GEODESICS ON THE SYMPLECTOMORPHISM GROUP

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In memory of Jerry Marsden

1. Introduction

In [E-M], after the celebrated paper of V. I. Arnol'd [A], we proved the existence of geodesics for a right invariant L^2 Riemannian metric on the group of volume preserving diffeomorphisms (volumorphisms) of a compact manifold M. These geodesics are important because they describe all possible motions of an incompressible inviscid fluid which fills M. In the present work we shall assume that M has a symplectic form ω and prove the same result for the group of diffeomorphisms of M which preserve ω (symplectomorphisms). The method of proof is essentially the same in both cases. We find a vector field on the tangent bundle to the group whose integral curves are (lifts of) the geodesics.

We introduce a Hilbert manifold structure on the group of diffeomorphisms and therefore on its tangent bundle with respect to which this vector field is smooth. Then the local existence of its integral curves follows from the fundamental theorem of ordinary differential equations on a Hilbert manifold; that is, they can be constructed by a standard Picard iteration.

If M is two-dimensional (in which case the volumorphism and symplectomorphism groups are the same) one can find estimates to show that geodesics exist globally, or for all time (see [W] and [K] or [E2]). We shall use a similar estimate to show that in the symplectic case, the solutions exist globally for higher dimensional M as well.

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2. A Hilbert Manifold structure for the diffeomorphism group

We begin by describing the Hilbert manifold structure that was used in [E-M] and introduced previously in [E]. Our description will be brief and we refer the reader to [E-M] and [E] for a more complete account.

Let M be a compact oriented manifold of dimension m without boundary. In the symplectic case, m must be even so we shall write $m = 2n$. Endow M with a Riemannian metric g and let μ be the volume element given by g and the orientation. Let TM be the tangent bundle of M, and let $H^s(TM)$ be the space of all sections of this bundle which have L^2 derivatives up to order s. We define the inner product

$$(v, w)_s = \sum_{k \leq s} \int_M g(\nabla^k v, \nabla^k w)\mu$$

where ∇^k means the kth-order covariant derivative given by the Riemannian metric. With this inner product, $H^s(TM)$ is a Hilbert space. We shall assume that s is greater than $m/2$ so that by the Sobolev lemma all sections in $H^s(TM)$ are continuous and the topology of $H^s(TM)$ is stronger than that of uniform convergence (C^0).

Let $E : TM \to M$ be the exponential map which comes from g and define $\tilde{E} : TM \to M \times M$ by $\tilde{E}(v) = (\pi(v), E(v))$ where $\pi : TM \to M$ is the bundle map. As is well known, $\tilde{E}$ is a diffeomorphism from U, a neighbourhood of the zero section of TM, to $\tilde{E}(U)$, which is a neighbourhood of the diagonal in $M \times M$. Let $H^s(U) = \{v \in H^s(TM) | v(M) \subseteq U\}$ and let $H^s(M,M)$ be the space of all maps from M to itself which have L^2 derivatives up to order s, endowed with the H^s topology. Then $\Omega_E : H^s(U) \to H^s(M,M)$, defined by $\Omega_E(v) = E \circ v$ is a bijection from $H^s(U)$ to a neighbourhood of the identity in $H^s(M,M)$. It will be a chart for $H^s(M,M)$ about the identity map.

Similarly for any $\zeta \in H^s(M,M)$, we define $H^s_\zeta(TM)$ to be the space of sections of TM over ζ; that is, $H^s_\zeta(TM) = \{u : M \to TM | \pi \circ u = \zeta\}$. Then $\Omega_E : H^s_\zeta(U) \to H^s(M,M)$ is a bijection onto a neighbourhood of ζ. The pairs $(H^s_\zeta(U), \Omega_E)$ give a set of charts for $H^s(M,M)$ which makes it a smooth manifold, since the transition functions are just composition with smooth maps.

Now assume that $s > m/2 + 1$ so the H^s topology is stronger than C^1. Let $C^1\mathcal{D}$ be the group of C^1 diffeomorphisms of M and let $\mathcal{D}^s = H^s(M,M) \bigcap C^1\mathcal{D}$. $C^1\mathcal{D}$ is open in $C^1(M,M)$, the set of C^1 maps from M to itself, so $\mathcal{D}^s$ is open in $H^s(M,M)$ and hence inherits its manifold structure. It is a topological group. Right multiplication is smooth, but left multiplication is only continuous. In what follows we shall consider only the identity component of $\mathcal{D}^s$, which we shall still call $\mathcal{D}^s$.

