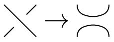
KHOVANOV HOMOLOGY

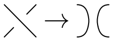
Cube of Resolutions

Let D_K be a tangle diagram. Assign to each crossing of D_K an integer $\{1, \ldots, n\}$.

$\{0,1\}^n$ forms the vertices of a hypercube with edges $\alpha \xrightarrow{k} \beta$ where $\alpha_j = \beta_j$ for $j \neq k$ and $\alpha_k = 0$, $\beta_k = 1$.

For each $\alpha \in \{0,1\}^n$, associate the smoothing D_α of D_K where each crossing j is α_j-smoothed. If K is a link then, D_α is a disjoint union of circles in the plane.

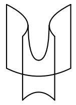
0-smoothing: 

1-smoothing: 

For each edge $\alpha \xrightarrow{k} \beta$, associate the cobordism of smoothings $D_{\alpha,\beta}$ which outside a neighborhood of the crossing change $N(k) \times I$ is the identity cobordism

$$(D_\alpha - N(k)) \times I = (D_\beta - N(k)) \times I$$

and within $N(k) \times I$ is a saddle cobordism. If K is a link then, $D_{\alpha,\beta}$ is a disjoint union of cylinders and one (possibly upside-down) pair of pants.

Saddle cobordism: 

For each $\alpha \in \{0,1\}$ we can associate $r_\alpha = \sum_{j=1}^{n} \alpha_j$ and for each edge $\alpha \xrightarrow{k} \beta$ associate $r_{\alpha,\beta} = \sum_{j=1}^{k-1} \alpha_j = \sum_{j=1}^{k-1} \beta_j$.

Khovanov Bracket

Define $[D_K] = \{[D_K]^i, d^i\}$ by

$$[D_K]^i = \bigoplus_{r_\alpha = i} D_\alpha$$

and $d^i : [D_K]^i \to [D_K]^{i+1}$ is the matrix of cobordisms

$$d^i_{\alpha,\beta} = \begin{cases} (-1)^{r_{\alpha,\beta}} D_{\alpha,\beta} & \alpha \xrightarrow{i} \beta \\ 0 & \text{otherwise} \end{cases}$$

Let $\begin{array}{ccc}\alpha & \to & \beta\\ \downarrow & & \downarrow\\ \gamma & \to & \delta\end{array}$ be a face of the hypercube. Then $D_{\beta,\delta}D_{\alpha,\beta} = D_{\gamma,\delta}D_{\alpha,\gamma}$ since these composite cobordisms are homeomorphic. Also

$$d^2 = \sum_{\begin{array}{ccc}\alpha & \to & \beta\\ \downarrow & & \downarrow\\ \gamma & \to & \delta\end{array}} (-1)^{r_{\alpha,\beta}}(-1)^{r_{\beta,\delta}}D_{\beta,\delta}D_{\alpha,\beta} + (-1)^{r_{\alpha,\gamma}}(-1)^{r_{\gamma,\delta}}D_{\gamma,\delta}D_{\alpha,\gamma}$$

But if $\alpha \xrightarrow{j} \beta$, $\gamma \xrightarrow{j} \delta$, $\beta \xrightarrow{k} \delta$ and $\alpha \xrightarrow{k} \gamma$, then if $j < k$ then $r_{\alpha,\beta} = r_{\gamma,\delta}$ and $r_{\beta,\delta} = r_{\alpha,\gamma} + 1$ and if $j > k$ then $r_{\alpha,\beta} = r_{\gamma,\delta} - 1$ and $r_{\beta,\delta} = r_{\alpha,\gamma}$. Thus,

$$d^2 = 0$$

We need to construct the category where $[D_K]$ lives. Given finite $A, B \subset (0,1)$, let $Cob(A,B)$ be the category with objects 1-manifolds D in $(0,1) \times [0,1]$ with endpoints $\partial D = A \times \{0\} \cup B \times \{1\}$ and with morphisms, cobordisms $D_\alpha \to D_\beta$. These cobordisms are 2-manifolds $D_{\alpha,\beta}$ with corners embedded in $(0,1)\times[0,1]\times[0,1]$ with boundary

$$\partial D_{\alpha,\beta} = \underbrace{A \times \{0\} \times [0,1] \cup B \times \{1\} \times [0,1]}_{y\text{-boundary}} \cup \underbrace{D_\alpha \times \{0\} \cup D_\beta \times \{1\}}_{z\text{-boundary}}$$

Cobordisms are considered up to boundary preserving ambient isotopies. Composition in $Cob(A,B)$, called vertical composition, is given by gluing along the z-boundary. There is a functor called horizontal composition $Cob(A,B)\times Cob(B,C) \to Cob(A,C)$ given by gluing along the y-boundary. There is another functor called the monoidal product $Cob(A,B) \times Cob(A',B') \to Cob(A \sqcup A', B \sqcup B')$ given by disjoint union, thought of as gluing along the (non-existent) x-boundary. This is the structure of a monoidal 2-category, Cob.

