QUANTUM INVARIANTS OF KNOTS

1. Braided Monoidal Categories

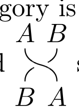
Braided monoidal categories are categories endowed with some extra structure. This structure is natural from the standpoint of category theory where it may be interpreted as an embedding of the category into a 3-dimensional category. Braided monoidal categories are also intimately connected with knot theory. We will use a notation which emphasizes this connection. We'll start by introducing a slightly different notation for categories.

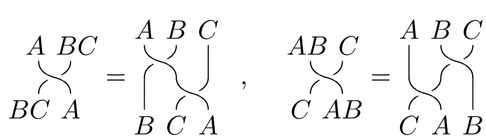
Notation. We'll denote a morphism f with source A and target A' by $\begin{matrix} A \\ f \\ A' \end{matrix}$. The object labels will be suppressed when clear. The composite for $\begin{matrix} A \\ f \\ A' \\ f' \\ A'' \end{matrix}$ will be denoted by $\begin{matrix} f \\ f' \end{matrix}$. The identity $\begin{matrix} A \\ 1_A \\ A \end{matrix}$ may be denoted by $\begin{matrix} A \\ \vert \\ A \end{matrix}$.

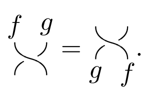
Definition. A (strict) monoidal category is a category with an object e and for any objects A, B an object AB and for any morphisms $\begin{matrix} A \\ f \\ A' \end{matrix}$ and $\begin{matrix} B \\ g \\ B' \end{matrix}$ a morphism $\begin{matrix} AB \\ fg \\ A'B' \end{matrix}$ such that:

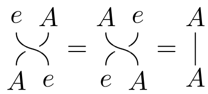
- The product is functorial: $\begin{matrix} AB \\ \vert \\ AB \end{matrix} = \begin{matrix} A & B \\ \vert & \vert \\ A & B \end{matrix}$ and $\begin{matrix} \overbrace{\underbrace{fg}} \\ \overbrace{\underbrace{f'g'}} \end{matrix} = \begin{Bmatrix} f \\ f' \end{Bmatrix} \begin{Bmatrix} g \\ g' \end{Bmatrix}$;
- The objects form a monoid: $(AB)C = A(BC)$ and $eA = A = Ae$;
- Morphisms also form a monoid: $(fg)h = f(gh)$ and $1_e f = f = f 1_e$.

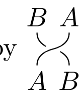
Notation. The properties of a (strict) monoidal category ensure that no parantheses are needed in writing compositions or products.

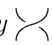
Definition. A monoidal category is braided iff it has a braiding, a natural family of isomorphisms $\begin{matrix} AB \\ x_{A,B} \\ BA \end{matrix}$ denoted  such that the braid relations hold



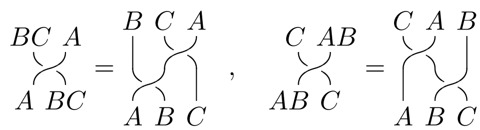
Naturality means that .

We also have  since $x_{e,A} = x_{ee,A} = \begin{matrix} 1_e & x_{e,A} \\ x_{e,A} & 1_e \end{matrix} = \begin{matrix} x_{e,A} \\ x_{e,A} \end{matrix}$ and since x is invertible and similarly for $x_{A,e}$.

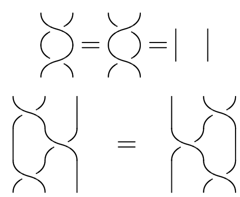
Notation. $x^{-1}_{A,B}$ will be denoted by .

Lemma. *In a braided monoidal category*  *forms another braiding.*

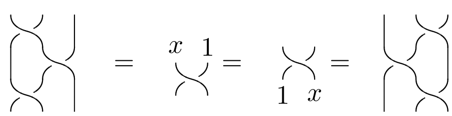
Since x is a natural family of isomorphisms, so is x^{-1}. We need only verify the braid relations. Inverting the braid relations for x gives the braid relations for x^{-1}

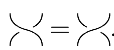


Lemma. *In a braided monoidal category we have*



The first equality just follows by invertibility. Using naturality and the braid relations we get the second equality,



Definition. A monoidal category is symmetric iff it has a symmetric braiding, a braiding such that .

Example. The braid groupoid has as objects the natural numbers and as morphisms

$$\begin{cases} \text{the braid group on } n \text{ strands, } B_n & \text{is } n \leftarrow n,\ n \in \mathbb{N} \\ \text{the empty set, } \{\} & \text{is } n \leftarrow m,\ n \neq m \in \mathbb{N} \end{cases}$$

It is a monoidal category with product being addition for objects $n+m$. For morphisms, the product $B_n \times B_m \to B_{n+m}$ sends generators (x_i, x_j) to $x_i x_{j+m}$.The braiding for the braid groupoid is

$$x_{n,m} = \Pi^n_{k=1} \Pi^m_{j=1} x_{n-k+j}$$

The braid groupoid is in fact the free braided monoidal category generated by one object.

The symmetric groupoid has as objects the natural numbers and as morphisms

$$\begin{cases} \text{the symmetric group on } n \text{ letters, } S_n & \text{is } n \leftarrow n,\ n \in \mathbb{N} \\ \text{the empty set, } \{\} & \text{is } n \leftarrow m,\ n \neq m \in \mathbb{N} \end{cases}$$

It has the structure of a symmetric monoidal category defined in the same way as for the braid groupoid with the transposition $(i, i+1)$ replacing x_i. The symmetric groupoid is in fact the free symmetric monoidal category generated by one object.

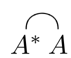
2. Tortile Monoidal Categories

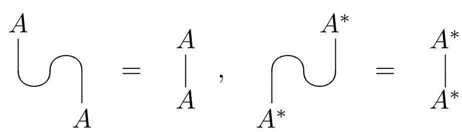
In category theory demanding equality is often too much. Instead one should ask for the weaker notion of isomorphism. Thus we introduce a twist which is a kind of isomorphism between the two braidings.