3. The subgroup $\mathcal{D}^s_\omega$

Given a symplectic form ω, we let $[\omega]_s$ be its H^s-comology class; that is,

$$[\omega]_s = \{\omega + d\alpha \,|\, \alpha \in H^{s+1}(T^*M)\}$$

where $H^{s+1}(T^*M)$ is the space of H^{s+1} co-vectors or one-forms. Then we define $\psi : \mathcal{D}^{s+1} \to [\omega]_s$ by $\psi(\zeta) = \zeta^*(\omega)$, the pullback of ω by ζ.

Proposition 3.1. *ψ is a submersion so $\mathcal{D}^{s+1}_\omega := \{\eta \,|\, \eta^*(\omega) = \omega\} = \psi^{-1}(\omega)$ is a submanifold of $\mathcal{D}^{s+1}$. Clearly it is also a subgroup.*

Proof. (The argument follows [Mos], Theorem 2) Given $\zeta \in \mathcal{D}^{s+1}$, let $\zeta(t)$ be a curve in $\mathcal{D}^{s+1}$ such that $\zeta(0) = id$, the identity, and $\zeta(1) = \zeta$. Then:

$$\zeta^*(\omega) = \omega + \int_0^1 \frac{d}{dt}\zeta(t)^*(\omega)\, dt$$

and $\frac{d}{dt}(\zeta(t)^*(\omega)) = \zeta(t)^*(\mathcal{L}_{v(t)}\omega)$ where $v(t) \in H^{s+1}(TM)$ is defined by the equation $\frac{d}{dt}\zeta(t) = v(t) \circ \zeta(t)$ and $\mathcal{L}_{v(t)}$ is the Lie derivative in direction $v(t)$. Letting i_v be the interior product, we get:

$$\begin{aligned} \zeta^*(\omega) &= \omega + \int_0^1 \zeta(t)^*(di_{v(t)}(\omega))\, dt \\ &= \omega + d\int_0^1 \zeta(t)^*(i_{v(t)}\omega)\, dt \end{aligned}$$

Hence we see that $\psi(\mathcal{D}^{s+1}) \subseteq [\omega]_s$.

We proceed to compute the tangent map to ψ at $\eta \in \mathcal{D}^{s+1}$, calling it $T_\eta\psi : T_\eta\mathcal{D}^{s+1} \to T_{\eta^*(\omega)}[\omega]_s$. For $\eta = id$ and $v \in T_{id}\mathcal{D}^{s+1} = H^{s+1}(TM)$, we have $T_{id}\psi(v) = \frac{d}{dt}(\zeta^*(t)(\omega))|_{t=0}$ where $\zeta(t)$ is a curve in $\mathcal{D}^{s+1}$ with $\zeta(0) = id$ and $\dot\zeta(0) = v$. (Here and below "$\cdot$" means $\frac{d}{dt}$.) Thus $T_{id}\psi(v) = \mathcal{L}_v(\omega) = di_v\omega$. But $T_\omega[\omega]_s = dH^{s+1}(T^*M)$ and since ω is non-degenerate, the map $v \to i_v\omega$ is an isomorphism from $H^{s+1}(TM)$ to $H^{s+1}(T^*M)$. Thus given $d\alpha \in dH^{s+1}(T^*M)$, there exists $v \in H^{s+1}(TM)$ such that $di_v\omega = d\alpha$, so $T_{id}\psi : T_{id}\mathcal{D}^{s+1} \to T_\omega[\omega]_s$ is onto.

Furthermore for any $\eta \in \mathcal{D}^{s+1}$ and $v \circ \eta \in T_\eta\mathcal{D}^{s+1}$ we find that $T_\eta\psi(v\circ\eta) = \eta^*(\mathcal{L}_v\omega)$ so given $d\alpha \in T_{\eta^*(\omega)}([\omega]_s)$ we can pick v so that $i_v\omega = (\eta^{-1})^*(d\alpha)$. Then we get $T_\eta\psi(v\circ\eta) = d\alpha$, so $T_\eta\psi$ is onto as well. The proposition follows. □

It is clear from the above that $T_\eta\mathcal{D}^{s+1}_\omega = \{v\circ\eta | \mathcal{L}_v\omega = 0\}$, the set of vector fields $v\circ\eta$ such that $i_v\omega$ is closed.