We can extend $Cob(A,B)$ by taking the preadditive closure $\overline{Cob(A,B)}$. Include all $\mathbb{Z}$-linear combinations of cobordisms $D_\alpha \to D_\beta$, and extend vertical composition linearly. Horizontal composition and monoidal product extend to $\overline{Cob}$ linearly.

Next take the additive closure $\overline{\overline{Cob(A,B)}}$; include direct sums of objects $\oplus_\alpha D_\alpha$ and a 0 object with matrices of (linear combinations of) cobordisms between them and do vertical composition like matrix multiplication. Horizontal composition and monoidal product extend to $\overline{Cob^*}$ by acting like tensor products.

Finally, let $Kob(A,B)$ be the category of cochain complexes in $\overline{\overline{Cob(A,B)}}$. Vertical composition in $Kob(A,B)$ is composition of chain maps. Horizontal composition and monoidal product extend to Kob by acting like tensor product of complexes. If D_K has source and target endpoints A, B, then $[D_K]$ is an object in $Kob(A,B)$.

LOCALITY AND SKEIN RELATION

Theorem. *The Khovanov bracket has the following properties:*

1. If D_K is the composition of two tangle diagrams D_0, D_1 then $[D_K]$ is the horizontal composition $[D_0] \otimes [D_1]$
2. If $D_K = D_0 \sqcup D_1$ then $[D_K]$ is the monoidal product $[D_0] \sqcup [D_1]$.

(3) If tangle diagram D_K has $0, 1$-resolutions D_0, D_1 at some crossing j, then $[D_K]$ is the cone of the chain map $[D_0] \to [D_1]$ induced by the cobordism which is a saddle in the neighborhood of j.

The first two properties comprise locality of the Khovanov bracket and the third property is the skein relation.

Proof. Choose an ordering of the crossings in D_K so that crossings in D_0 are first. Then, a smoothing D_α is a composition of smoothings $D_{\alpha^0} \otimes D_{\alpha^1}$ where $(\alpha_1, \dots, \alpha_n) = (\alpha_1^0, \dots, \alpha_j^0, \alpha_1^1, \dots, \alpha_{n-j}^1)$. We have that $r_\alpha = r_{\alpha^0} + r_{\alpha^1}$ so that

$$[D_K]^i = \bigoplus_{r_{\alpha=i}} D_\alpha = \bigoplus_{r_{\alpha^0}+r_{\alpha^1}=i} D_{\alpha^0} \otimes D_{\alpha^1} = ([D_0] \otimes [D_1])^i$$

Given an edge $\alpha \to \beta$, either there is an edge $\alpha^0 \to \beta^0$ and $\alpha^1 = \beta^1$ or $\alpha^0 = \beta^0$ and there is an edge $\alpha^1 \to \beta^1$. Writing id_α for identity cobordisms, we have that $D_{\alpha,\beta} = D_{\alpha^0,\beta^0} \otimes id_{\alpha^1}$ in the first case or $D_{\alpha,\beta} = id_{\alpha^0} \otimes D_{\alpha^1,\beta^1}$ in the second case. The matrix d_ϵ is $(-1)^{r_{\alpha^\epsilon,\beta^\epsilon}} D_{\alpha^\epsilon,\beta^\epsilon}$ if there is an edge $\alpha^\epsilon \to \beta^\epsilon$ and 0 otherwise. The matrix id_ϵ is id_{α^ϵ} if $\alpha^\epsilon = \beta^\epsilon$ and 0 otherwise. Thus,

$$d_0 \otimes id_1 + (-1)^{r_{\alpha^0}} id_0 \otimes d_1 =$$

$$\begin{cases} (-1)^{r_{\alpha^0,\beta^0}} D_{\alpha^0,\beta^0} \otimes id_{\alpha^1} \pm 0 \otimes 0, & \alpha_0 \to \beta_0, \alpha_1 = \beta_1 \\ 0 \otimes 0 + (-1)^{r_{\alpha^0}} id_{\alpha^0} \otimes (-1)^{r_{\alpha^1,\beta^1}} D_{\alpha^1,\beta^1} & \alpha_0 = \beta_0, \alpha_1 \to \beta_1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (-1)^{r_{\alpha,\beta}} D_{\alpha,\beta} & \alpha \to \beta \\ 0 & \text{otherwise} \end{cases} = d$$

So, $[D_K] = [D_0] \otimes [D_1]$. The proof for monoidal product is entirely analagous.

