

AXIOMATIC INTRODUCTION TO THE RIEMANN INTEGRAL

In elementary analysis texts and classes the Riemann integral is introduced constructively. This has the disadvantage that the constructions are complicated and poorly motivated. Another option would be to introduce the Riemann integral axiomatically. The axioms have very simple geometric motivation and the complication inherent in the construction can be deferred to a self-contained existence proof. Here will follow such an introduction.

Let (a, b) be an interval on the real line and let f be a real-valued function whose domain contains (a, b) . Define

$$M_a^b(f) = \sup f(a, b)$$

$$m_a^b(f) = \inf f(a, b)$$

A partition of an interval (a, b) is a sequence x_0, \dots, x_n with $a = x_0 < \dots < x_n = b$. Given $\epsilon > 0$, x_0, \dots, x_n is an (f, ϵ) -partition if and only if

$$\sum_{i=1}^n (x_i - x_{i-1}) \left(M_{x_{i-1}}^{x_i}(f) - m_{x_{i-1}}^{x_i}(f) \right) < \epsilon$$

f is Riemann integrable on (a, b) , $f \in R_a^b$, if and only if f is bounded on (a, b) , and for all $\epsilon > 0$ there exists an (f, ϵ) -partition of (a, b) . Then the Riemann integral \int is a set of maps $\int_a^b : R_a^b \rightarrow \mathbb{R}$ that obey the axioms of boundedness and additivity;

$$(b-a)m_a^b \leq \int_a^b \leq (b-a)M_a^b$$

$$\int_a^c + \int_c^b = \int_a^b$$

Note that by additivity and induction, if x_0, \dots, x_n is a partition of (a, b) , then $\int_a^b = \sum_{i=1}^n \int_{x_{i-1}}^{x_i}$.

It remains to show that the Riemann integral is unique, exists, and obeys all the familiar properties.

Proposition. \int is unique.

Proof. Let A and B be two Riemann integrals. Let $f \in R_a^b$, let $\epsilon > 0$, and let x_0, \dots, x_n be an (f, ϵ) -partition of (a, b) . Then,

$$\begin{aligned} |A_a^b(f) - B_a^b(f)| &= \left| \sum_{i=1}^n A_{x_{i-1}}^{x_i}(f) - B_{x_{i-1}}^{x_i}(f) \right| \\ &\leq \sum_{i=1}^n \left| A_{x_{i-1}}^{x_i}(f) - B_{x_{i-1}}^{x_i}(f) \right| \\ &\leq \sum_{i=1}^n (x_i - x_{i-1}) \left(M_{x_{i-1}}^{x_i}(f) - m_{x_{i-1}}^{x_i}(f) \right) \\ &< \epsilon \end{aligned}$$

But (a, b) , f , and ϵ are arbitrary so $A = B$. \square

Proposition. \int exists.

Proof. Given $f \in R_a^b$, let $S_a^b(f)$ be the set of

$$s(f; x_0, \dots, x_n) = \sum_{i=1}^n (x_i - x_{i-1}) m_{x_{i-1}}^{x_i}(f)$$

with x_0, \dots, x_n a partition of (a, b) . Then $\inf S_a^b(f) = (b-a)m_a^b(f)$ and $S_a^b(f)$ is bounded above by $(b-a)M_a^b(f)$. Let $\int_a^b(f) = \sup S_a^b(f)$. Then boundedness follows.

Let $s(f; x_0, \dots, x_n) \in S_a^b(f)$ and $c \in (a, b)$. If there exists j with $x_j = c$, then

$$s(f; x_0, \dots, x_n) = s(f; x_0, \dots, x_j) + s(f; x_j, \dots, x_n)$$

Otherwise, there exists j with $x_{j-1} < c < x_j$ so,

$$s(f; x_0, \dots, x_n) \leq s(f; x_0, \dots, x_{j-1}, c) + s(f; c, x_j, \dots, x_n)$$

So, $\int_a^c(f) \leq \int_a^b(f) + \int_b^c(f)$. Also,

$$s(f; a, \dots, c) + s(f; c, \dots, b) = s(f; a, \dots, c, \dots, b)$$

So, $\int_a^b(f) + \int_b^c(f) \leq \int_a^c(f)$. So additivity follows. \square

Proposition. *Linearity of integral part 1: If $k \in \mathbb{R}$, $f \in R_a^b$ then $kf \in R_a^b$ and $\int_a^b(kf) = k \int_a^b(f)$.*

Proof. If $k = 0$, then $kf = 0$ and for any $\epsilon > 0$, any partition of (a, b) is a (kf, ϵ) -partition. If $k \neq 0$, $\epsilon > 0$, then there exists an $(f, \frac{\epsilon}{|k|})$ -partition x_0, \dots, x_n of (a, b) which is also a (kf, ϵ) -partition. Hence $kf \in R_a^b$.