Note that $\psi(\mathcal{D}^{s+1})$ is open in $[\omega]_s$ and if $\tilde\omega$ is any other symplectic form in $[\omega]_s$, then $\tilde\psi(\mathcal{D}^{s+1})$ is also open in $[\omega]_s$, where $\tilde\psi(\zeta) := \zeta^*(\tilde\omega)$. One might ask the obvious question: are these two sets necessarily the same, or, put differently, are two cohomologous symplectic forms related by a diffeomorphism? If $\dim M = 2$ the answer is yes because the set of symplectic forms in $[\omega]_s$ is convex and thus connected. For $\dim M > 2$, the answer is in general unknown. For partial results see [M-S], section 10.4.

4. Geodesics on $\mathcal{D}^s$

Using a Riemannian metric g on M, we can define a weak Riemannian metric on $H^s(M,M)$ (and therefore on $\mathcal{D}^s$) by:

$$(v,w) = \int_M g(x)(v(x),w(x))\mu(x)$$

where μ is the volume element of g. We say that this metric is weak because it induces the L^2 or H^0 topology on each tangent space, rather than the H^s topology. However this weak metric gives geodesics which can be defined in the usual way as stationary curves with respect to energy; that is, we say that $\eta : [0,T] \to \mathcal{D}^s$ is a geodesic if it is a stationary curve of:

$$\mathcal{E}(\eta) := 1/2\int_0^T (\dot\eta(t),\dot\eta(t))\,dt$$

with respect to all variations which leave fixed the end points $\eta(0)$ and $\eta(T)$. Let $\eta(t,\tau)$ be such a variation with $\eta(t,0) = \eta(t)$ and let $\sigma(t) = \partial_\tau\eta(t,\tau)|_{\tau=0}$. Then for each t, $\sigma(t) \in T_{\eta(t)}\mathcal{D}^s$ and

$$\begin{aligned} 0 &= 1/2\int_0^T\int_M \partial_\tau g(\eta(t,\tau)(x))(\dot\eta(t,\tau)(x),\dot\eta(t,\tau)(x))\mu(x)\,dt \\ &= \int_0^T\int_M g(\eta(t)(x))(\dot\eta(t)(x),\dot\sigma(t)(x)) + \frac{1}{2}\partial_{\sigma(t)(x)}g(\eta(t)(x))(\dot\eta(t)(x),\dot\eta(t)(x))\mu(x)\,dt \end{aligned}$$

If we define the vector fields $v(t)$ and $w(t)$ by $\dot{\eta}(t) = v(t) \circ \eta(t)$ and $\dot{\sigma}(t) = w(t) \circ \eta(t)$ a straight forward calculation shows that we can rewrite this equation as:

$$0 = \int_0^T \int_M g(\partial_t v + \nabla_v v, w) \circ \eta(t) \mu \, dt$$

where ∇ is the conncection which comes from g. Since w can be any vector field, we find that $(\partial_t v + \nabla_v v)(t)(x) = 0$ for each t and x. From this it follows that for each fixed $x \in M$, $\eta(t)(x)$ is a geodesic on M. Thus geodesics $\eta(t)(x)$ on M combine to form a geodesic $\eta(t)$ on $\mathcal{D}^s$ or $H^s(M, M)$. The geodesics define an exponential map $Exp : TH^s(M, M) \to H^s(M, M)$ which is easily seen to be $Exp(w) = E \circ w$. Since M is compact, the metric g is complete, so $\eta(t)(x)$ exists for all t. It follows that $\eta(t) \in H^s(M, M)$ also exists for all t, but it possibly will not remain in $\mathcal{D}^s$ for all t. In fact if M is the circle and $\eta(0) = id$ with $\dot{\eta}(0)$ any non-constant vector field, then $\eta(t)$ will not remain in $\mathcal{D}^s$ for all t.

We proceed to look at (the lifts of) geodesics on $H^s(M, M)$ as integral curves of a vector field on $TH^s(M, M)$ which is naturally identified with $H^s(M, TM)$. Such a vector field will be a smooth map:

$$\mathcal{Z} : TH^s(M, M) \to TTH^s(M, M) \cong H^s(M, TTM)$$

with the property that $\Pi \circ \mathcal{Z}$ is the identity, where $\Pi : TTH^s(M, M) \to TH^s(M, M)$ is the bundle map. Note that with the natural identifications,
$\Pi : H^s(M, TTM) \to H^s(M, TM)$ is given by $\Pi(\xi) = \pi_1 \circ \xi$, where $\pi_1 : TTM \to TM$ is the bundle map of TTM.

