PRACTICE FINAL

(1) .

(a) Express $\underset{n\to\infty}{lim} \frac{\pi}{4n} \sum_{i=1}^{n} tan(\frac{i\pi}{4n})$ as an integral

In general we can express $\int_a^b f(x)dx = \underset{n\to\infty}{lim} \Delta x \sum_{i=1}^{n} f(x_i^*)$ where $\Delta x = \frac{b-a}{n}$ and $x_{i-1} \le x_i^* \le x_i$ and $x_i = a + i\Delta x$. If we take $\Delta x = \frac{\pi}{4n}$ and $x_i^* = x_i = \frac{i\pi}{4n}$ and $f(x) = tan(x)$ then we see $a = x_0 = 0$ and $b = x_n = \frac{\pi}{4}$. Thus, the limit can be expressed as

$$\int_0^{\frac{\pi}{4}} tan(x)dx$$

(b) Evaluate the integral from part (a).

We know that

$$\int tan(x)dx = ln|sec(x)| + C$$

since

$$tan(x) = \frac{tan(x)sec(x)}{sec(x)}$$

so if $u = sec(x)$ then $du = sec(x)tan(x)dx$ and

$$\int tan(x)dx = \int \frac{tan(x)sec(x)}{sec(x)}dx = \int \frac{du}{u}$$

$$= ln|u| + C = ln|sec(x)| + C$$

By the second fundamental theorem of calculus,

$$\int_0^{\pi/4} tan(x)dx = \int tan(x)dx\Big]_0^{\pi/4}$$

$$= ln|sec(x)|]_0^{\pi/4} = ln|sec(\pi/4)| - ln|sec(0)|$$

$$= ln\left|\frac{1}{cos(\pi/4)}\right| - ln\left|\frac{1}{cos(0)}\right| = ln\left|\frac{1}{1/\sqrt{2}}\right| - ln\left|\frac{1}{1}\right|$$

$$= ln\sqrt{2} - ln(1) = ln\sqrt{2} - 0 = ln\sqrt{2}$$

(2) Find the following.

(a) $\frac{d}{dx}\int_0^x sin(t^2)dt$

By the first fundamental theorem of calculus,

$$\frac{d}{dx}\int_0^x sin(t^2)dt = sin(x^2)$$

(b) $\frac{d}{dx}\int_0^{\sqrt{x}} ln(tan(t))dt$

By the first fundamental theorem of calculus and the chain rule

$$\frac{d}{dx}\int_0^{\sqrt{x}} ln(tan(t))dt = ln(tan\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{ln(tan\sqrt{x})}{2\sqrt{x}}$$

.

(c) $\frac{d}{dx}\int_{sin(x)}^{cos(x)} e^{\sqrt{t}}dt$
We have that for some real number a,

$$\frac{d}{dx}\int_{sin(x)}^{cos(x)} e^{\sqrt{t}}dt = \frac{d}{dx}\left(\int_{sin(x)}^{a} e^{\sqrt{t}}dt + \int_{a}^{cos(x)} e^{\sqrt{t}}dt\right)$$
$$= \frac{d}{dx}\left(\int_{a}^{cos(x)} e^{\sqrt{t}}dt - \int_{a}^{sin(x)} e^{\sqrt{t}}dt\right)$$
$$= \frac{d}{dx}\int_{a}^{cos(x)} e^{\sqrt{t}}dt - \frac{d}{dx}\int_{a}^{sin(x)} e^{\sqrt{t}}dt$$
$$= e^{\sqrt{cos(x)}}\left(\frac{d}{dx}cos(x)\right) - e^{\sqrt{sin(x)}}\left(\frac{d}{dx}sin(x)\right)$$
$$= -sin(x)e^{\sqrt{cos(x)}} - cos(x)e^{\sqrt{sin(x)}}$$

(3) Find the following.
(a) $\int e^{2sin(\theta)}cos(\theta)d\theta$
Use u-substitution with $u = sin(\theta)$. Then $du = cos(\theta)d\theta$ and,

$$\int e^{2sin(\theta)}cos(\theta)d\theta = \int e^{2u}du$$
$$= \frac{1}{2}e^{2u} + C = \frac{1}{2}e^{2sin(\theta)} + C$$

