Math 364 Problem Sheet 3: Due Oct. 2

1. In each of the following cases, determine whether or not the set A is *open in the set* X:

   (a) X is the interval $(-2, 2)$ in $\mathbb{R}$ and A is the subinterval $(-2, 1]$.

   (b) X is the interval $(-2, 2)$ in $\mathbb{R}$ and A is the subinterval $(-1, 2)$.

   (c) X is the closed disc $\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ and A is the set of points with positive first coordinate.

2. Let $X \subset \mathbb{R}^n$ and suppose that $U \subset \mathbb{R}^n$ is open. Show that $U \cap X$ is *open in the set* X.

3. Each of the following maps is discontinuous. In each case, find an open set in the target space whose preimage is not open in the domain:

   (a) $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} +1 & \text{if } x_2 \geq 0, \\ -1 & \text{if } x_2 < 0. \end{cases}$$

   (b) $f\colon \mathbb{R} \to \mathbb{R}^3$ defined by

   $$f(x) = \begin{cases} (\cos x, \sin x, +1) & \text{if } x \geq 0, \\ (\cos x, \sin x, -1) & \text{if } x < 0. \end{cases}$$

4. Show that every open interval (a, b) is homeomorphic to the open semi-circle $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1, x_1 > 0\}$. Show that the open semi-circle is homeomorphic to $\mathbb{R}$

5. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a map which decreases distances in the sense that

   $$d(f(x), f(x')) < d(x, x').$$

   Show that f must be continuous. (Hint: Fix an arbitrary $p \in \mathbb{R}^n$ and check continuity at p using the $\delta - \epsilon$ definition of continuity.)