Let $Z : TM \to TTM$ be the spray of the metric g, the vector field on TM whose integral curves are $\dot{\gamma}(t)$, where $\gamma(t)$ is a geodesic on M. If $x := (x^1, \ldots, x^n)$ are local coordinates on M and $v^i \frac{\partial}{\partial x^i}$ is a local vector field, then $Z(x, v) = (v, -\Gamma(x)(v, v))$ where $\Gamma(v, v) := \Gamma_{ij} v^i v^j$, Γ_{ij} being the Christoffel symbols which come from the connection ∇.

Then $\mathcal{Z}(v) = Z \circ v$ has integral curves $\dot{\eta}(t)$ where for each $x \in M$, the curve $\dot{\gamma}(t) := \dot{\eta}(t)(x)$ is the lift of a geodesic. Hence $\mathcal{Z}$ is the spray for the L^2 metric on $H^s(M, M)$

5. Geodesics on $\mathcal{D}^s_\omega$

We proceed to show the existence of geodesics on $\mathcal{D}^s_\omega$, regarded as a Riemannian submanifold of $\mathcal{D}^s$. Note that since the Riemannian metric is weak, the existence of geodesics does not follow automatically; that is, an affine connection on $\mathcal{D}^s$ does not automatically define a connection on $\mathcal{D}^s_\omega$.

To find geodesics on $\mathcal{D}^s_\omega$ we first make the requirement that g and ω are compatible, which we now explain. Since g and ω are both non-degenerate forms they both give at each $x \in M$ isomorphisms from $T_x M$ to $T_x^* M$ defined by $g^\flat(v) := g(v,\ )$, respectively $\omega^\flat(v) := \omega(v,\ )$. Then letting $g^\sharp$ be the inverse of $g^\flat$, we define $J := g^\sharp \omega^\flat : T_x M \to T_x M$, an isomorphism. Since ω is antisymmetric, J is skew-adjoint with respect to g; that is, for $v, w \in T_x M$

$$g(Jv, w) = \omega(v, w) = -\omega(w, v) = -g(Jw, v) = -g(v, Jw).$$

We say that g and ω are compatible if $-J^2 = I$, where I is the identity operator on $T_x M$. In this case J is said to give an almost complex structure to M. Also since

$$g(v, w) = g(-J^2 v, w) = g(Jv, Jw)$$

we see that J is an isometry on each T_xM. Furthermore if we define μ, the volume element of g, using the orientation given by ω, then we find by direct calculation that $\mu = \omega^n$.

We note that for any symplectic form ω, it is always possible to find a compatible metric g as we now show. Given any metric $\tilde{g}$, $\tilde{g}^\sharp\omega^\flat$ is a skew-adjoint isomorphism so $-(\tilde{g}^\sharp\omega^\flat)^2$ is self-adjoint positive definite. Let S be the positive fourth root of $-(\tilde{g}^\sharp\omega^\flat)^2$. Then one readily checks that $g(v,w) := \tilde{g}(Sv, Sw)$ is compatible with ω.

We proceed to look for geodesics on $\mathcal{D}^s_\omega$ as stationary curves of the energy as we did for $\mathcal{D}^s$. However this time we restrict the variation to curves in $\mathcal{D}^s_\omega$.
Hence we consider $\eta(t,\tau)$ and $\sigma(t) := \partial_\tau \eta(t,\tau)|_{\tau=0}$, but with
$\sigma(t) \in T_{\eta(t)}\mathcal{D}^s_\omega = \{w \circ \eta(t) | d\omega^\flat(w) = 0\}$. Thus computing as before and defining $v(t)$ by $\dot{\eta}(t) = v(t) \circ \eta(t)$ we get: $\int_0^T \int_M g(\partial_t v + \nabla_v v, w) \circ \eta(t) \mu \, dt = 0$. Since $\eta(t)^*\omega = \omega$ and $\mu = \omega^n$, we find that:

$$(\partial_t v + \nabla_v v, w) = \int_M g(\partial_t v + \nabla_v v, w)\mu = 0 \tag{5.1}$$

for all w with $d\omega^\flat(w) = 0$, or for all $w \in T_{id}\mathcal{D}^s_\omega$.