(b) $\int \frac{2+x^2}{\sqrt{6x+x^3}}dx$
Use u-substitution with $u = 6x + x^3$. Then $du = (6 + 3x^2)dx$. In particular, $(2 + x^2)dx = \frac{1}{3}(6 + 3x^2)dx = \frac{1}{3}du$, so

$$\int \frac{2+x^2}{\sqrt{6x+x^3}}dx = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-1/2}du$$
$$= \frac{1}{3}\cdot 2u^{1/2} + C = \frac{2}{3}\sqrt{u} + C$$

(c) $\int_0^{\sqrt{\pi}} xcos(x^2)dx$
Use u-substitution with $u = x^2$. Then $du = 2xdx$, so $xdx = \frac{1}{2}du$. Also we need to change our limits of integration. If $x = 0$, then $u = 0^2 = 0$. If $x = \sqrt{\pi}$, then $u = \sqrt{\pi}^2 = \pi$. So,

$$\int_0^{\sqrt{\pi}} xcos(x^2)dx = \frac{1}{2}\int_0^{\pi} cos(u)du = \frac{1}{2}\left[sin(u)\right]_0^{\pi}$$
$$= \frac{1}{2}(sin(\pi) - sin(0)) = \frac{1}{2}(0-0) = 0$$

(4) Find the following.
(a) $\int_{-2}^{2}(x^4 + x^2 + 3)dx$
$f(x) = x^4 + x^2 + 3$ is even since

$$f(-x) = (-x)^4 + (-x)^2 + 3$$
$$= x^4 + x^2 + 3 = f(x)$$

Thus,

$$\int_{-2}^{2}(x^4+x^2+3)dx = 2\int_{0}^{2}(x^4+x^2+3)dx = 2\left[\frac{1}{5}x^5+\frac{1}{3}x^3+3x\right]_0^2$$

$$= 2\left(\frac{1}{5}\cdot 2^5+\frac{1}{3}\cdot 2^3+3\cdot 2-\frac{1}{5}\cdot 0^5-\frac{1}{3}\cdot 0^3+3\cdot 0\right)$$

$$= 2\left(\frac{32}{5}+\frac{8}{3}+6\right) = 2\left(\frac{96+40+90}{15}\right) = \frac{452}{15}$$

(b) $\int_{-\pi}^{\pi} sin(x^5)dx$
$f(x) = sin(x^5)$ is odd since

$$f(-x) = sin((-x)^5) = sin(-x^5)$$

$$= -sin(x^5) = -f(x)$$

Thus,

$$\int_{-\pi}^{\pi} sin(x^5)dx = 0$$

(5) Find the following.

(a) $\int xe^{-x}dx$
Use integration by parts with $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ and $v = \int e^{-x}dx = -e^{-x}+C$; we'll set $C = 0$. So,

$$\int xe^{-x}dx = uv - \int vdu = -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C$$

(b) $\int arctan(1+x)dx$
Integrate by parts with $u = arctan(1+x)$ and $dv = dx$. Then $du = \frac{d(1+x)}{1+(1+x)^2} = \frac{dx}{x^2+2x+2}$ and $v = x + C$; we'll set $C = 0$. So,

$$\int arctan(1+x)dx = uv - \int vdu$$

$$= x\cdot arctan(1+x) - \int \frac{x}{x^2+2x+2}dx$$

We can do a substitution with $u = x^2+2x+2$, so $du = 2xdx$ and $xdx = \frac{1}{2}du$. Then,

$$\int \frac{x}{x^2+2x+2}dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}ln|u| + C$$

$$= \frac{1}{2}ln|x^2+2x+2| + C = \frac{1}{2}ln(x^2+2x+2) + C$$

since $x^2+2x+2 = 1+(1+x)^2 \geq 1 \geq 0$. Finally, setting $K = -C$

$$\int arctan(1+x)dx = x\cdot arctan(1+x) - \frac{1}{2}ln(x^2+2x+2) + K$$

(c) $\int x^2 cos(x)dx$

Integrate by parts with $u = x^2$, $dv = cos(x)dx$. We can have $du = 2xdx$ and $v = sin(x)$, so

$$\int x^2 cos(x)dx = x^2 sin(x) - 2\int x \cdot sin(x)dx$$

Integrate by parts again with $u = x$ and $dv = sin(x)dx$. Then $du = dx$ and $v = -cos(x)$, so

$$\int x \cdot sin(x)dx = -x \cdot cos(x) + \int cos(x)dx = -x \cdot cos(x) + sin(x) + C$$

