Paths

Let X be a topological space. Given $x, y \in X$, a path $\alpha : x \to y$ is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. $c_x : x \to x$ is defined to be the path $c_x(t) = x$ for $t \in [0,1]$. $\alpha^{-1} : y \to x$ is the path $\alpha^{-1}(t) = \alpha(1-t)$ for $t \in [0,1]$. And, given $\beta : y \to z$, $\beta\alpha : x \to z$ is the path

$$\beta\alpha(t) = \begin{cases} \alpha(2t) \text{ if } t \in [0, \frac{1}{2}] \\ \beta(2t-1) \text{ if } t \in [\frac{1}{2}, 1] \end{cases}$$

Iff there exists a path $\alpha : x \to y$, then x and y are path-connected, $x \sim_p y$ (by α). This is an equivalence relation since it is reflexive: $x \sim_p x$ by c_x; symmetric: if $x \sim_p y$ by α, then $y \sim_p x$ by α^{-1}; and transitive: if $x \sim_p y$ by α and $y \sim_p z$ by β, then $x \sim_p z$ by $\beta\alpha$. X is path-connected iff $\forall x, y \in X, x \sim_p y$.

Homotopy

Given continuous maps $f, g : X \to Y$, a homotopy $F : f \to g$ is a continuous map $F : X \times [0,1] \to Y$ such that $F(-,0) = f$ and $F(-,1) = g$. $c_f : f \to f$ is the homotopy $c_f(-,t) = f$ for $t \in [0,1]$. $F^{-1} : g \to f$ is the homotopy $F^{-1}(-,t) = F(-,1-t)$ for $t \in [0,1]$. And, given $G : g \to h$, $GF : f \to h$ is the homotopy

$$GF(-,t) = \begin{cases} F(-,2t) \text{ if } t \in [0, \frac{1}{2}] \\ G(-,2t-1) \text{ if } t \in [\frac{1}{2}, 1] \end{cases}$$

Iff there exists a homotopy $F : f \to g$, then f and g are homotopic, $f \sim_h g$ (by F). This is an equivalence relation since it is reflexive: $f \sim_h f$ by c_f; symmetric: if $f \sim_h g$ by F, then $g \sim_h f$ by F^{-1}; and transitive: if $f \sim_h g$ by F and $g \sim_h h$ by G, then $f \sim_h h$ by GF.

Path Homotopy

Given paths $\alpha, \beta : x \to y$ a path-homotopy $\chi : \alpha \to \beta$ is a homotopy through paths, i.e. a continuous map $\chi : [0,1] \times [0,1] \to X$ such that $\chi(-,0) = \alpha$, $\chi(-,1) = \beta$ and for all $t \in [0,1]$, $\chi(-,t)$ is a path $x \to y$. $c_\alpha : \alpha \to \alpha$ is the path-homotopy $c_\alpha(-,t) = \alpha$ for $t \in [0,1]$. $\chi^{-1} : \beta \to \alpha$ is the path-homotopy $\chi^{-1}(-,t) = \chi(-,1-t)$ for $t \in [0,1]$. And, given $\psi : \beta \to \gamma$, $\psi\chi : \alpha \to \gamma$ is the path-homtopy

$$\psi\chi(-,t) = \begin{cases} \chi(-,2t) \text{ if } t \in [0, \frac{1}{2}] \\ \psi(-,2t-1) \text{ if } t \in [\frac{1}{2}, 1] \end{cases}$$

Iff there exists a path-homotopy $\chi : \alpha \to \beta$, then α and β are path-homotopic, $\alpha \sim_{ph} \beta$ (by χ). This is an equivalence relation since it is reflexive: $\alpha \sim_{ph} \alpha$ by c_α; symmetric: if $\alpha \sim_{ph} \beta$ by χ, then $\beta \sim_{ph} \alpha$ by χ^{-1}; transitive: if $\alpha \sim_{ph} \beta$ by χ and $\beta \sim_{ph} \gamma$ by ψ, then $\alpha \sim_{ph} \gamma$ by $\psi\chi$. For X path-connected, X is simply connected iff $\forall x, y \in X, \alpha, \beta : x \to y, \alpha \sim_{ph} \beta$.

Fundamental Groupoid

The set of path-homotopy equivalence classes of paths $x \to y$ is $\pi_1(X, x, y)$. Given $[\alpha] \in \pi_1(X, x, y), [\beta] \in \pi_1(X, y, z)$, then $[\beta][\alpha] \in \pi_1(X, x, z)$ is defined by $[\beta][\alpha] = [\beta\alpha]$. This product or composition has the following properties:

- Well-defined: If $\alpha \sim \alpha'$ by χ and $\beta \sim \beta'$ by ψ, then $\beta\alpha \sim \beta'\alpha'$ by

  $$(s,t) \mapsto \begin{cases} \chi(2s,t) \text{ if } s \in [0,\frac{1}{2}] \\ \psi(2s-1,t) \text{ if } s \in [\frac{1}{2},1] \end{cases}$$

