

A category is a class of objects and for any objects  $A, B$  a set  $B \leftarrow A$  of morphisms and for any object  $A$  an identity  $A \xleftarrow{1_A} A$  and for any morphisms  $C \xleftarrow{g} B \xleftarrow{f} A$  a composite  $C \xleftarrow{gf} A$  such that:

- Identity Axiom: For any morphism  $B \xleftarrow{f} A$ ,  $1_B f = f = f 1_A$
- Associativity Axiom: For any morphisms  $D \xleftarrow{h} C \xleftarrow{g} B \xleftarrow{f} A$ ,  $(hg)f = h(gf)$ .

The identity is unique since given two identities  $1_A, \tilde{1}_A$  we have  $1_A = \tilde{1}_A 1_A = \tilde{1}_A$ . Inductively define  $n$ -ary composition

- The nullary composite for an object  $A$  is  $1_A$ .
- For morphisms  $A_{k+1} \xleftarrow{f_{k+1}} \dots \xleftarrow{f_1} A_0$ ,  $f_{k+1} \dots f_1 = f_{k+1}(f_k \dots f_1)$

Composition is “paranthesis independent”, i.e.  $f_n \dots f_1 = (f_n \dots f_{i+1})(f_i \dots f_1)$  for all  $i$ . Indeed, the nullary composite is trivially so. And, if  $f_k \dots f_1 = (f_k \dots f_{i+1})(f_i \dots f_1)$  then

$$\begin{aligned} f_{k+1} \dots f_k &= f_{k+1}(f_k \dots f_1) = f_{k+1}((f_k \dots f_{i+1})(f_i \dots f_1)) \\ &= (f_{k+1}(f_k \dots f_{i+1}))(f_i \dots f_1) = (f_{k+1} \dots f_{i+1})(f_i \dots f_1) \end{aligned}$$

Hence, paranthesis indepedence follows by induction.

Also, the unary composite  $B \xleftarrow{f} A$  is  $f 1_A = f$  and the binary composite  $C \xleftarrow{g} B \xleftarrow{f} A$  is  $gf$  as expected.

Iff  $f_n \dots f_1 = g_m \dots g_1$  we say that the following diagram commutes.

$$B \begin{array}{ccc} \xleftarrow{f_n} & \dots & \xleftarrow{f_1} \\ \xleftarrow{g_m} & \dots & \xleftarrow{g_1} \end{array} A$$

A commutative diagram can have any identities or composites or indeed  $n$ -ary composites added and remain commutative, thus we may assume their presence when undrawn. The identity axiom is thus equivalent to the commutativity of  $B \xleftarrow{f} A$  and the associativity axiom is equivalent to the commutativity of  $D \xleftarrow{h} C \xleftarrow{g} B \xleftarrow{f} A$ .

Say  $B \xleftarrow{f} A$  and  $B \xrightarrow{g} A$  are inverses iff  $gf = 1_A$  and  $fg = 1_B$ , i.e. the following diagram commutes.

$$B \begin{array}{c} \xleftarrow{f} \\ \xrightarrow{g} \end{array} A$$

The inverse is unique since given two inverses  $g, \tilde{g}$  for  $f$ ,  $g = 1_A g = (\tilde{g}f)g = \tilde{g}(fg) = \tilde{g}1_B = \tilde{g}$ . Write  $g = f^{-1}$ . A morphism which has an inverse is called an isomorphism. Iff there exists an isomorphism  $B \xleftarrow{f} A$  then say  $A$  and  $B$  are isomorphic.

A groupoid is a category in which every morphism is an isomorphism.

A monoid is a category with a unique object.

A group is a groupoid and monoid.

A functor between categories  $\mathfrak{B} \xleftarrow{F} \mathfrak{A}$  is an object  $F(A)$  in  $\mathfrak{B}$  for every object  $A$  in  $\mathfrak{A}$  and a morphism  $F(A') \xleftarrow{F(f)} F(A)$  for every morphism  $A' \xleftarrow{f} A$  such that

- $F(1_A) = 1_{F(A)}$  for any object  $A$
- $F(gf) = F(g)F(f)$  for any morphisms  $A'' \xleftarrow{g} A' \xleftarrow{f} A$ .

A natural family of morphisms  $G \xleftarrow{\tau} F$  between functors  $\mathfrak{B} \xleftarrow{F,G} \mathfrak{A}$  is a morphism  $G(A) \xleftarrow{\tau(A)} F(A)$  for every object  $A$  in  $\mathfrak{A}$  such that for any morphism  $A' \xleftarrow{f} A$ , the following diagram commutes.

$$\begin{array}{ccccc} & G(A) & \xleftarrow{\tau(A)} & F(A) & \\ G(f) & \downarrow & & \downarrow & F(f) \\ & G(A') & \xleftarrow{\tau(A')} & F(A') & \end{array}$$

It is a natural family of isomorphisms iff each  $\tau(A)$  is an isomorphism. Then say  $F$  and  $G$  are naturally isomorphic.

The opposite category  $Op(\mathfrak{C})$  has the same objects as  $\mathfrak{C}$  and morphisms  $A \xleftarrow{Op(f)} B = B \xleftarrow{f} A$  and  $Op(g)Op(f) = fg$ .  $Op$  is a functor.

The product category  $\mathfrak{A} \times \mathfrak{B}$  has objects ordered pairs of objects  $(A, B)$  and morphisms  $f \times f'$  with  $(g \times g')(f \times f') = gf \times g'f'$ .

The category  $Set$  has objects sets and morphisms maps.

Functors  $\mathfrak{B} \xrightleftharpoons[G]{F} \mathfrak{A}$  are called adjoint iff the functors  $Set \xleftarrow{\widehat{F}, \widehat{G}} Op(\mathfrak{A}) \times \mathfrak{B}$  are naturally isomorphic where

- $\widehat{F}(A, B) = B \leftarrow F(A)$
- $\widehat{G}(A, B) = G(B) \leftarrow A$
- $\widehat{F}(Op(f) \times g)(u) = guF(f)$
- $\widehat{G}(Op(f) \times g)(v) = G(g)vf$