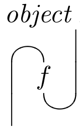
Definition. A twist is a natural family of automorphisms $\begin{matrix} A \\ \varphi_A \\ A \end{matrix}$ such that the twist relation holds: $\begin{matrix} \varphi_A\varphi_B \\ x_{A,B} \end{matrix} = \begin{matrix} \varphi_{AB} \\ x^{-1}_{B,A} \end{matrix}$. Naturality means that $\begin{matrix} f \\ \varphi_{A'} \end{matrix} = \begin{matrix} \varphi_A \\ f \end{matrix}$.

We also have that $\varphi_e = 1_e$ since $\begin{matrix} \varphi_e \\ \varphi_e \end{matrix} = \begin{matrix} 1_e & \varphi_e \\ \varphi_e & 1_e \end{matrix} = \varphi_e\varphi_e = \varphi_{ee} = \varphi_e$ and since φ is invertible.

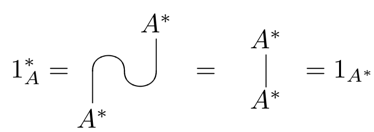
Another important concept for monoidal categories is duality.

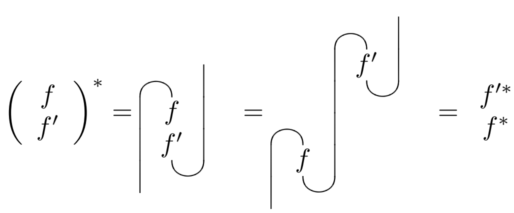
Definition. A right dual to an object A in a monoidal category is an object A^* with morphisms $\begin{matrix} AA^* \\ \cup_A \\ e \end{matrix}$ and $\begin{matrix} e \\ \cap_A \\ A^*A \end{matrix}$ denoted by  and  such that the duality relations hold



Lemma. *If every object has a right dual, then right duals extends to a contravariant functor with* $f^* =$  .

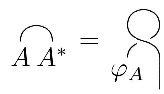
Using the duality relations, we verify

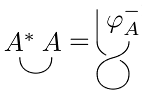


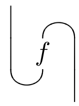


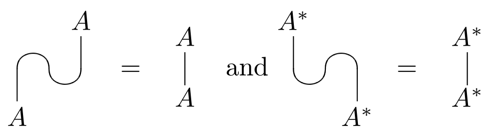
Definition. A monoidal category is tortile iff it has a braiding and a twist and every object has a right dual such that $\varphi^*_A = \varphi_{A^*}$.

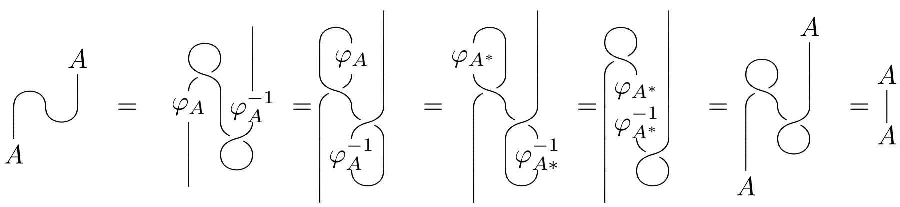
In a tortile monoidal category we may introduce left duality in a natural way.

Definition. The left dual of A is ${}^*A = A^*$. Define $\overset{e}{\underset{AA^*}{{}_A\cap}}$ =  and

$\overset{A^*A}{\underset{e}{{}_A\cup}}$ = .

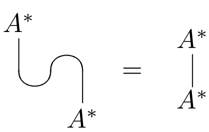
Lemma. *In a tortile monoidal category, left duals extend to a contravariant functor with* ${}^*f =$  *and the composition of left and right duals in either order is naturally isomorphic to the identity functor.*

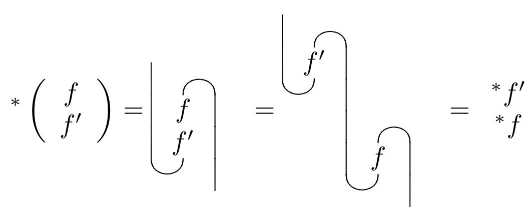
We first verify that . We have that



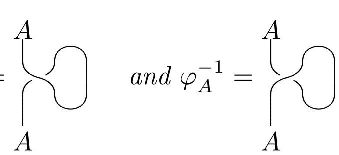
where the last equality is an exercise. The other assertion follows similarly.

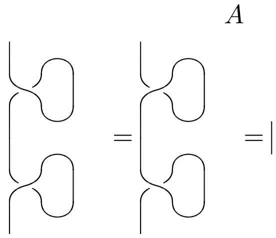
Next we verify that left dual is a contravariant functor.

$${}^*1_A =$$  $= 1_{A^*}$

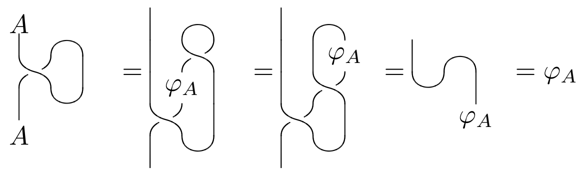


Now define α_A

Lemma. *In a tortile monoidal category* $\varphi_A =$  *and*



We have that



where the second to last equality uses naturality of the braiding. The last part follows by invertibility of φ.

3. Tangles

Tangles are a generalization of knots, links and braids. Just as braids formed a braided monoidal category, so should tangles form a tortile monoidal category. We will assume that all manifolds and maps of manifolds are in the piecewise-linear category.

Definition. A tangle is a 1-manifold, i.e. a disjoint union of arcs and circles, embedded in $(0,1)^2 \times [0,1]$. The boundary of a tangle is then contained in $(0,1)^2 \times \{0,1\}$. Two tangles are considered equivalent if and only if they are ambient isotopic relative to their boundaries.

Lemma. *Two tangles are equivalent iff they are connected by a sequence of moves of the following type*

Elementary move: If a triangle intersects the tangle at exactly 1 *or* 2 *of the triangle's edges (and not in its interior) then replace those edges on the tangle with the remaining* 2 *or* 1 *of its edges. (insert graph)*

I don't know a proof of this. Apparently it's in Reidemeister's Knottentheorie.

