal{U} are isometries on fibres (that is, for each bundle chart $h: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ the restrictions $h|_{\pi^{-1}(x)}: \pi^{-1}(x) \rightarrow x \times \mathbb{R}^k$ preserve lengths when \mathbb{R}^k is given the canonical metric).
- (a) Let $\mathbb{R} + \mathbb{R}$ be the disjoint union of two copies of \mathbb{R} , and let $f: \mathbb{R} + \mathbb{R} \rightarrow \mathbb{R}^2$ be the map with the components $f_1(x) = (x, 0)$ and $f_2(y) = (0, \exp(y))$. Show that f is an injective immersion, but not an embedding.

(b) Let the map $f: \mathbb{R} + S^1 \rightarrow \mathbb{C}$ be defined by

$$f(x) = (1 + \exp(x)) \cdot \exp(ix) \text{ for } x \in \mathbb{R}$$

and

$$f(\exp(it)) = \exp(it) \text{ for } \exp(it) \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Show that f is an injective immersion, but not an embedding.

8. Define the map $f: \mathbb{R} \rightarrow \mathbb{C} \times \mathbb{C}$ by

$$f(t) = (\exp(ait), \exp(bit))$$

for real numbers $a, b \in \mathbb{R}$. Show that f is an immersion if a and/or b is nonzero. If a/b is irrational show that f is an injective immersion and its image is dense in $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$. Is f an embedding?

9. Let A be a real symmetric $(n \times n)$ -matrix, and let $b \in \mathbb{R}$ be different from zero. Show that the *quadric*

$$M = \{x \in \mathbb{R}^n \mid x^t Ax = b\}$$

is an $(n - 1)$ -dimensional submanifold of \mathbb{R}^n .

10. Consider the smooth map $f: M(n, n) = \mathbb{R}^{n^2} \rightarrow S$ from the space of $n \times n$ -matrices to the space S of symmetric matrices defined by $f(A) = A^t A$. (here A^t denotes the transpose of A). Show that the set of real orthogonal $n \times n$ -matrices, $O(n)$, is a smooth compact submanifold of \mathbb{R}^{n^2} by showing that the identity matrix is a regular value of f .

11. A k -frame in \mathbb{R}^n is an orthonormal k -tuple (v_1, \dots, v_k) of vectors in \mathbb{R}^n . The set $V_n^k \subset \mathbb{R}^n \times \dots \times \mathbb{R}^n$ of k -frames is called a *Stiefel Manifold*. Show that V_n^k is a compact smooth manifold of dimension $nk - k(k + 1)/2$.