We now proceed to decompose $H^s(TM) = T_{id}\mathcal{D}^s$ into orthogonal summands. From the Hodge decomposition of one-forms we find that:

$$H^s(T^*M) = \mathcal{H} \oplus d\delta H^{s+2}(T^*M) \oplus \delta d H^{s+2}(T^*M) \tag{5.2}$$

where the d is the exterior derivative, δ is its adjoint and $\mathcal{H}$ is the finite dimensional space of harmonic one-forms. The three summands are orthogonal with respect to the L^2 inner product on $H^s(T^*M)$ defined by g and μ. Also the null-space of $\delta d : H^s(T^*M) \to H^{s-2}(T^*M)$ is the first two summands.

Proposition 5.1. $\omega^\sharp(d\delta H^{s+2} \oplus \mathcal{H}) = T_{id}\mathcal{D}^s_\omega$.

Proof. Let $\alpha = d\delta\beta + h$, h harmonic, and let $v = \omega^\sharp(\alpha)$. Then $d\omega^\flat(v) = d(d\beta + h) = 0$. Conversely assume v is such that $d\omega^\flat(v) = 0$. Then $\omega^\flat(v) = d\delta\beta + h$ for some β and h, so $v = \omega^\sharp(d\delta\beta + h)$. ☐

From (5.1) we find that $\partial_t v + \nabla_v v$ is perpendicular to $\omega^\sharp(d\delta H^{s+2}(T^*M) \oplus \mathcal{H})$ so from the Hodge decomposition we see that: $\partial_t v + \nabla_v v := F \in \omega^\sharp(\delta d H^{s+2}(T^*M))$. Hence $F = \omega^\sharp(\delta d\alpha)$ for some $\alpha \in H^{s+2}(T^*M)$.

We proceed to compute F as a function of v. Since $d\omega^\flat(\partial_t v) = \partial_t d\omega^\flat(v) = 0$, we find that $d\omega^\flat F = d\omega^\flat(\nabla_v v)$, so $d\delta d\alpha = d\omega^\flat(\nabla_v v)$. But $d\delta d\alpha = \Delta d\alpha$, where we consider $\Delta = \delta d + d\delta$ as an isomorphism from the orthogonal complement of $\mathcal{H}$ in $H^{s+1}(T^*M)$ to the orthogonal complement of $\mathcal{H}$ in $H^{s-1}(T^*M)$. Hence $d\alpha = \Delta^{-1} d\omega^\flat(\nabla_v v)$ and $F = \omega^\sharp\delta\Delta^{-1} d\omega^\flat(\nabla_v v)$. But since $d\omega^\flat v = 0$ we can write $d\omega^\flat(\nabla_v v)$ as $[d\omega^\flat, \nabla_v]v$ where $[\ ,\ ]$ means commutator. Thus we have $F = \omega^\sharp\delta\Delta^{-1}[d\omega^\flat, \nabla_v]v$. We note that as operators $\omega^\sharp$ has order zero, $\delta\Delta^{-1}$ has order minus one and $[d\omega^\flat, \nabla_v]$, being the commutator of two first order operators, has order one. Thus the composite map has order zero and F as a function of v is smooth from $H^s(TM)$ to itself.

Now we are ready to form the geodesic equation. As before we define v by $\dot{\eta} = v \circ \eta$, so

$$(\partial_t v + \nabla_v v) \circ \eta = F \circ \eta = (\omega^\sharp\delta\Delta^{-1}[d\omega^\flat, \nabla_v]v) \circ \eta. \tag{5.3}$$

Combining (5.3) with the spray for $\mathcal{D}^s$ we get the spray:

$$\tilde{\mathcal{Z}}(\eta, v\circ\eta) = (v\circ\eta, -\Gamma(v,v)\circ\eta + F\circ\eta) \tag{5.4}$$

where F is defined above.

Theorem 5.2 (Local Existence)**.** *Equation (5.4) defines a smooth vector field $\tilde{\mathcal{Z}}$ on $T\mathcal{D}^s_\omega$. Hence for any $u \in T\mathcal{D}^s_\omega$, there exist positive T_b, and T_e and a unique $\eta : (-T_b, T_e) \to \mathcal{D}^s_\omega$ such that $\dot\eta$ is an integral curve of $\tilde{\mathcal{Z}}$ and $\dot\eta(0) = u$. Furthermore η depends smoothly on u.*

Proof. From (5.3) and the definition of $\mathcal{Z}$ we see that:

$$\ddot\eta = \mathcal{Z}(\dot\eta) + (0, \omega^\sharp\delta\Delta^{-1}[d\omega^\flat, \nabla_{\dot\eta\circ\eta^{-1}}]\dot\eta\circ\eta^{-1}))\circ\eta. \tag{5.5}$$