Lemma. *A tangle has a representative in its equivalence class whose projection onto the plane segment* $\{\frac{1}{2}\} \times (0,1) \times [0,1]$ *has no self-intersections except double points where intersections are transverse.*

All non-transverse double-points can be reduced to the following cases.

(1) 2 pairs of coincident edges of the tangle have vertices which project to the same point: In this case perform an elementary move on one of the pairs of coincident edges to remove the double point. (insert graph)
(2) 2 edges of the tangle project to the same line: In this case perform an elementary move on one of the edges to remove the common projection except for the endpoints of the edge which may result in case 1. (insert graph)

All self-intersections of order 3 or more can be reduced to the following cases

(1) A triple point where transverse edges intersect: In this case perform an elementary move on one of the edges to remove the triple point. (insert graph)
(2) An edge of the tangle projects to a point: In this case perform an elementary move on the edge reducing this to case 2 of the non-transverse double points. (insert graph)

After removing all self-intersections save transverse double points in this way, the result is an equivalent tangle by the previous lemma.

Definition. A tangle diagram is a projection as in the previous lemma along with a "over-" or "under-crossing" mark at each double point indicating relative displacements from the plane $\{\frac{1}{2}\} \times (0,1) \times [0,1]$. (insert graph examples)

Theorem. *(Reidemeister) A tangle diagram determines an equivalence class of tangles. Two tangle diagrams determine the same equivalence class of tangles if and only if they are connected by a sequence of moves of the following type or their inverses:*

- 1st planar move: Make a straight line jagged. (insert graphs)
- 2nd planar move: Make a straight line crossing another straight line jagged (insert graphs)
- 1st Reidemeister move: Remove a kink in a branch. (insert graphs)
- 2nd Reidemeister move: Move a branch across another branch. (insert graphs)
- 3rd Reidemeister move: Move a branch past a crossing. (insert graphs)

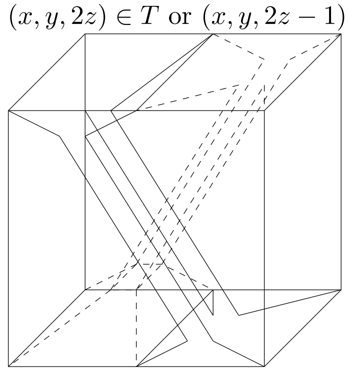
That diagrams determine tangles follows since an ambient isotopy can always be used to change heights, so that any tangle yielding the same diagram is equivalent.

It is clear that any tangle diagrams connected by a planar or Reidemeister moves will correspond to tangles connected by an elementary move. Conversely, it suffices to show that projection of an elementary move can be generated by planar and Reidemeister moves. We will show that a reduction to cases of projections of elementary moves is a rearrangment of exactly the planar and Reidemeister moves.

(Insert tons of figures)

Lemma. *Tangles form a category.*

The source of a tangle is its intersection with $(0,1)^2 \times \{0\}$ and its target is its intersection with $(0,1)^2 \times \{1\}$. If the target of tangle T is $A \times \{1\}$ and the source of tangle T' is $A \times \{0\}$, we can form their composite tangle $\begin{matrix} T \\ T' \end{matrix}$ with $(x,y,z) \in \begin{matrix} T \\ T' \end{matrix}$ iff $(x,y,2z) \in T$ or $(x,y,2z-1) \in T'$ as indicated in the figure.



These compositions are invariant under ambient isotopy since we can compose the ambient isotopies in the same way. Also associativity holds under ambient isotopy using the homotopy for associativity (insert graph). The identity tangle between a source $A \times \{0\}$ and target $A \times \{1\}$ is the class of $A \times [0,1]$ (insert graph). Composition with the identity is invariant under ambient isotopy by the homotopy for identities (insert graph).

Lemma. *Tangles form a monoidal category.*

We take the monoidal product TT' of any two tangles to be $(x, y, z) \in TT'$ iff $(x, 2y, z) \in T$ or $(x, 2y - 1, z) \in T$ (insert graph).

Lemma. *Tangles form a braided monoidal category.*

The braiding between two objects is the embedding of the disjoint union of identity tangles of those objects embedded in $(0, 1)^2 \times [0, 1]$ as in the following figure (insert graph).

Lemma. *Tangles form a tortile monoidal category.*

The duality tangles for an object result from embedding the identity tangle as in the following figure (insert graph).

The twist tangle results from embedding the identity tangle as in the following figure (insert graph).

4. Orientation, Framing, Labels

Tangles may be given some more ornaments such as orientation and framing. These are important topologically as when framing for a link is needed for Dehn surgery. They are also important in category theory, where orientation corresponds to duality and framing corresponds to twists. Labeling tangles with morphisms in some category is a simple extension which allows for a more general freeness result (Shum's theorem) with no additional effort.

Definition. A tangle may be given an orientation. A tangle may also be given a framing, i.e. an extension of the embedding of the tangle to an embedding of its product with an interval. The framings of the boundary points of the tangle must point along the same direction.

The definitions and properties for oriented tangles are the obvious analogs as those for unoriented tangles (orient the Reidemeister moves, etc). The definitions and properties for framed tangles are also analagous to those for framed tangles, except for the following:

Lemma. *A framed tangle has a representative in its equivalence class whose projection onto the plane segment $\{\frac{1}{2}\} \times (0, 1) \times [0, 1]$ has no self-intersections except double points where intersections are transverse and such that the framing projects nondegenerately (i.e. the interval doesn't project down to a point).*

If the framing must project degenerately an even number of times since otherwise the framing of the boundary points of the tangle will point in different directions. We can remove opposite half-twists by an ambient isotopy of the framing which leaves the tangle itself unchanged. We can remove the remaining pairs of degeneracies by performing elementary moves introducing a kink for each full-twist.

Definition. A blackboard diagram of a framed tangle is such a projection along with crossing information.

