A lattice is an additive subgroup $\Lambda \subset \mathbb{C}$ generated by $\omega_1, \omega_2 \in \mathbb{C}$ with ω_1/ω_2 nonreal.

An elliptic function with respect to Λ is a meromorphic function $f : \mathbb{C} \to \mathbb{C}_\infty$ such that $\forall z \in \mathbb{C}, \omega \in \Lambda, f(z+\omega) = f(z)$. Alternatively, since f is constant on cosets of Λ, it can be thought of as a meromorphic function $f \in \mathcal{M}(\mathbb{C}/\Lambda)$.

Proposition: An elliptic function with no poles or no zeroes is constant.

Proof: If f has no poles then $|f|$ is bounded on the entire complex plane so this follows by Liouville's theorem. If f has no zeroes, then $1/f$ has no poles, so is constant, so f is.

Proposition: $\sum_{z\in P(f)} res_z(f) = 0$.

Proof: Given $a \in \mathbb{C}$, let $P_a = \{a + t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\}$. By ∂P_a, is meant a parametrized curve traced positively around P_a. Choose a so that none of the poles of f fall on the boundary of P_a. Then $\sum_{z\in P(f)\cap P_a} res(f; z) = (2\pi i)^{-1} \int_{\partial P_a} f$. By the periodicity of f, the four boundary component integrals cancel in pairs. The full sum then must be zero.

Corollary: $\sum_{z\in P(f)\cup Z(f)} ord_z(f) = 0$. Here ord is the order of the zero or minus the order of the pole.

Proof: This is the previous proposition applied to $f'/f \in \mathcal{M}(\mathbb{C}/\Lambda)$.

The Weierstrass $\wp$-function is the series $\wp_\Lambda(z) = z^{-2} + \sum_{\omega\in\Lambda'}(z-\omega)^{-2} - \omega^{-2}$ where $\Lambda' = \Lambda - 0$.

Proposition: The series $\wp_\Lambda$ converges absolutely on $\mathbb{C} - \Lambda$ and uniformly on compact subsets of $\mathbb{C} - \Lambda$.

Proof: If $|\omega| > 2|z|$, then $|(z-\omega)^{-2} + \omega^{-2}| \leq 10|z|/|\omega^3|$, so the series converges if $\sum_{\omega\in\Lambda'} \omega^{-3}$ converges. Since ω_2/ω_1 is nonreal, there exists a $k > 0$ such that $|n_1\omega_1 + n_2\omega_2| \geq k(|n_1| + |n_2|)$ for all real n_1, n_2. There are $4n$ integer pairs (n_1, n_2) with $|n_1| + |n_2| = n$. So, $\sum_{\omega\in\Lambda'} 1/|\omega^3| \leq 4k^{-3} \sum_{i=1}^{\infty} n^{-2}$ which converges.

Proposition: $P(\wp_\Lambda) = \Lambda$, i.e. the poles of $\wp_\Lambda$ are exactly Λ.

Proof: Since $\wp$ converges off of Λ, it has no poles off of Λ. Further, it has a pole at 0 so by periodicity it has poles everywhere on Λ.

Proposition: All poles of $\wp$ have order 2 and residue 0.

Proof: 0 is a pole of order 2 and residue 0, so follows from periodicity.

Proposition: $\wp$ is even.

Proof: $\wp(z) = \wp(-z)$, just substitute $-\omega$ for ω.

Proposition: $\wp_\Lambda \in \mathcal{M}(\mathbb{C}/\Lambda)$, i.e. $\wp_\Lambda$ is an elliptic function with respect to Λ.

Proof: Termwise differentiation gives $\wp'(z) = -2\sum_{\omega\in\Lambda'}(z-\omega)^{-3}$. From this expression, $\wp'$ is elliptic. Integrating gives $\wp(z+\omega) = \wp(z) + C$. Substitute $z = -\omega/2$, then $C = 0$ by evenness. Hence $\wp$ is elliptic.

Proposition: $\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp_\Lambda, \wp'_\Lambda)$, i.e. every elliptic function is a rational combination of $\wp$ and $\wp'$.

Proof: Any elliptic function f is the sum of an even and odd function $f(z) = 1/2(f(z) + f(-z)) + 1/2(f(z) - f(-z))$. Further if f is odd then $\wp' f$ is even, so it suffices to prove this for even elliptic functions. Then something else.

The Eisenstein series is $G_k(\Lambda) = \sum_{\omega\in\Lambda'} \omega^{-2k}$.

Proposition: The Laurent series for $\wp$ about 0 is $\wp(z) = z^{-2} + \sum_{k=1}^{\infty}(2k+1)G_{k+1}z^{2k}$.

Proof: $(z-\omega)^{-2} - \omega^{-2} = \sum_{j=1}^{\infty}(j+1)z^j\omega^{-(j+2)}$. Plugging into the formula for $\wp$, get $\wp(z) = z^{-2} + \sum_{i=1}^{\infty}(k+1)z^k \sum_{\omega\in\Lambda'}\omega^{-(k+2)}$. But $\wp$ is even, so the coefficients of odd powers must be 0. Hence, the formula follows.

Proposition: $(\wp')^2 = 4\wp^3 - 60G_2\wp - 140G_3$.

Proof: Expanding, $(\wp')^2(z) = 4z^{-6} - 24G_4z^{-2} - 80G_6 + \dots, \wp^3(z) = z^{-6} + 9G_4z^{-2} + 15G_6 + \dots, \wp(z) = z^{-2} + 3G_4z^2 + \dots$. So, $(\wp')^2 - 4\wp^3 + 60G_2\wp + 140G_3$ is holomorphic, vanishes at 0 and is elliptic, hence identically equals 0.

Hence, the image of the map $\mathbb{C}/\Lambda \to \mathbb{P}^2$, $z \mapsto [\wp(z), \wp'(z), 1]$ is an elliptic curve.

Theorem (Uniformization): Every complex elliptic curve is the image of such a map for some Λ.