In the case $D_K = \times$, the skein relation is easy to verify.

$$C \begin{pmatrix} [\rangle\langle] \\ \uparrow \\ [\asymp] \end{pmatrix} = C \begin{pmatrix} 0 & \leftarrow & \rangle\langle & \leftarrow & 0 \\ & & \uparrow & D_{0,1} & \\ 0 & \leftarrow & \asymp & \leftarrow & 0 \end{pmatrix}$$

$$= 0 \leftarrow 0 \oplus \rangle\langle \xleftarrow{\begin{pmatrix} 0 & 0 \\ D_{0,1} & 0 \end{pmatrix}} \asymp \oplus 0 \leftarrow 0$$

$$= 0 \leftarrow \rangle\langle \xleftarrow{D_{0,1}} \asymp \leftarrow 0 = [\times]$$

The general case then follows by locality and the properties of cones of tensor products. □

GRADING

For $D_{\alpha,\beta}$ a cobordism in $Cob(A, B)$ let $deg(D_{\alpha,\beta}) = \chi(D_{\alpha,\beta}) - \frac{|A|+|B|}{2}$.

We can see that vertical and horizontal composition and monoidal product are degree additive. For vertical composition,

$$deg(D_{\beta,\gamma} D_{\alpha,\beta}) = \chi(D_{\beta,\gamma} D_{\alpha,\beta}) - \frac{|A| + |B|}{2}$$

$$= \chi(D_{\beta,\gamma}) + \chi(D_{\alpha,\beta}) - \chi(D_\beta) - \frac{|A| + |B|}{2}$$

$$= \chi(D_{\beta,\gamma}) - \frac{|A| + |B|}{2} + \chi(D_{\alpha,\beta}) - \frac{|A| + |B|}{2}$$

$$= deg(D_{\beta,\gamma}) + deg(D_{\alpha,\beta})$$

For horizontal composition,

$$\begin{aligned} deg(D_{\alpha,\beta} \otimes D_{\alpha',\beta'}) &= \chi(D_{\alpha,\beta} \otimes D_{\alpha',\beta'}) - \frac{|A|+|C|}{2} \\ &= \chi(D_{\alpha,\beta}) + \chi(D_{\alpha',\beta'}) - |B| - \frac{|A|+|C|}{2} \\ &= \chi(D_{\alpha,\beta}) - \frac{|A|+|B|}{2} + \chi(D_{\alpha',\beta'}) - \frac{|B|+|C|}{2} \\ &= deg(D_{\alpha,\beta}) + deg(D_{\alpha',\beta'}) \end{aligned}$$

For monoidal product,

$$\begin{aligned} deg(D_{\alpha,\beta} \sqcup D_{\alpha',\beta'}) &= \chi(D_{\alpha,\beta} \sqcup D_{\alpha',\beta'}) - \frac{|A \sqcup A'|+|B \sqcup B'|}{2} \\ &= \chi(D_{\alpha,\beta}) - \frac{|A|+|B|}{2} + \chi(D_{\alpha',\beta'}) - \frac{|A'|+|B'|}{2} \\ &= deg(D_{\alpha,\beta}) + deg(D_{\alpha',\beta'}) \end{aligned}$$

Thus, the set of morphisms $D_\alpha \to D_\beta$ in $\overline{Cob(A,B)}$ is a graded Abelian group $\bigoplus_i G_i$. Let $Cob_0(A,B)$ be the category with objects $D_\alpha\{a\}$, $a \in \frac{1}{2}\mathbb{Z}$ and morphisms $D_\alpha\{a\} \to D_\beta\{b\}$ equal to $\begin{cases} G_{b-a} & b-a \in \mathbb{Z} \\ \{0\} & \text{otherwise} \end{cases}$. Also, let $Kob_0(A,B)$ be the category of chain complexes in the additive closure, $\overline{Cob_0(A,B)}$. Cob_0 and Kob_0 are monoidal 2-categories, by degree additivity.