Lemma. *Two blackboard diagrams determine the same equivalence class of framed tangles if and only if they are connected by a sequence of planar moves, second and third Reidemeister moves and the following move:*

- Framed 1st Reidemeister move: Remove opposite kinks in a branch (insert figure).

It is simple to see that planar moves, and second and third Reidemeister moves correspond to ambient isotopies which extend to ambient isotopies for the framing. The first Reidemeister move will correspond to an ambient isotopy which introduces a twist in the framing. We can only allow such combinations of Reidemeister moves which introduce opposite pairs of twists. These are generated by the framed 1st Reidemeister move.

Definition. Given a category, we can label each component of a framed oriented tangle with a morphism. If the component is a circle then the label must be an endomorphism where the two composite endomorphisms fg and fg will be considered equivalent. Endpoints of an arc can be labelled with the source or target object of the morphism according to the orientation of the tangle (i.e. follow orientation from source to target). An object label will be marked as a dual if the orientation is upward near the corresponding endpoint (insert graph). Labelled oriented tangles may be composed if matching points are labelled my matching objects and dual markers. Then they must be labelled by their composite morphism.

The definitions and properties for labeled framed oriented tangles are analagous to those for framed tangles.

Theorem. *(Shum) The free tortile monoidal category generated over a category C is equivalent to the category of framed oriented tangles labeled by C.*

A sketch of the proof of Shum's theorem follows. Here equivalence means an equivalence of categories which preserves the monoidal structure, braiding, dual structure and twist. First you can show that the category of framed oriented tangles labeled by C is equivalent to a full subcategory, FTC. Objects in FTC are restricted to certain discrete sets defined inductively and lying on the midline of $(0,1)^2$ and their framings are restricted to have specific widths and to be directed along the midline to the right or left according to orientation. Such an equivalence to a full subcategory is called a retraction. Similarly there is a retraction of the free tortile monoidal category generated by C to a full subcategory RFC. Objects in RFC are restricted to being monoidal products of objects of the form A or A^* where A is an object in C, i.e. no objects like $(AB)^*$ or A^{**}. All the morphisms in RFC can be shown to be compositions of products of the structure morphisms (like the braiding, twist and duality unit and counit) as well as morphisms in C. Finally, FTC and RFC can be shown to be equivalent. This follows by constructing a bijection between their objects, and a functor from RFC to FTC which is determined by the inclusion of RFC in the free monoidal category generated over C and the universal functor from this category. Finally it must be shown that this functor is "injective" in the sense that all relations in FTC follow from its being a tortile monoidal category. This comes down to showing that ambient isotopies corresponding to planar and (framed) Reidemeister moves are induced by relations which hold in a tortile monoidal category, but this should be clear from lemmas which were proved in section 1.

Definition. An object A in a tortile monoidal category is self-dual iff $A = A^*$ and unframed iff $\varphi_A = 1_A$.

Corollary. *The category of framed, oriented tangles is the free tortile monoidal category generated by one object. The category of oriented tangles (unframed) is the tortile monoidal category generated by one unframed object. The category of* *framed tangles (unoriented) is the free tortile monoidal category generated by one self-dual object. The category of tangles (unframed, unoriented) is the free tortile monoidal category generated by one unframed, self-dual object.*

These follow since if, for the generator, $A = A^*$ then we lose any information on orientation and since if $\varphi_A = 1_A$ then we gain the unframed first Reidemeister move.

5. Kauffman Bracket

Definition. The Kauffman bracket is an invariant of framed links defined on blackboard diagrams by the Kauffman skein relation

$$\left\langle \diagup\!\!\!\!\diagdown \right\rangle = A \left\langle \,)\,(\, \right\rangle + A^{-1} \left\langle \asymp \right\rangle$$

the circle relation, that for the disjoint union of a diagram L and the circle

$$\left\langle L \sqcup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle L \rangle$$

as well as the normalization condition that for the empty link $\langle\rangle = 1$.

Note. This is not the standard normalization condition which is $\left\langle \bigcirc \right\rangle = 1$. Evidently the standard bracket can be recovered by dividing ours by $-A^2 - A^{-2}$.

The Kauffman skein relation should be interpreted as three blackboard diagrams which all agree outside a small neighborhood where they look as in the relation. From induction on the number of crossings we see that the Kauffman bracket is well defined on blackboard diagrams. Now we can examine the effect of planar and framed Reidemeister moves on the Kauffman bracket to show that it is a framed link invariant. However, let's proceed instead to demonstrate that the bracket extends to a functor from the category of framed tangles.

Definition. The category of planar tangles is like the category of tangles except we embed them in $(0,1) \times [0,1]$. Similar as with Shum's theorem the category of planar tangles can be shown to be equivalent to the monoidal category with duals generated by 1 self-dual object. The objects of this category are therefore the natural numbers and morphisms are compositions of products of $\underset{2}{\overset{0}{\cap}}$, $\underset{0}{\overset{2}{\cup}}$ and identities. The skein category has the same objects and has as morphisms $m \leftarrow n$ between objects the module over $\mathbb{Z}[A, A^{-1}]$ generated by the planar tangles $m \leftarrow n$. The skein category inherits a composition and monoidal structure from the cobordism category by demanding that they both act linearly. We also mod out by the circle relation

$$\bigcirc = (-A^2 - A^{-2}) 1_0$$

Finally, define $\underset{2}{\overset{2}{x}}$ to be $A \underset{\frown}{\smile} + A^{-1} |\ \ |$.

