Proof of (*):

$$
\begin{aligned}
d(x_0, tx + (1-t)y) &= \| x_0 - tx - (1-t)y \| \\
&= \| \underbrace{(1-t+t)}_{=1} x_0 - tx - (1-t)y \| \\
&= \| (1-t)x_0 - (1-t)y + tx_0 - tx \| \\
&= \| (1-t)(x_0 - y) + t(x_0 - x) \| \\
&\leq \| (1-t)(x_0-y) \| + \| t(x_0 - x) \| \\
&\qquad \text{(triangle inequality)} \\
&= (1-t)\|x_0 - y\| + t\|x_0 - x\| \\
&< (1-t)r + tr \quad \text{since } x, y \in U_r(x_0) \\
&= r
\end{aligned}
$$

The proof that the closed ball is convex is the same except that we use $\|x_0 - x\| \leq r$ and $\|y - x_0\| \leq r$.