Thus,

$$\int x^2 cos(x)dx = x^2 sin(x) - 2\left(-x \cdot cos(x) + sin(x) + C\right)$$

$$= (x^2 - 2)sin(x) + 2x \cdot cos(x) + K$$

(d) $\int_0^\pi t \cdot sin(3t)dt$

Integrate by parts with $u = t$, $dv = sin(3t)dt$. Then $du = dt$, $v = -\frac{1}{3}cos(3t)$, so

$$\int_0^\pi t \cdot sin(3t)dt = \int t \cdot sin(3t)dt\Big]_0^\pi$$

$$= \left[-\frac{1}{3}t \cdot cos(3t) + \frac{1}{3}\int cos(3t)dt\right]_0^\pi$$

$$= \left[-\frac{1}{3}t \cdot cos(3t) + \frac{1}{9}sin(3t)\right]_0^\pi$$

$$= -\frac{1}{3}\pi \cdot cos(3 \cdot \pi) + \frac{1}{9}sin(3 \cdot \pi) + \frac{1}{3}0 \cdot cos(3 \cdot 0) - \frac{1}{9}sin(3 \cdot 0)$$

$$= \frac{\pi}{3} + 0 + 0 + 0 = \pi/3$$

(6) Find the following.

(a) $\int sin^2(x)dx$

Using the half-angle identity

$$\int sin^2(x)dx = \int \frac{1}{2}(1 - cos(2x))dx$$

$$= \frac{1}{2}x - \frac{1}{4}sin(2x) + C$$

(b) $\int \frac{\sqrt{x^2-1}}{x^4}dx$

Use a trigonometric substitution $x = sec(\theta)$, so $dx = sec(\theta)tan(\theta)d\theta$ and

$$\sqrt{x^2 - 1} = \sqrt{sec^2(\theta) - 1} = \sqrt{tan^2(\theta)} = tan(\theta)$$

Then,

$$\int \frac{\sqrt{x^2 - 1}}{x^4}dx = \int \frac{tan(\theta)}{sec^4(\theta)}sec(\theta)tan(\theta)d\theta$$

And then, ummm, I donno.

(c) $\int_0^{\pi/4} tan^2(x)sec^4(x)dx$

$$\int_0^{\pi/4} tan^2(x)sec^4(x)dx = \int_0^{\pi/4} tan^2(x)sec^2(x)sec^2(x)dx$$
$$= \int_0^{\pi/4} tan^2(x)(1+tan^2(x))sec^2(x)dx$$

Let $u = tan(x)$, $du = sec^2(x)dx$. $x = 0 \Rightarrow u = tan(0) = 0$ and $x = \pi/4 \Rightarrow u = tan(\pi/4) = 1$

$$= \int_0^1 u^2(1+u^2)du = \int_0^1 (u^2+u^4)du$$
$$= \left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

(7) Find the following.

(a) $\int \frac{x^2+2x-1}{x^3-x}dx$

Use partial fraction decomposition.

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$
$$\Rightarrow x^2+2x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$
$$= (A+B+C)x^2 + (B-C)x - A$$
$$\Rightarrow 1 = A+B+C, \quad 2 = B-C, \quad -1 = -A$$
$$\Rightarrow A = 1, B = 1, C = -1$$
$$\Rightarrow \int \frac{x^2+2x-1}{x^3-x}dx = \int \left(\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}\right)dx$$
$$= ln|x| + ln|x-1| - ln|x+1| + C, \quad x \neq 0, 1, -1$$

(b) $\int \frac{10}{(x-1)(x^2+9)}dx$

x^2+9 is not factorable over the reals, so

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$
$$\Rightarrow 0x^2 + 0x + 10 = A(x^2+9) + (Bx+C)(x-1)$$
$$= (A+B)x^2 + (C-B)x + (9A-C)$$
$$\Rightarrow 0 = A+B, \quad 0 = C-B, \quad 10 = 9A-C$$
$$\Rightarrow A = 1, B = -1, C = -1$$
$$\Rightarrow \int \frac{10}{(x-1)(x^2+9)} = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right)dx$$
$$= ln|x-1| - \frac{1}{2}ln|x^2+9| - \frac{1}{3}arctan(\frac{x}{3}) + C$$

(8) Approximate the integral $\int_1^4 \frac{1}{x}dx$.