If $k \geq 0$, then

$$(b-a)m_a^b(kf) = k(b-a)m_a^b(f) \leq k \int_a^b(f) \leq k(b-a)M_a^b(f) = (b-a)M_a^b(kf)$$

and if $k < 0$, then

$$(b-a)m_a^b(kf) = k(b-a)M_a^b(f) \leq k \int_a^b(f) \leq k(b-a)m_a^b(f) = (b-a)M_a^b(kf)$$

Also, for any $k \in \mathbb{R}$,

$$k \int_a^c + k \int_c^b = k \left(\int_a^c + \int_c^b \right) = k \int_a^b$$

Hence by uniqueness, $k \int_a^b(f) = \int_a^b(kf)$. \square

Proposition. *Linearity of integral part 2: If $f, g \in R_a^b$ then $f + g \in R_a^b$ and $\int_a^b(f + g) = \int_a^b(f) + \int_a^b(g)$.*

Proof. If $\epsilon > 0$ then there is an $(f, \frac{\epsilon}{2})$ -partition and a $(g, \frac{\epsilon}{2})$ -partition of (a, b) and their union forms an $(f + g, \epsilon)$ -partition. Hence $f + g \in R_a^b$.

Let x_0, \dots, x_n be a (g, ϵ) -partition of (a, b) . Then,

$$\begin{aligned} \int_a^b (f+g) - \int_a^b (g) &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f+g) - \int_{x_{i-1}}^{x_i} (g) \\ &\leq \sum_{i=1}^n (x_i - x_{i-1}) \left(M_{x_{i-1}}^{x_i} (f+g) - m_{x_{i-1}}^{x_i} (g) \right) \\ &\leq \sum_{i=1}^n (x_i - x_{i-1}) \left(M_a^b (f) + M_{x_{i-1}}^{x_i} (g) - m_{x_{i-1}}^{x_i} (g) \right) \\ &< (b-a) M_a^b (f) + \epsilon \end{aligned}$$

But $\epsilon > 0$ is arbitrary so $\int_a^b (f+g) - \int_a^b (g) \leq (b-a) M_a^b (f)$. Similarly, $\int_a^b (f+g) - \int_a^b (g) \geq (b-a) m_a^b (f)$. Also,

$$\begin{aligned} &\left(\int_a^c (f+g) - \int_a^c (g) \right) + \left(\int_c^b (f+g) - \int_c^b (g) \right) \\ &= \int_a^c (f+g) + \int_c^b (f+g) - \left(\int_a^c (g) + \int_c^b (g) \right) \\ &= \int_a^b (f+g) - \int_a^b (g) \end{aligned}$$

Hence by uniqueness, $\int_a^b (f+g) - \int_a^b (g) = \int_a^b (f)$. \square

Proposition. *Mean value theorem for integrals: If f is continuous in $[a, b]$, then $f \in R_a^b$ and there exists $c \in [a, b]$ such that $\int_a^b (f) = (b-a) f(c)$*

Proof. By the extreme value theorem, f is bounded on $[a, b]$. By the uniform continuity theorem, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in [a, b]$, if $|x - x'| < \delta$ then $|f(x) - f(x')| < \frac{\epsilon}{b-a}$. Let $n \in \mathbb{N}$ with $n > \frac{b-a}{\delta}$ and let x_0, \dots, x_n be the partition of (a, b) with $x_i = a + i \frac{b-a}{n}$. Then

$$\sum_{i=1}^n (x_i - x_{i-1}) \left(M_{x_{i-1}}^{x_i} (f) - m_{x_{i-1}}^{x_i} (f) \right) < \sum_{i=1}^n \frac{b-a}{n} \frac{\epsilon}{b-a} = \epsilon$$

Hence, x_0, \dots, x_n is an (f, ϵ) -partition and $f \in R_a^b$.

By the intermediate value theorem, for any $y \in [m_a^b (f), M_a^b (f)]$ there is some $c \in [a, b]$ with $y = f(c)$. Hence by boundedness, $\int_a^b (f) = (b-a) f(c)$. \square

Proposition. *Fundamental theorem of calculus part 1: If f is continuous in $[a, b]$ and $F(x) = \int_a^x (f)$ for $x \in (a, b)$ then F is continuously differentiable and $F' = f$ in (a, b) .*

Proof. By the mean value theorem for integrals, if $h > 0$ then $\int_x^{x+h} (f) = hf(c)$ for some $c \in (x, x+h)$. Thus,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h} &= \lim_{h \rightarrow 0^+} \frac{\int_a^{x+h} (f) - \int_a^x (f)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{1}{h} \int_x^{x+h} (f) \\ &= \lim_{h \rightarrow 0^+} f(c) \\ &= f(x) \end{aligned}$$

Similarly, $\lim_{h \rightarrow 0^-} \frac{F(x+h) - F(x)}{h} = f(x)$. □

Proposition. *Fundamental theorem of calculus part 2: If f is differentiable in (a, b) and $f' \in R_a^b$ then $\int_a^b (f') = f(b) - f(a)$.*

Proof. By the mean value theorem for derivatives there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$ and hence

$$(b - a) m_a^b(f') \leq f(b) - f(a) \leq (b - a) M_a^b(f')$$

Also

$$(f(c) - f(a)) + (f(b) - f(c)) = f(b) - f(a)$$

Hence by uniqueness, $f(b) - f(a) = \int_a^b (f')$. □