Theorem. *x induces a braiding on the skein category, making it a tortile monoidal category.*

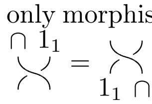
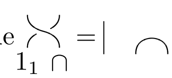
x is invertible with $x^{-1} = A^{-1} \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} + A\,|\;|$ since

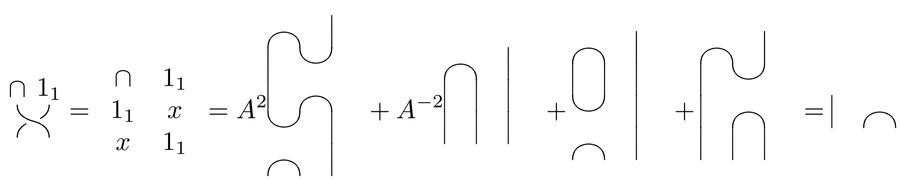
$$\begin{matrix} x \\ x^{-1} \end{matrix} = \begin{matrix} x^{-1} \\ x \end{matrix} = \begin{matrix}\cup\\ \bigcirc \\ \cap\end{matrix} + (A^2 + A^{-2}) \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} + |\;| = |\;|$$

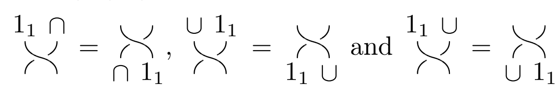
We can define $\begin{matrix} m+n \\ x_{m,n} \\ n+m \end{matrix}$, using double induction, $x_{1,n+1} = \begin{matrix} x_{1,n} & 1_1 \\ 1_n & x \end{matrix}$ and $x_{1+m,n} = \begin{matrix} 1_1 & x_{m,n} \\ x_{1,n} & 1_m \end{matrix}$ with base cases $x_{0,0} = 1_0$ and $x_{1,0} = x_{0,1} = 1_1$. We must verify the braid relations. Let's see that $x_{k+m,n} = \begin{matrix} 1_k & x_{m,n} \\ x_{k,n} & 1_m \end{matrix}$. It is true in the cases $k = 0, 1$ and if it is true for a given k then,

$$x_{1+k+m,n} = \begin{matrix} 1_1 & x_{k+m,n} \\ x_{1,n} & 1_{k+m} \end{matrix} = \begin{matrix} 1_1 & 1_k & x_{m,n} \\ 1_1 & x_{k,n} & 1_m \\ x_{1,n} & 1_k & 1_m \end{matrix} = \begin{matrix} 1_{1+k} & x_{m,n} \\ x_{1+k,n} & 1_m \end{matrix}$$

So the statement follows from induction. Similarly, we have $x_{m,n+k} = \begin{matrix} x_{m,n} & 1_k \\ 1_n & x_{m,k} \end{matrix}$ and $x_{0,n} = x_{n,0} = 1_n$. Also, $x_{m,n}$ is invertible since its a composition of products of invertible morphisms.

We must also verify naturality. x is natural since the only morphisms with source or target 1 are multiples of the identity. Let's see that . We have that for the right-hand side  and for the left-hand side,



Similarly, we have . Naturality in general will follow from the fact that all morphisms are generated over $\mathbb{Z}[A, A^{-1}]$ by $\cup, \cap$ and identities.

Finally, the skein category has duals $n^* = n$ with duality morphisms $\cup_n, \cap_n$ inherited from the cobordism category and the morphism $\varphi_n = \begin{matrix} 1_n & \cap_n \\ x_n & 1_n \\ 1_n & \cup_n \end{matrix}$ provides a twist.

Corollary. *There is a unique tortile functor from the category of framed tangles to the skein category which sends the generating object to 1.*

By a tortile functor we mean a functor which preserves the monoidal structure, braiding, dual structure and twist. The statement just follows from the fact that framed tangles are generated by a self-dual object so use the unique universal functor.

Corollary. *The Kauffman bracket is a framed link invariant.*

The image of the previous functor on a framed link L is $< L > 1_0$ the Kauffman bracket. The skein relation comes from the braiding in the skein category. The circle relation is part of the definition of the skein category. The normalization condition is also contained in the functor. Since the functor respects framed tangle equivalence the Kauffman bracket must respect framed link equivalence.

6. Hopf Algebras

An example of a symmetric monoidal category is the category of vector spaces with the tensor product which is why, in the literature, monoidal categories are often called tensor categories. The category of vector spaces also has a duality structure with the usual notion of dual space. In order to weaken the symmetry we consider modules over ribbon Hopf algebras. We will fix a ground field k for any vector spaces and make the identifications $k \otimes V = V = V \otimes k$ and $U \otimes (V \otimes W) = (U \otimes V) \otimes W$.

Definition. An algebra is a vector space A with a multiplication $\mu : A \otimes A \to A$ and a unit $\eta : k \to A$ such that if $1 : A \to A$ is the identity then the following hold:

- Associativity: $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$
- Unitality: $\mu(\eta \otimes 1) = 1 = \mu(1 \otimes \eta)$

Notation. We also write $\mu(a \otimes b) = ab$ and $\eta(1_k) = 1_\eta$

Definition. A coalgebra is a vector space A with a comultiplication $\Delta : A \to A \otimes A$ and a counit $\epsilon : A \to k$ such that if $1 : A \to A$ is the identity then the following hold:

- Coassociativity: $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$
- Counitality: $(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta$

Definition. A homomorphism of algebras is a linear map $h : A \to B$ such that $\mu_B(h \otimes h) = h\mu_A$ and $h\eta_A = \eta_B$. A homomorphism of coalgebras is a linear map $h : A \to B$ such that $(h \otimes h)\Delta_A = \Delta_B h$ and $\epsilon_B h = \epsilon_A$.

Lemma. *k has the structure of an algebra and coalgebra with μ the standard multiplication, Δ the standard diagonal map $\Delta(a) = a \otimes a$, and $\eta = \epsilon = 1_k$. Also, if A is an algebra then $A \otimes A$ is an algebra with $\mu' = \mu \otimes \mu$ and $\eta' = \eta \otimes \eta$. If A is a coalgebra then $A \otimes A$ is a coalgebra with $\Delta' = \Delta \otimes \Delta$ and $\epsilon' = \epsilon \otimes \epsilon$. Also, if A were an algebra and coalgebra then μ is a homomorphism of coalgebras if and only if Δ is a homomorphism of algebras and η is a homomorphism of coalgebras if and only if ϵ is a homomorphism of algebras.*

Verifying any of these statements is trivial. For instance, μ is a homomorphism of coalgebras if and only if $(\mu \otimes \mu)(\Delta \otimes \Delta) = \Delta \otimes \mu$ which is true if and only if Δ is a homomorphism of algebras.

