Math 566 Problem Sheet 1: Due Sept. 23

Please do problem #1 and choose **four** other problems to do. Try to think about how you might solve the problems you don't choose.

1. The complex projective space $\mathbb{C}P^n$ can be defined as follows: on the complex vector space $\mathbb{C}^{n+1}$, one has the equivalence relation

   $$x \sim y \iff \text{there is a number } \lambda \in \mathbb{C}, \lambda \neq 0, \text{ so that } \lambda x = y.$$

   The quotient space $(\mathbb{C}^{n+1} - \{0\})/\sim$ is defined to be $\mathbb{C}P^n$.

   Provide $\mathbb{C}P^n$ with the structure of a $2n$-dimensional differentiable manifold (i.e., a smooth atlas).

2. Let M be a differentiable manifold and let $f\colon N \to M$ be a homeomorphism. Prove that N possesses exactly one differentiable structure for which the map f is a diffeomorphism.

3. Show that every (differentiable) manifold possesses a countable (differentiable) atlas.

4. Using stereographic projection show that the sphere S^n possesses an atlas with precisely two charts.

5. (a) Show that the Cartesian product $f_1 \times f_2\colon N_1 \times N_2 \to M_1 \times M_2$ of two embeddings is an embedding.

   (b) If $N = S^l \times S^k$, show that there is an embedding of N into $\mathbb{R}^{l+k+1}$. (Hint: first describe an embedding of S^l into $\mathbb{R}^{l+1}$ and then use part (a).)

6. Show that S^n is a submanifold of $\mathbb{R}^{(n+1)}$.

7. Prove that $\mathbb{R}^k$ can be embedded into an n-dimensional (differentiable or topological) manifold M if $k \leq n$.

8. Let $C^\infty(M)$ be the space of smooth functions on the smooth manifold M. This set is an algebra under the natural addition and multiplication of functions.

   (a) A smooth map $f\colon M \to N$ induces a map

   $$f^*\colon C^\infty(N) \to C^\infty(M),\ \phi \mapsto \phi \circ f.$$

   Show that this is an algebra homomorphism.

   (b) For the identity map on M, $Id_M\colon M \to M$, show that Id_M^* acts as the identity on $C^\infty(M)$.

   (c) Show that $(f \circ g)^* = g^* \circ f^*$.

(d) Deduce from these properties that if $f\colon M \to N$ is a diffeomorphism then $f^*\colon C^\infty(N) \to C^\infty(M)$ must be an algebra isomorphism.

9. For a point $p \in M$ consider the subset

$$\mathcal{M}_p = \{\phi \in C^\infty(M) \mid \phi(p) = 0\}.$$

   (a) Show that $\mathcal{M}_p$ is a maximal ideal of $C^\infty(M)$.

   (b) If M is compact, show that every maximal ideal $\mathcal{M}$ of $C^\infty(M)$ is of the form $\mathcal{M}_p$ for some $p \in M$.

10. Show that $M = \{x \in \mathbb{R}^n \mid x_1^2 = x_2^2 + \cdots + x_n^2 \text{ and } x_1 \geq 0\}$ is not a smooth submanifold of $\mathbb{R}^n$.

11. (a) Let E be a vector bundle over X, and let X_0 be a submanifold of X with $i\colon X_0 \to X$ the inclusion map. Show that i^*E and $E|_{X_0}$ are canonically isomorphic.

   (b) Show that if E is a trivial bundle over X and $f\colon Y \to X$, then f^*E over Y is also trivial.

12. Recall that $\mathbb{R}P^n$ can be described as the quotient of S^n by the antipodal map $x \to -x$. The canonical real line bundle over $\mathbb{R}P^n$ is defined to be

$$\eta_n := \{([x], \lambda x) \mid x \in S^n, \lambda \in \mathbb{R}\}.$$

   (a) Show that η_n is a nontrivial subbundle of the trivial bundle $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. (Hint: If it were trivial then the set $\eta_n - \{zero - section\}$ would have two connected components.)

   (b) Prove that every rank one vector bundle is either trivial or is isomorphic to the bundle η_1. (Recall that S^1 is diffeomorphic to $\mathbb{R}P^1$.)

13. (a) Show that the tangent bundle of S^2 has an atlas with two bundle charts.

   (b) Construct a vector field on S^2 with exactly two zero points.

   (c) Construct a vector field on S^2 with exactly one zero point.

14. Consider

$$E := \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \cdots + z_n^2 = 1\},$$

as a submanifold of $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$. Show that E is diffeomorphic to the total space of the tangent bundle of S^n.

15. Show that for every submanifold $M \subset \mathbb{R}^n$, the Whitney sum $TM \oplus \perp M$ of the tangent and normal bundles of M is trivial.