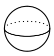
Choose an orientation on D_K. Let n_-, n_+ be the number of negative, and positive crossings respectively. We define the graded Khovanov bracket

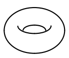
$$[D_K]^i = \bigoplus_{r_\alpha = n_- + i} D_\alpha\{\frac{n_+ - n_-}{2} - i\}$$

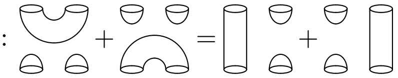
The coboundary still has the same definition. Identity cobordisms are disjoint unions of cylinders which have degree 0 and squares which also have degree 0, while saddle cobordisms have degree -1. Indeed, the coboundary components are in $D_\alpha\{\frac{n_+-n_-}{2} - i\} \to D_\beta\{\frac{n_+-n_-}{2} - i - 1\} = G_{-1}$. Thus the coboundary is a morphism in $\overline{Cob_0(A,B)}$ and $[D_K]$ is an object in $Kob(A,B)$.

INVARIANCE

In order to get homotopy equivalences from Reidemeister moves we have to quotient by certain local relations.

Sphere relation:  $= 0$

Torus relation:  $= 2$

Four tube relation: 

These relations make sense in $\overline{Cob}$, so in Kob_0. Modding out by local relations and homotopy equivalences we get $Kob_{0,l,h}$.

Theorem. *The isomorphism class of the graded Khovanov bracket* $[D_K]$ *in* $Kob_{0,l,h}$ *are invariants of the framed oriented tangle* K *represented by* D_K *as a blackboard diagram.*

Some preliminary work must proceed the proof.

Permutation of Crossings. Let σ be a permutation of $\{1,\dots,n\}$. Let σD_K be the tangle diagram D_K with crossings relabeled according to σ. Given $\alpha \in \{0,1\}^n$, let $\sigma\alpha$ be given by $(\sigma\alpha)_i = \alpha_{\sigma(i)}$. Then, $r_{\sigma\alpha} = r_\alpha$ so, given $\alpha \xrightarrow{k} \beta$, and working in the ungraded case

$$[\sigma D_K]^i = \bigoplus_{r_\alpha = i} D_\alpha$$

$$(\sigma d)^i_{\alpha,\beta} = \begin{cases} (-1)^{r_{\sigma\alpha,\sigma\beta}} D_{\alpha,\beta} & \alpha \xrightarrow{i} \beta \\ 0 & \text{otherwise} \end{cases}$$

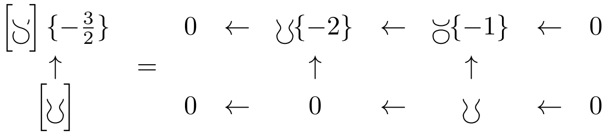
Let $R_\alpha = \{j : \alpha_j = 1\}$. The natural ordering on $\{1,\dots,n\}$ induces an ordering on R_α. σ induces a permutation σ_α on the ordering of R_α. Define $\begin{array}{c}[\sigma D_K]\\ \uparrow \Sigma\\ [D_K]\end{array}$ where Σ^i is a diagonal matrix with $\Sigma^i_{\alpha,\alpha} = sign(\sigma_\alpha) id_{D_\alpha}$. Then,

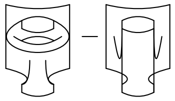
$$(\sigma d)^i_{\alpha,\beta}\Sigma^i_{\alpha,\alpha} = (-1)^{r_{\sigma\alpha,\sigma\beta}} sign(\sigma_\alpha) D_{\alpha,\beta}$$

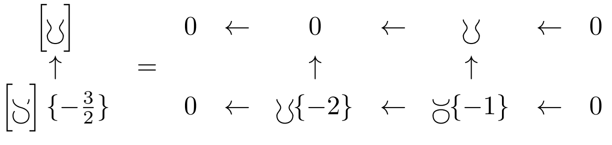
$$\Sigma^{i+1}_{\beta,\beta} d^i_{\alpha,\beta} = sign(\sigma_\beta)(-1)^{r_{\alpha,\beta}} D_{\alpha,\beta}$$

