

Fundamental Theorem of Riemannian Geometry

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A connection on a smooth manifold M is a map $\Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$, $(X, Y) \mapsto \nabla_X Y$ with

- Tensoriality: For scalar field f , $\nabla_{fX} Y = f \nabla_X Y$ and $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$.
- Linearity: For a scalar c , $\nabla_X cY = c \nabla_X Y$ and $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.
- Product Rule: For a scalar field f , $\nabla_X fY = X(f)Y + f \nabla_X Y$.

The torsion is $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

The curvature is $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.

Both the torsion and the curvature are tensorial.

The connection is torsion-free iff $[X, Y] = \nabla_X Y - \nabla_Y X$.

If M is a Riemannian manifold with metric g then the connection is metric-compatible iff $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

The fundamental theorem of Riemannian geometry is that there exists a unique torsion-free, metric-compatible connection, the Levi-Civita connection. Let's prove it.

Permuting X , Y , and Z we get the following three equations from metric-compatibility:

- $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
- $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$
- $Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$

Combining these gives

$$X(g(Y, Z)) - Z(g(X, Y)) + Y(g(Z, X)) = \\ g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

So, by symmetry and tensoriality of g

$$X(g(Y, Z)) - Z(g(X, Y)) + Y(g(Z, X)) = \\ g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) - g(X, \nabla_Z Y - \nabla_Y Z)$$

So, by torsion-freedom

$$X(g(Y, Z)) - Z(g(X, Y)) + Y(g(Z, X)) = \\ g([Y, X] + 2\nabla_X Y, Z) + g(Y, [X, Z]) - g(X, [Z, Y])$$

Rearranging,

$$g(\nabla_X Y, Z) = \frac{1}{2}(X(g(Y, Z)) - Z(g(X, Y)) + Y(g(Z, X)) \\ + g(Z, [X, Y])) + g(Y, [Z, X]) - g(X, [Y, Z])$$

The theorem now follows by the invertibility of g .