THE TRANSPOSE

Definition. The transpose of an $m \times n$-matrix A, is the $n \times m$-matrix A^T which exchanges rows for columns. If A has entries a_{ij} then A^T has entries a_{ji}.

Example. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix}^T = \begin{pmatrix} 1 & 6 & 11 \\ 2 & 7 & 12 \\ 3 & 8 & 13 \\ 4 & 9 & 14 \\ 5 & 10 & 15 \end{pmatrix}$

Proposition. *The dot product is* $v \cdot w = v^T w$

Proof.

$$\begin{aligned} v^T w &= \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}^T \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\ &= \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\ &= v_1 w_1 + \cdots + v_n w_n = v \cdot w \end{aligned}$$

□

Proposition. $(A^T)^T = A$

Proof. Exchange rows for columns. Then do it again. You get back what you started with. □

Example. $\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T\right)^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Proposition. $(AB)^T = B^T A^T$

Proof. Remember that the i, j-entry of AB is the dot product of the ith row of A and the jth column of B. Thus the i, j-entry of $(AB)^T$ is the dot product of the jth row of A and the ith column of B. Thus the i, j-entry of $(AB)^T$ is the dot product of jth column of A^T and the ith row of B^T. By symmetry of the dot product, the i, j-entry of $(AB)^T$ is the dot product of the ith row of B^T and the jth column of A^T. Thus the i, j-entry of $(AB)^T$ is the i, j-entry of $B^T A^T$. Thus $(AB)^T = B^T A^T$. □

Example.

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right)^T = \begin{pmatrix} 50 \\ 122 \end{pmatrix}^T = \begin{pmatrix} 50 & 122 \end{pmatrix}$$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 50 & 122 \end{pmatrix}$$

Proposition. *Given $v \in \mathbb{R}^n, w \in \mathbb{R}^m$ and an $n \times m$-matrix A, $v \cdot (Aw) = \left(A^T v\right) \cdot w$*

Proof.

$$v \cdot (Aw) = v^T A w = v^T \left(A^T\right)^T w = \left(A^T v\right)^T w = \left(A^T v\right) \cdot w$$

□

Definition. A system of linear equations is what it sounds like

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1m}x_m &= c_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m &= c_n \end{aligned}$$

We say it is homogeneous iff $c_1 = \cdots = c_n = 0$.

Note that we can express this as a matrix equation

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Example. $\begin{matrix} x + y + z = 4 \\ x - y + z = 2 \\ x + z = 3 \end{matrix}$ can be expressed as $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

It may be that a system of linear equations $Ax = c$ has no solution. In this case, we want to find an approximate solution called the least squares approximation. To do this we try to find an x which minimizes $|Ax - c|^2$.

Theorem. *The least squares approximation to $Ax = c$ is the solution to the linear equation $A^T A x = A^T c$.*

Example. Say you wanted to fit a line through 3 points $(0, 1), (1, 3), (2, -1)$. These points are not collinear so there is no such line. Instead we find a least squares approximation. The line is $y = mx + b$, so if it went through all the points, then

$$1 = m \cdot 0 + b$$

$$3 = m \cdot 1 + b$$

$$-1 = m \cdot 2 + b$$

or as a matrix equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

Multiplying by the transpose gives

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{array}{c} 5m + 3b = 1 \\ 3m + 3b = 3 \end{array}$$

$$\Rightarrow m = -1, b = 2$$

Thus the least squares approximation is $y = -x + 2$.