(a) Use the left endpoint rule with $n = 3$ subintervals
Set $\Delta x = \frac{b-a}{n} = \frac{4-1}{3} = 1$ and

$$x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$$

So,

$$A \approx \Delta x(f(x_0) + f(x_1) + f(x_2)) = 1(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}) = \frac{11}{6}$$

(b) Use the right endpoint rule with $n = 3$ subintervals
Use same Δx and x_i from (a)

$$A \approx \Delta x(f(x_1) + f(x_2) + f(x_3)) = 1(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{13}{12}$$

(c) Use the midpoint rule with $n = 3$ subintervals
Use $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$, so

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{5}{2}, \quad \bar{x}_3 = \frac{7}{2}$$

$$A \approx \Delta x(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3)) = 1(\frac{2}{3} + \frac{2}{5} + \frac{2}{7}) = \frac{70+42+30}{105} = 142/105$$

(d) Use the trapezoid rule with $n = 3$ subintervals
Use the formula $T_n = \frac{1}{2}(L_n + R_n)$, so $T_3 = \frac{1}{2}(\frac{11}{6} + \frac{13}{12}) = \frac{35}{24}$

(e) Use Simpson's rule with $n = 6$ subintervals
Use the formula $S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n$, so $S_6 = \frac{2}{3}\frac{142}{105} + \frac{1}{3}\frac{35}{24} = yuck$

(9) .

(a) Find $\int_4^\infty e^{-\frac{y}{2}} dy$
Improper integral so,

$$\int_4^\infty e^{-\frac{y}{2}} dy = \lim_{t \to \infty} \int_4^t e^{-y/2} dy = \lim_{t \to \infty} \left[(-2)e^{-y/2}\right]_4^t$$

$$= \lim_{t \to \infty} \left((-2)e^{-t/2} + 2e^{-2}\right) = 0 + 2e^{-2} = 2e^{-2}$$

(b) Find $\int_0^3 \frac{2}{\sqrt{x}} dx$
Improper integral since $2/\sqrt{x}$ has an asymptote at $x = 0$, so,

$$\int_0^3 \frac{2}{\sqrt{x}} dx = \lim_{t \to 0^+} \int_t^3 \frac{2}{\sqrt{x}} dx = \lim_{t \to 0^+} \left[4\sqrt{x}\right]_t^4$$

$$= \lim_{t \to 0^+} \left(8 - 4\sqrt{t}\right) = 8 - 0 = 8$$

(c) Determine whether $\int_1^\infty \frac{sin^2(x)}{x^2} dx$ converges
I suspect it converges since $\int_1^\infty x^{-2} dx = [-x^{-1}]_1^\infty = 0 + 1 = 1$. We have that

$$0 \le sin^2(x) \le 1 \Rightarrow 0 \le \frac{sin^2(x)}{x^2} \le \frac{1}{x^2}$$

So $\int_1^\infty \frac{sin^2(x)}{x^2} dx$ converges by the comparison theorem.

(10) Find the area enclosed by the curves.

(a) $y = 1 + \sqrt{x}$ and $y = \frac{3+x}{3}$
I can't really sketch here. Let's find the intersection points.

$$1 + \sqrt{x} = 1 + \frac{x}{3} \Rightarrow \sqrt{x} = \frac{x}{3} \Rightarrow x = \frac{x^2}{9}, x \geq 0$$

$$\Rightarrow x^2 - 9x = 0 \Rightarrow x(x-9) = 0 \Rightarrow x = 0, 9$$

For $0 \leq x \leq 9$, $1 + \sqrt{x} \geq \frac{3+x}{3}$ so

$$A = \int_0^9 \left(1 + \sqrt{x} - \left(\frac{3+x}{3}\right)\right) dx = \int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\right]_0^9 = 18 - \frac{27}{2} - 0 + 0 = \frac{9}{2}$$