Definition. A module over an algebra A is a vector space V with a product $A \otimes V \to V$, $a \otimes v \mapsto av$ such that $a(bv) = (ab)v$ and $1_\eta v = v$. A bialgebra is an algebra (A, μ, η) and coalgebra (A, Δ, ϵ) such that μ and η are homomorphisms of coalgebras, or Δ and ϵ are homomorphisms of algebras.

Lemma. *Modules over a bialgebra form a monoidal category.*

We need to see that given modules V, W over a bialgebra A, $V \otimes W$ has the structure of an A-module. But, $V \otimes W$ has the structure of an $A \otimes A$-module taking $(a \otimes b)(v \otimes w) = av \otimes bw$. So given $v \otimes w \in V \otimes W$ and $a \in A$, define $a(v \otimes w) = \Delta(a)(v \otimes w)$. This is an A-module structure since

$$(ab)(v \otimes w) = \Delta\mu(a \otimes b)(v \otimes w) = (\mu \otimes \mu)(\Delta \otimes \Delta)(a \otimes b)(v \otimes w)$$
$$= \Delta(a)\Delta(b)(v \otimes w) = a(b(v \otimes w));$$
$$1_\eta(v \otimes w) = \Delta(1_\eta)(v \otimes w) = 1_\eta \otimes 1_\eta(v \otimes w) = v \otimes w$$

With this structure, the isomorphism $U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ is A-linear since

$$a(u \otimes (v \otimes w)) = (1 \otimes \Delta)\Delta(a)(u \otimes v \otimes w) = (\Delta \otimes 1)\Delta(a)(u \otimes v \otimes w) = a((u \otimes v) \otimes w)$$

We also need to see that k is an A-module. So given $c \in k$ and $a \in A$, define $ac = \epsilon(a)c$. This is an A-module structure since

$$(ab)c = \epsilon\mu(a \otimes b)c = \mu_k(\epsilon \otimes \epsilon)(a \otimes b)c = \epsilon(a)\epsilon(b)c = a(bc)$$

With this structure, the isomorphisms $k \otimes V \to V \leftarrow V \otimes k$ are A-linear since

$$a(c \otimes v) = (\epsilon \otimes 1)\Delta(a)(c \otimes v) = 1(a)(cv) = acv,$$
$$a(v \otimes c) = (1 \otimes \epsilon)\Delta(a)(v \otimes c) = 1(a)(cv) = acv$$

Thus, the category of modules over A into a monoidal category with product $\otimes$ and identity k.

Definition. A bialgebra A is quasitriangular if and only if it has some element $R = \sum_i s_i \otimes t_i \in A \otimes A$ such that R is invertible and if $\tau : A \otimes A \to A \otimes A$ is given by $\tau(a \otimes b) = b \otimes a$ then

$$\tau\Delta(a) = R\Delta(a)R^{-1}$$
$$(\Delta \otimes 1)(R) = \left(\sum_i s_i \otimes 1_\eta \otimes t_i\right)\left(\sum_j 1_\eta \otimes s_j \otimes t_j\right)$$
$$(1 \otimes \Delta)(R) = \left(\sum_i s_i \otimes 1_\eta \otimes t_i\right)\left(\sum_j s_j \otimes t_j \otimes 1_\eta\right)$$

Lemma. *Modules over a quasitriangular bialgebra form a braided monoidal category with braiding $x_{V,W}(v \otimes w) = \tau(R)(w \otimes v) = \sum_i t_i w \otimes s_i v$.*

Let $\tau_{V,W} : V \otimes W \to W \otimes V$ be given by $\tau_{V,W}(v \otimes w) = w \otimes v$. Then $x_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w))$. We have that $x_{V,W}$ is A-linear since

$$x_{V,W}(a(v \otimes w)) = x_{V,W}(\Delta(a)(v \otimes w)) = \tau_{V,W}(R\Delta(a)(v \otimes w))$$
$$= \tau_{V,W}(\tau(\Delta(a))R(v \otimes w)) = \Delta(a)\tau_{V,W}(R(v \otimes w)) = ax_{V,W}(v \otimes w)$$

Furthermore $x_{V,W}$ is invertible with $x^{-1}_{V,W}(w \otimes v) = R^{-1}(v \otimes w)$. The braid relations follow by checking

$$x_{U,V\otimes W}(u \otimes v \otimes w) = \tau_{U,V\otimes W}((1 \otimes \Delta)(R)(u \otimes v \otimes w))$$
$$= \tau_{U,V\otimes W}\left(\left(\sum_i t_i \otimes 1_\eta \otimes s_i\right)\left(\sum_j t_j \otimes s_j \otimes 1_\eta\right)(u \otimes v \otimes w)\right)$$

$$= \tau_{U,V\otimes W}\left(\sum_i\sum_j t_it_ju\otimes s_jv\otimes s_iw\right) = \sum_i\sum_j s_jv\otimes s_iw\otimes t_it_ju =$$

$$(1_V\otimes x_{U,W})\sum_j s_jv\otimes t_ju\otimes w = (1_V\otimes x_{U,W})(x_{U,V}\otimes 1_W)(u\otimes v\otimes w)$$

So, $x_{U,V\otimes W} = (1_V\otimes x_{U,W})(x_{U,V}\otimes 1_W)$ and similarly $x_{U\otimes V,W} = (x_{U,W}\otimes 1_V)(1_U\otimes x_{V,W})$. Naturality follows since given $f : V\to V'$, $g : W\to W'$

$$x_{V',W'}(f\otimes g)(v\otimes w) = x_{V',W'}(f(v)\otimes g(w)) = \sum_i t_ig(w)\otimes s_if(v)$$

$$= (g\otimes f)\sum_i t_iw\otimes s_iv = (g\otimes f)x_{V,W}(v\otimes w)$$

So $x_{V',W'}(f\otimes g) = (g\otimes f)x_{V,W}$. Thus, $x_{V,W}$ is a braiding.

