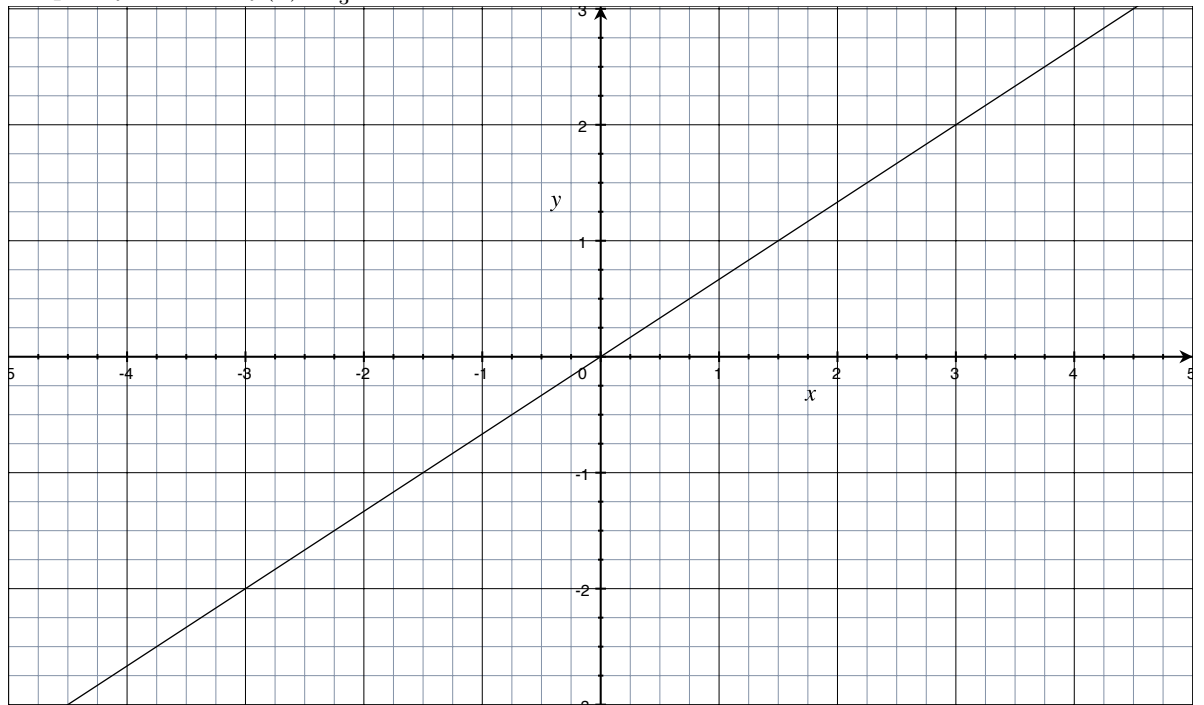


LINEAR ALGEBRA

In linear algebra we study linear maps between vector spaces.

Example. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{2}{3}x$



Definitions will come later. For now notice that the graph is that of a line and that the line goes through the origin. Notice also that the function f is just multiplication by a number, specifically $\frac{2}{3}$.

The pedagogy of our class will split into three parts:

- (1) Theory
- (2) Computation
- (3) Applications

We will emphasize theory but all are important. Without further ado, let's begin.

FIELDS

Fields are algebraic gadgets which formalize our notion of being able to add, subtract, multiply and divide numbers.

Definition. A field is a set F with two binary operations, addition denoted by $+$, and multiplication denoted by juxtaposition which obey the following axioms:

Field Axioms	Addition	Multiplication
Associativity	$\forall a, b, c \in F,$ $a + (b + c) = (a + b) + c$	$\forall a, b, c \in F,$ $a(bc) = (ab)c$
Commutativity	$\forall a, b \in F,$ $a + b = b + a$	$\forall a, b \in F,$ $ab = ba$
Distributivity/Linearity	$\forall a, b, c \in F,$ $(a + b)c = ac + bc,$ and	$a(b + c) = ab + ac$
Identity	$\exists 0 \in F : \forall a \in F,$ $0 + a = a = a + 0$	$\exists 1 \in F : \forall a \in F,$ $1a = a = a1$
Inverse	$\forall a \in F, \exists -a \in F :$ $(-a) + a = 0 = a + (-a)$	$\forall a \neq 0, \exists a^{-1} \in F :$ $(a^{-1})a = 1 = a(a^{-1})$

Example. Examples of fields are provided by the rational numbers \mathbb{Q} , the real numbers \mathbb{R} , and the complex numbers \mathbb{C} , which we will discuss.

We can define subtraction and division by $a - b = a + (-b)$ and $a/b = a(b^{-1})$. However, in order to do this we must be certain that $-b$ and b^{-1} are unique.

Proposition. 0 and 1 are unique and for given $a \in F$, $-a$ and a^{-1} are unique.

Proof. Suppose we had two additive identities 0 and $\tilde{0}$. Then using our additive identity axiom we have

$$\tilde{0} = \tilde{0} + 0 = 0$$

So in fact, $\tilde{0} = 0$ and indeed the additive identity is unique.

Suppose we had two multiplicative identities 1 and $\tilde{1}$. Then,

$$\tilde{1} = \tilde{1}1 = 1$$

Suppose we had two additive inverses $-a$ and $\sim a$ of a . Then,

$$\sim a = (\sim a) + 0 = (\sim a) + (a + (-a))$$

$$= ((\sim a) + a) + (-a) = 0 + (-a) = -a$$

Suppose we had two multiplicative inverses a^{-1} and $a^{\sim 1}$. Then,

$$\begin{aligned} a^{\sim 1} &= (a^{\sim 1})1 = (a^{\sim 1})(a(a^{-1})) \\ &= ((a^{\sim 1})a)(a^{-1}) = 1(a^{-1}) = a^{-1} \end{aligned}$$

□

Proposition. For all $a \in F$, $0a = a0 = 0$ and $(-1)a = -a$

Proof. Using the additive identity axiom and distributivity we can see that

$$0a = (0 + 0)a = 0a + 0a$$

Now subtract $0a$ from both sides. By definition of subtraction we get

$$0 = 0a - 0a = (0a + 0a) - 0a = 0a + (0a - 0a) = 0a + 0 = 0a$$

Then by commutativity of multiplication we also get $a0 = 0a = 0$.

□