$$\begin{array}{ccccccccc} & & M_1 & \xrightarrow{u_1} & M_2 & \xrightarrow{u_2} & M_3 & \to & 0 \\ & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\ 0 & \to & N_1 & \xrightarrow{v_1} & N_2 & \xrightarrow{v_2} & N_3 & & \end{array}$$

Snake Lemma: If the above is a commutative diagram of R-modules with exact rows, then there exists an exact sequence:

$$\ker f_1 \xrightarrow{\hat{u}_1} \ker f_2 \xrightarrow{\hat{u}_2} \ker f_3 \xrightarrow{\delta} \operatorname{coker} f_1 \xrightarrow{\hat{v}_1} \operatorname{coker} f_2 \xrightarrow{\hat{v}_2} \operatorname{coker} f_3$$

Proof: Let $\delta(x) = v_1^{-1} f_2 u_2^{-1}\{x\}$. We need to show that δ is a well-defined homomorphism. First find that

$$\delta(x) \text{ is non-empty}$$

since,

$$\begin{aligned} \delta(x) &= v_1^{-1} f_2 u_2^{-1}(\text{nonempty} \subset \ker f_3) \\ &= v_1^{-1} f_2(\text{nonempty} \subset \ker f_3 u_2) \text{ [since } u_2 \text{ is surjective]} \\ &= v_1^{-1} f_2(\text{nonempty} \subset \ker v_2 f_2) \\ &= v_1^{-1} f_2 f_2^{-1}(\text{nonempty} \subset \ker v_2) \\ &= v_1^{-1}(\text{nonempty} \subset \ker v_2) \\ &= v_1^{-1}(\text{nonempty} \subset \text{im } v_1) \\ &= \text{nonempty} \end{aligned}$$

Also, find that

$$\delta(0) = \text{im } f_1$$

since,

$$\begin{aligned} \delta(0) &= v_1^{-1} f_2 u_2^{-1}\{0\} \\ &= v_1^{-1} f_2 \ker u_2 \\ &= v_1^{-1} f_2 \text{ im } u_1 \\ &= v_1^{-1} \text{ im } f_2 u_1 \\ &= v_1^{-1} \text{ im } v_1 f_1 \\ &= v_1^{-1} v_1 \text{ im } f \\ &= \text{im } f_1 \text{ [since } v_1 \text{ is injective]} \end{aligned}$$

Next, find that

$$\delta(x+ry) \supset \delta(x) + r\delta(y)$$

since,

$$\begin{aligned}
\delta(x+ry) &= v_1^{-1} f_2 u_2^{-1}\{x+ry\} \\
&\supset v_1^{-1} f_2 (u_2^{-1}\{x\} + r u_2^{-1}\{y\}) \\
&= v_1^{-1} (f_2 u_2^{-1}\{x\} + r f_2 u_2^{-1}\{y\}) \\
&\supset v_1^{-1} f_2 u_2^{-1}\{x\} + r v_1^{-1} f_2 u_2^{-1}\{y\} \\
&= \delta(x) + r\delta(y)
\end{aligned}$$

Now, find that

$$\delta \text{ is well-defined}$$

since, $\forall z \in \delta(x), \delta(x) = z + \text{im } f_1$, since,

$$\begin{aligned}
z + \text{im } f_1 &\subset \delta(x) + \delta(0) \\
&\subset \delta(x+0) \\
&= \delta(x)
\end{aligned}$$

so $\delta(x) \supset z + \text{im } f_1$, and, given $w \in \delta(x)$,

$$\begin{aligned}
w - z &\in \delta(x) - \delta(x) \\
&\subset \delta(x-x) \\
&= \delta(0) \\
&= \text{im } f_1
\end{aligned}$$

so, $w \in z + \text{im } f_1$, so $\delta(x) \subset z + \text{im } f_1$.
Finally, find that

$$\delta \text{ is a homomorphism}$$

since, $\delta(x) + r\delta(y) \supset \delta(x+ry)$, since

$$\begin{aligned}
\delta(x) + r\delta(y) &= \delta(x + ry - ry) + r\delta(y) \\
&\supset \delta(x + ry) - r\delta(y) + r\delta(y) \\
&= \delta(x + ry) \text{ [since } \delta \text{ is well-defined]}
\end{aligned}$$