- Associative: $\gamma(\beta\alpha) \sim (\gamma\beta)\alpha$; let $u = \frac{1}{4}(t+1)$, $v = \frac{1}{4}(t+2)$; the path-homotopy is

  $$(s,t) \mapsto \begin{cases} \alpha(\frac{s}{u}) \text{ if } s \in [0,u] \\ \beta(\frac{s-u}{v-u}) \text{ if } s \in [u,v] \\ \gamma(\frac{s-v}{1-v}) \text{ if } s \in [v,1] \end{cases}$$

- Right-identity: $\alpha c_x \sim \alpha$; let $u = \frac{1}{2}(1-t)$; the path-homotopy is

  $$(s,t) \mapsto \begin{cases} x \text{ if } s \in [0,u] \\ \alpha(\frac{s-u}{1-u}) \text{ if } s \in [u,1] \end{cases}$$

- Left-identity: $c_y\alpha \sim \alpha$; let $u = \frac{1}{2}(t+1)$; the path-homotopy is

  $$(s,t) \mapsto \begin{cases} \alpha(\frac{s}{u}) \text{ if } s \in [0,u] \\ y \text{ if } s \in [u,1] \end{cases}$$

  Recall that a category is a class of morphisms for which the axioms of associativity, and right and left-identities hold. Therefore, $\pi_1(X,x,y)$ form the morphisms $x \to y$ of a category whose objects are points in X.

An inverse can also be defined: $[\alpha]^{-1} \in \pi_1(X,y,x)$ is defined by $[\alpha]^{-1} = [\alpha^{-1}]$. The inverse then has the following properties:

- Well-defined: If $\alpha \sim \alpha'$ by χ, then $\alpha^{-1} \sim {\alpha'}^{-1}$ by $(s,t) \mapsto \chi(1-s,t)$.
- Left-inverse: $\alpha^{-1}\alpha \sim c_x$; let $u = \frac{1}{2}(1-t), v = \frac{1}{2}(t+1)$; note that $u = 1-v$; the path-homotopy is

  $$(s,t) \mapsto \begin{cases} \alpha(s) \text{ if } s \in [0,u] \\ \alpha(u) \text{ if } s \in [u,v] \\ \alpha(1-s) \text{ if } s \in [v,1] \end{cases}$$

- Right-inverse: $\alpha\alpha^{-1} \sim c_y$; let $u = \frac{1}{2}(1-t), v = \frac{1}{2}(t+1)$; note that $v = 1-u$; the path-homotopy is

  $$(s,t) \mapsto \begin{cases} \alpha(1-s) \text{ if } s \in [0,u] \\ \alpha(v) \text{ if } s \in [u,v] \\ \alpha(s) \text{ if } s \in [v,1] \end{cases}$$

  Recall that a groupoid is a category all of whose morphisms are invertible. Therefore, $\pi_1(X,x,y)$ are morphisms of a groupoid. It is called the fundamental groupoid of X.

Fundamental Group

Recall that endomorphisms in a groupoid form a group. Thus $\pi_1(X,x) = \pi_1(X,x,x)$ is a group, called the fundamental group of X with basepoint x. Any $[\alpha] \in \pi_1(X,x,y)$ induces an isomorphism $\pi_1(X,x) \to \pi_1(X,y)$ given by $[\gamma] \mapsto [\alpha][\gamma][\alpha]^{-1}$. That this is a homomorphism is a simple exercise and the inverse is induced by $[\alpha]^{-1}$. It is another simple exercise to see that if $\pi_1(X,x)$ is Abelian then this isomorphism is natural, that is, any two $[\alpha],[\beta] \in \pi_1(X,x,y)$ induce the same isomorphisms.

A continuous map $f : X \to Y$ with $f(x) = y$ induces a homomorphism of groups $f^* : \pi_1(X,x) \to \pi_1(Y,y)$ given by $f^*[\gamma] = [f \circ \gamma]$.

The induced homomorphism f^* has the following properties:

- Well-defined: f^* is well-defined since if $\gamma \sim \gamma'$ by χ then $f \circ \gamma \sim f \circ \gamma'$ by $f \circ \chi$.
- Homomorphism: f^* is a homomorphism since $f \circ (\gamma\gamma') = (f \circ \gamma)(f \circ \gamma')$. All that was needed was path homotopy equivalence but in face equality holds.
- Homotopy invariance: Given $g : X \to Y$ with $g(x) = y$, $g \sim f$ by F with $F(-,x) = y$ then $g^* = f^*$. Here $g \circ \gamma \sim f \circ \gamma$ by $(t,s) \mapsto F(t,\gamma(s))$.
- Identity preserving: It is a simple exercise to see that $id^*_{(X,x)} = id_{\pi_1(X,x)}$.
- Composition preserving: It is a simple exercise to see that given $f : (X,x) \to (Y,y)$ and $g : (Y,y) \to (Z,z)$, then $(g \circ f)^* = g^* \circ f^*$.

Thus, we have a functor $\pi_1 : Top_0 \to Grp$ with $(X,x) \mapsto \pi_1(X,x)$ and $f \mapsto f^*$. Here, Top_0 is the category of topological spaces with basepoints where morphisms are continuous maps which take basepoints to basepoints and Grp is the category of groups.