Since F was constructed based on the fact that $\eta(t)$ is a curve in $\mathcal{D}^s_\omega$, or more precisely that $\dot\eta(t)$ is a curve in $T\mathcal{D}^s_\omega$, we see that $\tilde{\mathcal{Z}}$ is a vector field on $T\mathcal{D}^s_\omega$; that is, the image of $\tilde{\mathcal{Z}}$ is contained in $TT\mathcal{D}^s_\omega$. Also $\mathcal{Z}$ is clearly smooth in η so it remains to check that $F\circ\eta$. is smooth. To do this we introduce the notation L_η for any linear operator L on functions or vector fields on M. We define $L_\eta u := L(u\circ\eta^{-1})\circ\eta$. Then

$$F\circ\eta = \omega^\sharp_\eta(\delta\Delta^{-1})_\eta[d\omega^\flat, \nabla_v]_\eta\dot\eta. \tag{5.6}$$

Given any partial derivative ∂_i, a direct computation shows that $(\partial_i)_\eta$ is smooth in η. We have $(\partial_i)_\eta f = (\partial_j\eta^i)^{-1}\partial_j f$ where $^{-1}$ means matrix inverse, so $(\partial_i)_\eta$ is smooth as a map from $H^s(M,\mathbb{R})\times\mathcal{D}^s \to H^{s-1}(M,\mathbb{R})$. From this we readily see that $\omega^\sharp_\eta$ and $[d\omega^\flat, \nabla_v]_\eta$ are also smooth in η. In [E-M], appendix A, Lemma 6, it is shown that $(\delta\Delta^{-1})_\eta$ is smooth in η as well. □

6. Existence of geodesics for all time

Theorem 6.1 (Global Existence)**.** *As in the beginning of section 5, we assume that M has a symplectic form ω and a compatible Riemannian metric g. Using these structures, we get a vector field $\tilde{\mathcal{Z}}$ defined in (5.4). The integral curves of this vector field (or equivalently the solutions of equation (5.5)) extend for all time; that is, the limits T_b and T_e of the local existence theorem can both be taken to be infinity so geodesics on $\mathcal{D}^s_\omega$ extend globally.*

The proof will constitute this entire section.

If $(-T_b, T_e)$ is the maximum interval on which $\dot\eta$ exists and T_e is finite, then from standard o. d. e. theory we know that as $t \nearrow T_e$, $\dot\eta(t)$ must grow without bound in the H^s topology. We shall show that that cannot happen. We extend the H^s inner product on $T_{id}\mathcal{D}^s_\omega$ by right invariance. Then $\|\dot\eta\|_s = \|v\|_s$ where $v = \dot\eta\circ\eta^{-1}$, so it suffices to show that $\|v\|_s$ remains bounded. Since right multiplication preserves geodesics, it suffices to restrict to geodesics which at time zero equal the identity.

We shall use (5.3) to estimate the growth of $\|v\|_s$.

$$\frac{1}{2}\frac{d}{dt}(v,v)_s = (\nabla^s(\nabla_v v + F), \nabla^s v) \tag{6.1}$$

$$= (\nabla_v\nabla^s v, \nabla^s v) + ([\nabla^s, \nabla_v]v, \nabla^s v) + (\nabla^s v, \nabla^s F) \tag{6.2}$$

and

$$(\nabla_v\nabla^s v, \nabla^s v) = \frac{1}{2}\int_M g(\nabla^s v, \nabla^s v) di_v\mu = 0$$

since $\mu = \omega^{n/2}$ and $di_v\omega = 0$.

From[T], Chapter 13, Proposition 3.7 we have the basic estimates:

$$||fg||_k \leq K(\|f\|_{C^0}\|g\|_k + \|g\|_{C^0}\|f\|_k) \tag{6.3}$$

$$\|\nabla^k(fg) - f\nabla^k g\|_0 \leq K(\|f\|_k\|g\|_{C^0} + \|\nabla f\|_{C^0}\|g\|_{k-1}) \tag{6.4}$$

Applying (6.4) to $\|[\nabla^s, \nabla_v]v\|_0$ with $f = v$ and $g = \nabla v$ we get:

$$\|[\nabla^s, \nabla_v]v\|_0 \leq K\|v\|_{C^1}\|v\|_s$$

Also since $\delta\Delta^{-1} : H^{s-1} \to H^s$ is continuous, we find

$$\|F\|_s \leq K\|[d\omega^\flat, \nabla_v]v\|_{s-1}$$

Computing $[d\omega^\flat, \nabla_v]v$ in coordinates (and suppressing indices) we get only terms of the form: $(\partial\omega)v(\partial v)$ and $\omega(\partial v)\partial v$ since the terms with $\partial^2 v$ cancel one another. But by (6.3) the H^{s-1} norm of such terms is bounded by $K\|v\|_{C^1}\|v\|_s$.