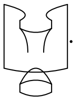
The permutation on $R_\beta = R_\alpha \sqcup \{k\}$ can be gotten as the permutation on R_α followed by moving k. Thus, if $sign(\sigma_\alpha) = sign(\sigma_\beta)$, then σ moves k an even number of places in the ordering of R_β. But $r_{\alpha,\beta}$ is the number of elements j of $R_{\alpha,\beta}$ with $j < k$. So, $r_{\alpha,\beta} = r_{\sigma\alpha,\sigma\beta} \bmod 2$. Alternatively, if $sign(\sigma_\alpha) \neq sign(\sigma_\beta)$, then σ moves k an odd number of places and $r_{\alpha,\beta} \neq r_{\sigma\alpha,\sigma\beta} \bmod 2$. Thus, $(\sigma d)^i\Sigma^i = \Sigma^{i+1}d^i$ so Σ is a chain map. Indeed, Σ is invertible since σ is. Thus, $[D_K]$ and $[\sigma D_K]$ are isomorphic. This works in the graded case too since shifts have no effect.

Positive Writhe Reidemeister 1. Define chain maps:

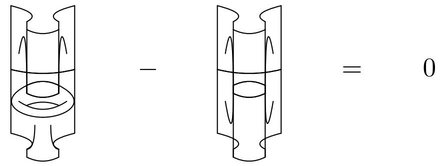


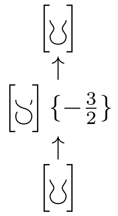
where  is the difference of degree −1 cobordisms  and



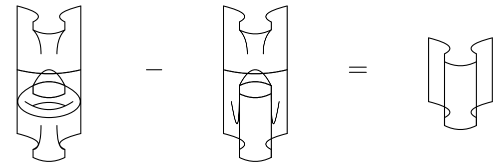
where  is the degree 1 cobordism .

We can see that these are chain maps. The only thing to check is that

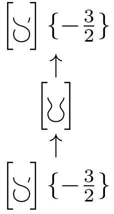


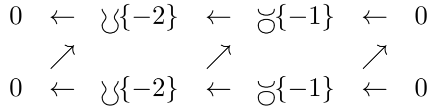
since the cobordisms are isotopic. We can also see that the composition 

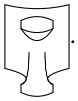
is the identity since

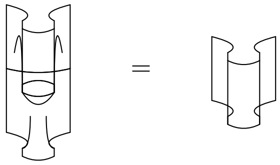


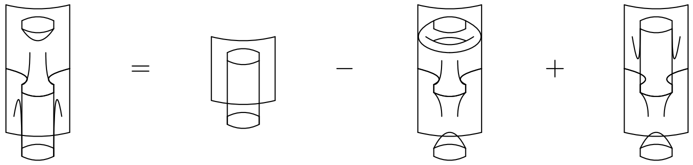
by the torus relation. We can construct a homotopy between the composition

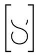
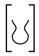
 and the identity:



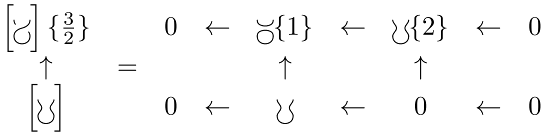
where  is the degree 1 cobordism . To see it is a homotopy we need to check two things.

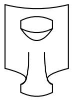


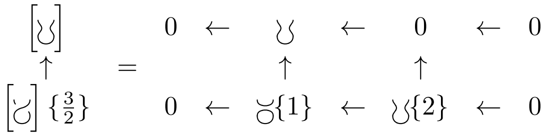


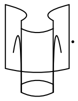
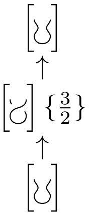
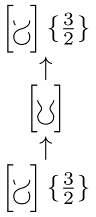
The first is trivial and the second follows from the four tube relation. Thus,  $\{-1\}$ and  are homotopy equivalent.

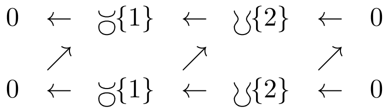
Negative Writhe Reidemeister 1. Define chain maps:

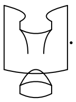
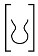


where  is the degree 1 cobordisms  and

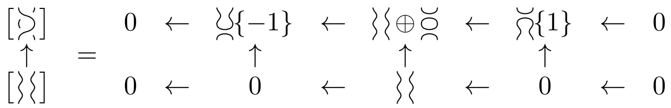


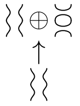
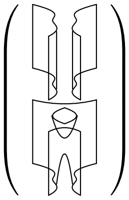
where  is the difference of degree −1 cobordisms  − . Once again, these are chain maps and the composition  is the identity by the torus relation. Also, by the four tube relation, the composition  is homotopic to the identity by

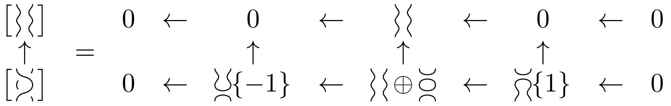


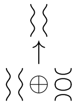
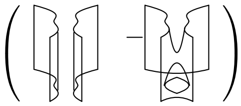
where  is the degree 1 cobordism . Thus  $\{\frac{3}{2}\}$ and  are homotopy equivalent.

