HERMITIAN INNER PRODUCTS

Recall that we defined the absolute value of a complex number $z = a + bi$ to be $|z| = \sqrt{a^2 + b^2}$.

Definition. Given a complex number $z = a + bi$, its conjugate is $\bar{z} = a - bi$.

The conjugate is well-behaved under addition and multiplication $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$ and it is idempotent $\overline{\overline{z}} = z$. A complex number x is real iff $\overline{x} = x$.

Proposition. $|z|^2 = z\bar{z}$

Proof. $z\bar{z} = (a + bi)(a - bi) = (a^2 + b^2) + (ba - ab)i = a^2 + b^2 = |z|^2$ ☐

We wish to generalize this to $\mathbb{C}^n$. Define the Hermitian dot product to be

$$(z_1, \ldots, z_n) \cdot (w_1, \ldots, w_n) = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

Definition. Given a complex vector space V, an Hermitian inner product is a complex number $\langle v|w\rangle$ for every $v, w \in V$ such that

1. Positivity: $\langle v|v\rangle$ is a nonnegative real number, i.e. $\langle v|v\rangle \geq 0$.
2. Definiteness: If $\langle v|v\rangle = 0$, then $v = 0$.
3. Linearity in the first argument: $\langle u + v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ and $\langle cu, v\rangle = c\,\langle u, v\rangle$
4. Conjugate symmetry: $\langle u|v\rangle = \overline{\langle v|u\rangle}$

Example. Obviously, the Hermitian dot product is an Hermitian inner product on $\mathbb{C}^n$. If we consider the space of continuous complex valued functions of a real variable, we have an Hermitian inner product given by $\langle f(x)|g(x)\rangle = \int_0^1 f(x)\overline{g(x)}dx$.

Definition. We again define the norm of a vector to be $|v| = \sqrt{\langle v|v\rangle}$ and say that a vector is normalized iff $|v| = 1$ and two vectors are orthogonal iff $\langle v|w\rangle = 0$. An orthonormal basis is once again a basis whose elements are normalized and pairwise orthogonal.

Note that Hermitian inner products are not bilinear. They fail to be linear in the second argument since

$$\langle u|cv\rangle = \overline{\langle cv|u\rangle} = \overline{c\,\langle v|u\rangle} = \bar{c}\ \overline{\langle v|u\rangle} = \bar{c}\ \overline{\overline{\langle u|v\rangle}} = \bar{c}\ \ \langle u|v\rangle$$

We say that it is conjugate-linear in the second argument.

We can write the Hermitian dot product as $v \cdot w = v^T\overline{w}$.

Consider an linear transformation of complex vector spaces $T : V \to W$. Its adjoint $T^* : W \to V$ is the linear map which obeys $\langle T(v)|w\rangle = \langle v|T^*(w)\rangle$. For instance, if A is an $n \times m$-matrix, then thinking of it as a linear transformation $A : \mathbb{C}^m \to \mathbb{C}^n$, its adjoint is gotten by transposing and conjugating all its entries $\overline{A}^T$ since

$$Av \cdot w = (Av)^T\overline{w} = v^T A^T\overline{w} = v^T\overline{\overline{A}^T}\,\overline{w} = v^T\overline{(\overline{A}^T w)} = v \cdot (\overline{A}^T w)$$

Example. The conjugate-transpose of $\begin{pmatrix} i & 1+i & 2 \\ 4 & 3 & -i \end{pmatrix}$ is $\begin{pmatrix} -i & 4 \\ 1-i & 3 \\ 2 & i \end{pmatrix}$.

Definition. We call a linear map $T : V \to W$ unitary iff $T^* = T^{-1}$ and we call a linear operator $S : V \to V$ self-adjoint iff $S^* = S$.

Proposition. *Every eigenvalue of a self-adjoint operator is real.*

Proof. Let λ be an eigenvalue for a self-adjoint operator $S : V \to V$. This means that there is some nonzero vector $v \in V$ with $S(v) = \lambda v$. Then

$$\lambda \langle v|v\rangle = \langle \lambda v|v\rangle = \langle S(u)|v\rangle = \langle u|S^*(v)\rangle = \langle u|S(v)\rangle = \langle v|\lambda v\rangle = \overline{\lambda} \langle v|v\rangle$$

Since $v \neq 0$, $\langle v|v\rangle \neq 0$ and we can divide it from both sides leaving $\lambda = \overline{\lambda}$. Thus, λ is real. □

Theorem. *(Spectral theorem) Given a self-adjoint operator $S : V \to V$, there is an orthonormal basis of eigenvectors of S.*

Example. Consider $\begin{pmatrix} \frac{3}{2} & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 2 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & \frac{5}{2} \end{pmatrix}$. It is self-adjoint. It's characteristic polynomial is

$$det \begin{pmatrix} \frac{3}{2}-\lambda & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 2-\lambda & 0 \\ \frac{\sqrt{3}}{2}i & 0 & \frac{5}{2}-\lambda \end{pmatrix}$$

$$\begin{aligned} &= (\frac{3}{2}-\lambda)(2-\lambda)(\frac{5}{2}-\lambda) - (\frac{\sqrt{3}}{2}i)(2-\lambda)(-\frac{\sqrt{3}}{2}i) \\ &= (2-\lambda)((\frac{3}{2}-\lambda)(\frac{5}{2}-\lambda) + (\frac{\sqrt{3}}{2}i)^2) \\ &= (2-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= (2-\lambda)(\lambda-3)(\lambda-1) \end{aligned}$$

The characteristic polynomial has roots $1, 2, 3$ so these are the eigenvalues. We need to find eigenvectors belonging to them. Let's look for a normalized eigenvector belonging to 1.

$$\begin{pmatrix} \frac{3}{2} & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 2 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{3}{2}x - \frac{\sqrt{3}}{2}iz \\ 2y \\ \frac{\sqrt{3}}{2}ix + \frac{5}{2}z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solving gives $y = 0$ and $z = -\frac{\sqrt{3}}{3}ix$ so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ (\frac{-\sqrt{3}}{3}i)x \end{pmatrix}$$

We want a normalized eigenvector so divide it by its length, or simply plug in $x = \frac{\sqrt{3}}{2}$ and get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2}i \end{pmatrix}$$

Next find a normalized eigenvector belonging to 2.

$$\begin{pmatrix} \frac{3}{2} & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 2 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{3}{2}x - \frac{\sqrt{3}}{2}iz \\ 2y \\ \frac{\sqrt{3}}{2}ix + \frac{5}{2}z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Solving gives $x = z = 0$ so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$$

Or normalized,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$$

A normalized eigenvector belonging to 3 is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$