(b) $4x + y^2 = 12$ and $x = y$
First express both curves as functions of y, in particular, $x = 12 - \frac{1}{4}y^2$.
Now find the intersection points.

$$y^2 + 4y - 12 = 0 \Rightarrow (y-2)(y+6) = 0 \Rightarrow y = 2, 6$$

For $2 \leq y \leq 6$, $y \leq 12 - \frac{1}{4}y^2$ so

$$A = \int_2^6 (12 - \frac{1}{4}y^2 - y)dy = \int_2^6 (-\frac{1}{4}y^2 - y + 12)dy$$

$$= \left[-\frac{1}{12}y^3 - \frac{1}{2}y^2 + 12y\right]_2^6 = -18 - 18 + 72 + \frac{2}{3} + 2 - 24 = 44/3$$

(11) Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{4}x^2$ and $y = 5 - x^2$ about the x-axis.
Find intersections.

$$\frac{1}{4}x^2 = 5 - x^2 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$

For $-2 \leq x \leq 2$, $\frac{1}{4}x^2 \leq 5 - x^2$, so using evenness of the integrand,

$$V = \pi \int_{-2}^2 \left(\left(5 - x^2\right)^2 - \left(\frac{1}{4}x^2\right)^2\right) dx$$

$$= 2\pi \int_0^2 \left(25 - 10x^2 + \frac{15}{16}x^4\right) dx$$

$$= 2\pi \left[25x - \frac{10}{3}x^2 + \frac{3}{16}x^5\right]_0^2$$

$$= 2\pi(50 - \frac{40}{3} + 6) = \frac{256\pi}{3}$$

(12) Find the arc length of the curves
(a) $y = \frac{x^2}{2} - \frac{ln(x)}{4}$ with $2 \leq x \leq 4$

$$\frac{dy}{dx} = x - \frac{1}{4x} \Rightarrow$$

$$L = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_2^4 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}} dx$$

$$= \int_2^4 \left(x + \frac{1}{4x} \right) dx = \left[x^2/2 + ln(x)/4 \right]_2^4 = 4 + ln(4)/4 - ln(2)/4 = 4 + ln(2)/4$$

(b) $x = e^t + e^{-t}$, $y = 5 - 2t$ with $0 \le t \le 4$

$$\frac{dx}{dt} = e^t - e^{-t}, \quad \frac{dy}{dt} = -2$$

$$L = \int_0^4 \sqrt{(e^t - e^{-t})^2 + (-2)^2} dt = \int_0^4 (e^t + e^{-t}) dt = e^4 + e^{-4} - 2$$

(13) What is the work required to empty a conical tank with radius 1 and height 1 which is filled with a liquid with force density 1?

The work is given by

$$dW = xdF = xdV = xdV = xd(Ax) = xAdx$$

Since the tank is a cone, the area of a slice is πr^2 where $r = 1 - x$ so

$$dW = \pi x(1-x)^2 dx \Rightarrow W = \pi \int_0^1 x(1-x)^2 dx$$

$$= \pi \int_0^1 (x^3 - 2x^2 + x) dx = \pi(1/4 + 2/3 + 1/2) = 17\pi/12$$

(14) Find the mean of the exponential distribution function $f(x) = \begin{cases} 0 & x < 0 \\ ce^{-cx} & x \ge 0 \end{cases}$, where $c > 0$.

The mean is

$$\mu = \int_{-\infty}^{\infty} x f(x) dx = c \int_0^{\infty} x e^{-cx} dx$$

Integrate by parts with $u = x$, $dv = e^{-cx} dx$, so $du = dx$ and $v = -\frac{1}{c} e^{-cx}$.

$$\mu = c \left[-\frac{1}{c} x e^{-cx} + \frac{1}{c} \int e^{-cx} dx \right]_0^{\infty}$$

$$= c \left[-\frac{1}{c} x e^{-cx} - \frac{1}{c^2} e^{-cx} \right]_0^{\infty}$$

$$= \frac{1}{c}$$