Reidemeister 2. We can also construct chain maps

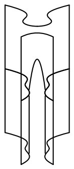
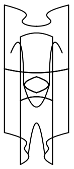


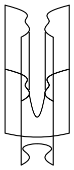
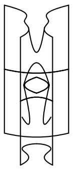
where  is the matrix of degree 0 cobordisms  and

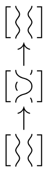


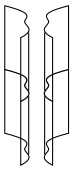
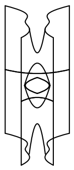
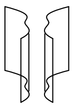
where  is the matrix of signed degree 0 cobordisms .

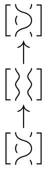
These are chain maps since

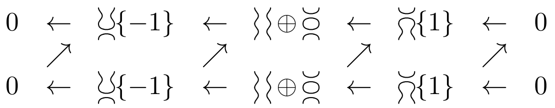
 $-$  $= \quad 0$

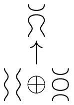
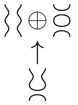
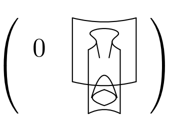
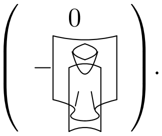
 $-$  $= \quad 0$

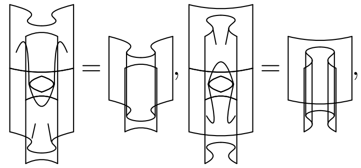
The composition  is the identity since by the sphere relation

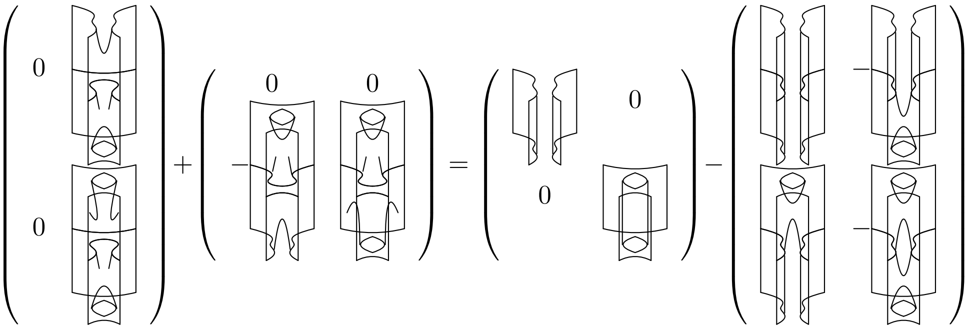
 $-$  $=$ 

There is a homotopy between  and the identity given by

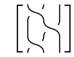
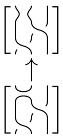
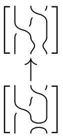
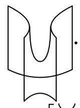
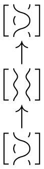
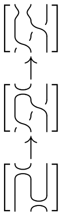
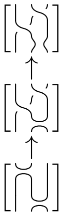


where  and  are degree 1 matrices  and .

This is a homotopy since , and



The only nontrivial part is the bottom right entry of the matrix equation which follows from the four tube relation.

Reidemeister 3. By the skein relation we have that  and  are cones of the chain maps  and  induced by . The Reidemeister 2 homotopy equivalence is a strong deformation retract:  is the identity and  is homotopic to the identity. Thus, by the properties of cones,  and  are homotopy equivalent to the cones of  and  which are equal as chain maps.

Invariance Proof. Since permutations induce isomorphisms, the isomorphism class of Khovanov brackets is independent of ordering. Two blackboard diagrams for a tangle K are related locally by Reidemeister moves. Reidemeister 2 and 3 moves induce local isomorphisms. Framed Reidemeister 1 moves, that is elimination of adjacent positive and negative writhe twists, induce a local isomorphism of the Khovanov bracket: that isomorphism is the horizontal composite of the 2 Reidemeister 1 isomorphisms obtained above. Thus, by locality, the theorem follows.