A (discrete) rigid body is a finite set of particles with masses m_i and positions $x_{i\mu}$ such that $|x_i - x_j|$ is constant in time. Greek indices are spatial and Latin indices correspond to the different particles. The total mass is $M = \sum_i m_i$, the center of mass is $X_\mu = (\sum_i m_i x_i)/M$, and the inertia tensor is $I_{\mu\nu} = \sum_i m_i(|x_i|^2\delta_{\mu\nu} - x_{i\mu}x_{i\nu})$. $I_{\mu\nu}$ is symmetric. Therefore, there is a right-handed orthonormal basis of eigenvectors of $I_{\mu\nu}$. The eigenvectors $\phi_{\mu 1}, \phi_{\mu 2}, \phi_{\mu 3}$ are called the principle axes. The eigenvalues I_1, I_2, I_3 are called the moments of inertia.

$SO(n)$, the special orthogonal group, is the group of $n \times n$ matrices A such that $AA^T = 1$ and $det\, A = 1$. Such matrices can be interpreted either as right-handed orthonormal bases or as linear maps preserving the inner product and handedness (i.e. rotations). The principles axes $\phi_{\mu\nu}$ is in $SO(3)$.

There are two coordinate systems of interest, the spatial coordinate system that has been assumed given, and a body coordinate system which is fixed with respect to the body. The coordinate system is parametrized by $\mathbb{R}^3 \times SO(3)$. An example of a body coordinate system is the principle axes with origin at the center of mass, $(X_\mu, \phi_{\mu\nu})$. A path $(x_\mu(t), \theta_{\mu\nu}(t)) \in \mathbb{R}^3 \times SO(3)$ with $x_\mu(0) = 0$ and $\theta_{\mu\nu}(0) = \delta_{\mu\nu}$ describes the motion of the rigid body in time.

In order to actually compute this path it is best to let $x_\mu(t) = X_\mu(t)$ and $\theta_{\mu\nu}(t) = \phi_{\mu\nu}(t)$. Choosing so decouples the equations of motion. Then the equation of motion for $X_\mu(t)$ is easy, $X''_\mu(t) = 0$. The equation of motion for $\phi_{\mu\nu}(t)$ is not so easy, however. It is quite a bitch to calculate it. Moreover, the equations hide what is really going on. To see what's going on, we've got to learn about Lie groups.

A Lie group is a smooth manifold G which is also a group such that the group operation $G \times G \to G$ and the inversion operation $G \to G$ are smooth. A Lie algebra is a vector space A with a bilinear operation $[,] : A \times A \to A$ such that for all $x, y, z \in A$, $[x, x] = 0$ and $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$. To each Lie group there is an associated Lie algebra. There are many equivalent constructions of this. Here is one. Let G be a Lie group. Let $g = TeG$, where e is the identity of G. The derivative at e of the map $G \times G \to G$, $(x, y) \mapsto xyx^{-1}y^{-1}$ gives the Lie bracket $g \times g \to g$.

$SO(3)$ is an example of a Lie group with group operation given by matrix multiplication. Let's figure out its associated Lie algebra $so(3)$. Physically, this is the space of angular velocities. It is just the set of $\theta'(0)$ where $\theta(t)$ is a path through $SO(3)$ with $\theta(0) = 1$. So, differentiating the condition $\theta\theta^T = 1$ gives $\theta'(0)\theta^T(0) + \theta(0)\theta^{T\prime}(0) = 0$. Let $\omega = \theta'(0)$. $\theta(0) = 1$ so, $\omega + \omega^T = 0$. Differentiating the condition $det\, \theta = 1$ gives $tr\, \omega = 0$. So, $\omega \in so(3)$ can be written as $\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$. Differentiating $\theta\phi\theta^{-1}\phi^{-1}$ gives $[\omega, \psi]_1 = \omega_2\psi_3 - \psi_2\omega_3$, $[\omega, \psi]_2 = \omega_3\psi_1 - \psi_3\omega_1$, $[\omega, \psi]_3 = \omega_1\psi_2 - \psi_1\omega_2$, i.e. the cross product. This gives us a Lie algebra isomorphism between $so(3)$ and $\mathbb{R}^3$ with the cross product given by $so(3) \to \mathbb{R}^3$, $\omega \mapsto (\omega_1, \omega_2, \omega_3)$.

Remember the inertia tensor I? It turns out you can extend I to be a

Riemannian metric on $SO(3)$. It acts on $so(3)$ by $I(\omega, \psi) = \sum_\mu \sum_\nu I_{\mu\nu}\omega_\mu\psi_\nu$. To be a tensor on $SO(3)$ though, it must act on all of the tangent spaces $T_\phi SO(3)$. It can be shown that in fact $T_\phi SO(3) = \phi so(3) = so(3)\phi$, giving two canonical isomorphisms between $T_\phi SO(3)$, one for the space frame, one for the body frame. The second gives the angular velocity of the body in the space frame. I was defined using spatial coordinates so it should act on spatial angular velocity in the obvious way. I is bilinear and symmetric since $I_{\mu\nu}$ is a symmetric matrix. Also, I is positive definite since mass is positive. Yay! Now $SO(3)$ is a Riemannian manifold and we can calculate curvature and geodesics and fun stuff.

We have a metric which means we have a notion of length, but what the hell is the length of an angular velocity? Well, half the length squared is $(1/2)I(\omega, \omega)$, the kinetic energy. By the least action principle, we expect the time integral of kinetic energy is minimized along the actual path (when there's no torque). But, geometrically a length minimizing path is just a geodesic, so the orientation of a rigid body traces out a geodesic in $SO(3)$ with respect to its inertia tensor! Hence, the equation of motion is just the geodesic equation.