Definition. A Hopf algebra is a bialgebra A with an antipode $S : A\to A$ such that

$$\mu(1\otimes S)\Delta = \eta\epsilon = \mu(S\otimes 1)\Delta$$
$$S\mu = \mu(S\otimes S)\tau$$
$$S\eta = \eta$$

A ribbon Hopf algebra is a quasitriangular Hopf algebra with a central unit $\theta\in Z(A^\times)$ such that

$$S(\theta) = \theta$$
$$\epsilon(\theta) = 1$$
$$\tau\Delta(\theta) = R\tau(R)(\theta\otimes\theta)$$

Theorem. *Modules over ribbon Hopf algebras, which are finite dimensional over k, form a tortile monoidal category.*

By the previous lemma, finite dimensional modules over ribbon Hopf algebras form a braided monoidal category. We must find a dual structure and a twist. Let A be a Hopf algebra and let V be an A-module. Given $\lambda\in V^*, a\in A$ let $(a\lambda)(v) = \lambda(S(a)v)$. This gives an A-module on V^* since

$$S(ab) = S\mu(a\otimes b) = \mu(S\otimes S)\tau(a\otimes b) = S(b)S(a);$$
$$S(1_\eta) = S\eta(1_k) = \eta(1_k) = 1_\eta$$

so that

$$((ab)\lambda)(v) = \lambda(S(ab)v) = \lambda(S(b)S(a)v) = (b\lambda)(S(a)v) = (a(b\lambda))(v);$$
$$(1_\eta\lambda)(v) = \lambda(S(1_\eta)v) = \lambda(1_\eta v) = \lambda(v)$$

The evaluation map $\cup_V : V^*\otimes V\to k$ is $\cup_V(\lambda\otimes v) = \lambda(v)$. It is A-linear since over $A\otimes A$,

$$\cup_V((a\otimes b)(\lambda\otimes v)) = \cup_V(a\lambda\otimes bv) = (a\lambda)(bv) = \lambda(S(a)bv)$$
$$= \lambda(\mu(S\otimes 1)(a\otimes b)v) = \cup_V(\lambda\otimes\mu(S\otimes 1)(a\otimes b)v)$$

so that over A,

$$\cup_V(a(\lambda\otimes v)) = \cup_V(\Delta(a)(\lambda\otimes v)) = \cup_V(\lambda\otimes\mu(S\otimes 1)\Delta(a)v)$$
$$= \cup_V(\lambda\otimes\eta\epsilon(a)v) = \cup_V(\epsilon(a)(\lambda\otimes v)) = \epsilon(a)\cup_V(\lambda\otimes v) = a\cup_V(\lambda\otimes v)$$

The coevalauation map $\cap_V : k\to V\otimes V^*$ is the composition $k\to End(V) = V\otimes V^*$ where $k\to End(V)$ is $1_k\mapsto 1_V$ and $V\otimes V^* = End(V)$ is the usual identification

$v \otimes \lambda = f$ where $f(w) = \lambda(w)v$. Choosing a basis e_i for V with dual basis e^i for V^*, we see that $\cap_V(1_k) = \sum_i e_i \otimes e^i$, since $\sum_i e^i(w)e_i = w$. The coevaluation is A-linear since over $A \otimes A$

$$((a \otimes b) \cup_V (1_k))(w) = ((a \otimes b) \sum_i e_i \otimes e^i)(w)$$

$$= (\sum_i ae_i \otimes be^i)(w) = \sum_i (be^i)(w)(ae_i) = \sum_i e^i(S(b)w)ae_i$$

$$= a \sum_i (e^i(S(b)w))e_i = aS(b)w = \sum_i e^i(aS(b)w)e_i$$

$$= \sum_i e^i(\mu(1 \otimes S)(a \otimes b)w)e_i = (\sum_i e_i \otimes \mu(1 \otimes S)(a \otimes b)e^i)(w)$$

so that over A,

$$(a \cap_V (1_k))(w) = (\Delta(a) \cap_V (1_k))(w) = (\sum_i e_i \otimes \mu(1 \otimes S)\Delta(a)e^i)(w)$$

$$= (\sum_i e_i \otimes \eta\epsilon(a)e^i)(w) = (\sum_i \epsilon(a)(e_i \otimes e^i))(w) = \epsilon(a)(\sum_i e_i \otimes e^i)(w)$$

$$= \cap_V(\epsilon(a)1_k)(w) = \cap_V(a1_k)(w)$$

We also get the duality relations since

$$(1_V \otimes \cup_V)(\cap_V \otimes 1_V)(v) = (1_V \otimes \cup_V)(\cap_V \otimes 1_V)(1_k \otimes v)$$

$$= (1_V \otimes \cup_V)(\sum_i e_i \otimes e^i \otimes v) = \sum_i e_i \otimes e^i(v) = v;$$

$$(\cup_V \otimes 1_{V^*})(1_{V^*} \otimes \cap_V)(\lambda) = (\cup_V \otimes 1_{V^*})(1_{V^*} \otimes \cap_V)(\lambda \otimes 1_k)$$

$$= (\cup_V \otimes 1_{V^*})(\sum_i \lambda \otimes e_i \otimes e^i) = \sum_i \lambda(e_i) \otimes e^i = \lambda$$

This means that our category has left duals. Define the twist by $\varphi_V : V \to V$ by $\varphi_V(v) = \theta v$. It is linear since θ is central and invertible with $\varphi_V^{-1}(v) = \theta^{-1}v$. Also, $\varphi_k = 1_k$ since $\epsilon(\theta) = 1_k$. Let's show the twist relation holds.

$$x_{V,W}(\varphi_V \otimes \varphi_W)(v \otimes w) = x_{V,W}(\theta v \otimes \theta w) = \tau(R)(\theta w \otimes \theta v)$$

$$= \tau(R)(\theta \otimes \theta)(w \otimes v) = R^{-1}\tau\Delta(\theta)(w \otimes v)$$

$$= x_{W,V}^{-1}\Delta(\theta)(v \otimes w) = x_{W,V}^{-1}\varphi_{V\otimes W}(v \otimes w);$$