Thus we find that $\|[\nabla^s, \nabla_v]v + \nabla^s F\|_0$ is bounded by $K\|v\|_{C^1}\|v\|_s$ and from this it follows that

$$\frac{1}{2}\frac{d}{dt}(v, v)_s \leq K\|v\|_{C^1}(v, v)_s$$

for some constant K. Hence if $\|v\|_{C^1}$ remains bounded on any finite interval, then so does $\|v\|_s$.

We proceed to estimate the C^1-norm of v. By proposition 5.1, $v(t) = \omega^\sharp(df(t) + h(t))$ where $h(t)$ is harmonic. The L^2-norm of $v(t)$ is constant and $\omega^\sharp(df)$ is perpendicular to $\omega^\sharp(h)$ in the L^2 inner product. Thus the L^2-norm of h is bounded as a function of t. But since $\mathcal{H}$ is finite dimensional, all norms on it are equivalent so any H^s- or C^k-norm is also bounded as a function of t.

It remains to consider $df(t)$. To do so we shall use a version of Noether's theorem:

Theorem 6.2. *(Noether) Let $X\,\langle\,,\,\rangle$ be a Riemannian manifold with a Lie Group G of isometries of X. Then for $\gamma(t)$ a geodesic in X and for $\xi(s)$ a one parameter subgroup of G, the quantity $\langle\dot\gamma(t), \frac{d}{ds}\xi(s)(\gamma(t))|_{s=0}\rangle$ is constant along the geodesic.*

Proof. see [A2], page 88 □

In our case the group $\mathcal{D}^s_\omega$ acts on itself isometrically by right multiplication. Thus we find that $\int_M g(\dot\eta(t), D\eta(w))\mu$ is constant for each $w \in T_{id}\mathcal{D}^s_\omega$. Therefore $\int_M \eta^*(g^\flat v)(w)\mu$ is constant as well, so applying $\omega^\flat g^\sharp$ to the decomposition (5.2), we find that there exists a two-form $\lambda(t)$ such that:

$$g^\flat v(0) = \eta(t)^*(g^\flat v(t)) + g^\flat\omega^\sharp(\delta\lambda(t))$$

Therefore

$$\delta\omega^\flat v(0) = \delta\omega^\flat g^\sharp\eta^* g^\flat v(t) \tag{6.5}$$

But by Proposition 5.1 $\omega^\flat v(0) = df(0) + h(0)$ so $\delta\omega^\flat v(0) = \Delta f(0)$

We proceed to compute the right side of (6.5). To do so we use the fact that for any one-form α, $\delta\alpha = *di_{g^\sharp\alpha}\mu$ where $*$ is the identification of n-forms with functions; that is, $*(f\mu) = f$. Thus if we let $\beta = \eta^* g^\flat v(t)$, then the right side of

(6.5) will equal

$$\begin{aligned}
*di_{(g^\sharp\omega^\flat g^\sharp\beta)}\mu &= *di_{(Jg^\sharp\beta)}\omega\wedge\omega^{n/2-1}\\
&= *d\omega^\flat Jg^\sharp\beta\wedge\omega^{n/2-1}\\
&= *dg^\flat J^2g^\sharp\beta\wedge\omega^{n/2-1}\\
&= -*d\beta\wedge\omega^{n/2-1}\\
&= -*d\eta(t)^*g^\flat v(t)\wedge\omega^{n/2-1}\\
&= -*\eta(t)^*dg^\flat\omega^\sharp(df(t)+h(t))\wedge\omega^{n/2-1}\\
&= \eta(t)^*\delta df(t)\\
&= \Delta f(t)\circ\eta(t)
\end{aligned}$$

Therefore

$$\Delta f(0) = (\Delta f(t))\circ\eta(t), \tag{6.6}$$

so $\|\Delta f(t)\|_{C^0}$ is bounded independently of t.

To recover *df* from Δf, we work in coordinate charts $U_i\subset M$. In [M], chapter 5, it is shown that on each such chart U_i there exists a function $k_i(x,y)$ such that

$$df(x) = \sum_i\int_{U_i}k_i(x,y)(\Delta f)(y)dy$$

Furthermore each k_i is smooth off of the diagonal of $U_i\times U_i$ and has a bound of the form

$$k_i(x,y)\le K|x-y|^{1-n}$$

for some constant K. Also the x-derivatives of k obey an inequality

$$\partial_{x_j}k_i(x,y)\le K|x-y|^{-n}$$

From these inequalities we get the following:

Lemma 6.3. *If $\Delta f = q$ where q is a continuous function on M with $||q||_{C^0}\le Q$, then df is quasi-Lipschitz; that is in each chart U, df obeys the inequality*

$$|df(x)-df(y)|\le K|x-y|\log(\frac{1}{|x-y|})$$

if $|x-y|\le 1$. Furthermore K depends only on the bound Q. Since M is covered by finitely many charts, we can pick a single K that will work for all of them.

Proof. See[K], Lemma 1.4. Actually the proof in [K] is given only for $n=2$, but exactly the same argument works for general n. Alternatively see [M], Theorem 2.5.1. □

Note that $(\Delta f)(t) = (\Delta f)(0)\circ\eta(t)^{-1}$ so the C^0, bound for Δf is uniform in t, so the quasi-Lipschitz bound for *df* is uniform in t as well. Also since h is uniformly bounded in the C^1-norm, we find that v is also quasi-Lipschitz, uniformly in t.

Lemma 6.4. *For v uniformly quasi-lipschitz on $(-T_b,T_e)$ with flow $\eta(t)$, working in charts, we can find positive constants K and α such that $|\eta(t)(x)-\eta(t)(y)|\le K|x-y|^\alpha$ for x,y in the same chart and for $\eta(t)(x),\eta(t)(y)$ in the same chart and for all $t\in(T_b,T_e)$*

Proof. see [K], section 2.5 □

Since v is in H^s with $s > n/2 + 1$, there exists a positive α such that $v(0)$ is in $C^{1+\alpha}$ and thus $\Delta f(0)$ is in C^α. From the above lemma and (6.6) we find that Δf is uniformly bounded in C^α (possibly with smaller α) Thus by standard elliptic estimates we find that f is uniformly bounded in $C^{2+\alpha}$, so df and hence v are bounded in $C^{1+\alpha}$ and therefore in C^1.

But now from our previous estimates, we find that if T_e is finite, $\|v(t)\|_s$ remains bounded as $t \nearrow T_e$ so T_e is not maximal. By the same reasoning we find that T_b cannot be finite either.

7. The C^∞ Case

Since we are generally working in a C^∞ manifold M, it is natural to concern ourselves with C^∞ diffeomorphisms rather than H^s diffeomorphisms. This provides no difficulty because the exponential map on $T\mathcal{D}^s_\omega$ retricts to the same map on $T\mathcal{D}_\omega$. Any $v_0 \in T\mathcal{D}_\omega$ is also in $T\mathcal{D}^s_\omega$ so we can construct its geodesic $\eta(t)$ in $\mathcal{D}^s_\omega$. But for any $\tilde{s} > s$ $v_0 \in T\mathcal{D}^{\tilde{s}}_\omega$, so we get a geodesic $\tilde{\eta}(t)$ in $\mathcal{D}^{\tilde{s}}_\omega$ as well, and it is also a geodesic in $\mathcal{D}^s_\omega$. But then by uniqueness, $\eta(t)$ and $\tilde{\eta}(t)$ must coincide. Thus $\eta(t)$ is in $\mathcal{D}^{\tilde{s}}_\omega$ for any $\tilde{s} > s$ and hence it is in $\mathcal{D}_\omega$.

8. Further questions

In [E-M-P], we showed that in the case of two-dimensional fluid flows (geodesics on the volumorphism group), the exponential map is a non-linear Fredholm map; that is, the tangent map at each point is Fredholm. We suspect that this is the case for symplectomorphisms for M of any dimension. It might be interesting to study the nullspaces of this tangent map to see if they tell anything about the structure of (M, ω).

Another avenue would be the Vlasov equations, whose solutions are similar to our geodesics. Can one solve the Vlasov equations by the methods of this paper or by similar methods?

Finally there is the obvious question: can one extend the results above to the case of a manifold with boundary?

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