We must also show compatibility between the twist and duality. We dealt before with right duals but left duals are entirely analagous, also extending to a contravariant functor.

$$\varphi_V^*(\lambda)(v) = (\cup_V \otimes 1_{V^*})(1_{V^*} \otimes \varphi_V \otimes 1_{V^*})(1_{V^*} \otimes \cap_V)(\lambda)(v)$$

$$= (\cup_V \otimes 1_{V^*})(1_{V^*} \otimes \varphi_V \otimes 1_{V^*})(1_{V^*} \otimes \cap_V)(\lambda \otimes 1_k)(v)$$

$$= \sum_i (\cup_V \otimes 1_{V^*})(1_{V^*} \otimes \varphi_V \otimes 1_{V^*})(\lambda \otimes e_i \otimes e^i)(v)$$

$$= \sum_i (\cup_V \otimes 1_{V^*})(\lambda \otimes \theta e_i \otimes e^i)(v) = \sum_i \lambda(\theta e_i)e^i(v)$$

$$= \lambda(\theta v) = \lambda(S(\theta)v) = (\theta\lambda)(v) = \varphi_{V^*}(\lambda)(v)$$

This concludes the proof.

Corollary. *Given a ribbon Hopf algebra A and an A-module V there is a unique tortile monoidal functor from the category of framed oriented tangles to the category of finite dimensional A-modules which sends the generating object to V.*

7. Jones' Polynomial

The Jones' polynomial is an invariant of framed oriented links $V(L)$ defined on oriented blackboard diagrams with values in $\mathbb{C}[q^{1/2}, q^{-1/2}]$ by the Conway skein relation

$q^{-1/4}V\Big($  $\Big) - q^{1/4}V\Big($  $\Big) = (q^{1/2} - q^{-1/2})V\Big($  $\Big)$

the circle relation, that for the disjoint union of a diagram L and the circle

$$V\left(L \sqcup \bigcirc\right) = (-q^{-1/2} - q^{1/2})\,\langle L\rangle$$

and the normalization condition on the unlink $V() = 1$.

The Jones' polynomial is related to the representation theory of the Lie algebra sl_2.

Definition. The algebra $U = U_q sl_2$ is an algebra over $\mathbb{C}(q)$ generated by E, F, K such that K is invertible and

$$KEK^{-1} = qE$$
$$KFK^{-1} = qF$$
$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

U can be given a topology of a power series ring in h with $q = exp(h)$. We also consider H with $q^{H/2} = K$.

Proposition. *U is a Hopf algebra with*

$$\Delta(E) = E \otimes 1 + K \otimes E,\ \Delta(F) = F \otimes K^{-1} + 1 \otimes F,\ \Delta(K) = K \otimes K$$
$$\epsilon(E) = \epsilon(F) = 0,\ \epsilon(K) = 1$$
$$S(E) = -K^{-1}E,\ S(F) = -FK,\ S(K) = K^{-1}$$

Definition. A topological ribbon Hopf algebra is a topological Hopf algebra A with $\theta \in \hat{A}$, the completion of A, and $R \in A\hat{\otimes}A$, the completion of $A \otimes A$, such that θ and R obey the usual properties in a ribbon Hopf algebra.

Proposition. *U is a topological ribbon Hopf algebra with*

$$R = q^{H \otimes H/4} exp_q\left((q^{1/2} - q^{-1/2})E \otimes F\right)$$
$$\theta = ?$$

using the notation

$$exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{[n]!} x^n$$
$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$
$$[n]! = [n][n-1]\cdots[1]$$

Definition. The fundamental representation of U is the module generated by $v_\pm$ with

$$Ev_+ = 0, Ev_- = v_+$$
$$Fv_+ = v_-, Fv_- = 0$$
$$Kv_+ = q^{1/2}v_+, Kv_- = q^{-1/2}v_-$$

Theorem. *There is a unique tortile monoidal functor from the category of framed oriented tangles to the category of U-modules which sends the generating object to the fundamental representation and which on links is multiplication by the Jones' polynomial.*

That there is such a functor follows from the fact that U is a topological ribbon Hopf algebra. We get that $Hv_+ = v_+$ and $Hv_- = -v_-$ since then

$$q^{H/2}v_+ = exp(hH/2)v_+ = exp(h/2)v_+ = q^{1/2}v_+ = Kv_+$$
$$q^{H/2}v_- = exp(hH/2)v_- = exp(-h/2)v_- = q^{-1/2}v_- = Kv_-$$

So taking a basis $(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-)$, we get $q^{H \otimes H/4}$ acts like

$$exp\left(\frac{h}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & q^{-1/4} & 0 & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix}$$

Also, since $E^n = F^n = 0$ for $n \geq 2$, $exp_q\left((q^{1/2} - q^{-1/2})E \otimes F\right)$ acts like

$$1 \otimes 1 + (q^{1/2} - q^{-1/2})E \otimes F$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (q^{1/2} - q^{-1/2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q^{1/2} - q^{-1/2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, R acts like

$$\begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & q^{-1/4} & q^{1/4} - q^{-3/4} & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix}$$

The braiding is τR which acts like

$$\begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & q^{-1/4} & q^{1/4} - q^{-3/4} & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix}$$

And the inverse braiding is $R^{-1}\tau$ which acts like

$$\begin{pmatrix} q^{-1/4} & 0 & 0 & 0 \\ 0 & -q^{3/4} + q^{-1/4} & q^{1/4} & 0 \\ 0 & q^{1/4} & 0 & 0 \\ 0 & 0 & 0 & q^{-1/4} \end{pmatrix}$$

Thus,

$$q^{1/4}\tau R - q^{-1/4}R^{-1}\tau = (q^{1/2} - q^{-1/2})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which implies the skein relation for the Jones' polynomial. The circle relation comes ? The normalization condition follows from monoidality.