Solutions to selected problems of homework 3

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0. (If part) Assume, $\mathbb{C}\backslash D$ is open. We shall show, D contains all it's boundary points. We shall show, $z_0 \in \mathbb{C}\backslash D \Rightarrow z_0$ is not a boundary point of D. Since, $\mathbb{C}\backslash D$ is open, $\exists\epsilon > 0,$ such that$\{z : |z - z_0| < \epsilon\} \subseteq \mathbb{C}\backslash D$.

(Only if part) Suppose D contains all it's boundary points. Let, $z_0 \in \mathbb{C}\backslash D$. We want to find $\epsilon > 0$such that$D(z_0, \epsilon) := \{z : |z-z_0| < \epsilon\} \subseteq \mathbb{C}\backslash D$.We know that, since z_0 is not a bounday point of $D, \exists\epsilon > 0$,such that

$$D(z_0, \epsilon) \cap D = \emptyset \text{ or } D(z_0, \epsilon) \cap \mathbb{C}\backslash D = \emptyset$$

. But $D(z_0, \epsilon) \cap \mathbb{C}\backslash D \neq \emptyset, \forall\epsilon > 0$. We conclude that, for that ϵ, $D(z_0, \epsilon) \subseteq \mathbb{C}\backslash D$.

4. a. $2 + i\frac{\pi}{3}$ **b.** $\frac{1}{3} - i\frac{3\pi}{4}$ **c.** $\frac{-3}{2} + i\pi$

1a. Let $A = \{\frac{i}{n} : n \in \mathbb{N}\}$. Then $\partial A = A \cup \{0\}$

Part (i)

$$A \cup \{0\} \subseteq \partial A$$

Obvious.

Part (ii)

$$\partial A \subseteq A \cup \{0\}$$

Observe that, A is a closed set and hence contains all it's boundary points. A is closed since every limit point is in the set (Show that, a set is closed iff it contains all it's limit points). This is because any sequence in A is essentially a subsequence of $\{\frac{i}{n}\}$.

1b.$\mathbb{C}$. Use the fact that topologically $\mathbb{C}$ is same as $\mathbb{R}^2$, and $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$.

1c.$\{z : Re(z^2) = 0 \text{ or } 1\}$.

1d. Let $D = \{e^{2\pi icn} : n \in \mathbb{N}\}$, where $c \in \mathbb{R}$. It is an interesting result due to Kronecker that D is dense in $\mathbb{T}$, where $\mathbb{T}$ is the unit circle, iff c is not a rational number. Hence $\partial D = \mathbb{T}$ where $c = \sqrt{2}$.

3. We know that a circle has the following equation

$$z\overline{z} + az + \overline{az} + b = 0, \ a \in \mathbb{C}, b \in \mathbb{R}$$

. Substitute, $w = f(z) = \frac{1}{z}$ to get

$$1 + a\overline{w} + \overline{a}w + bw\overline{w} = 0$$

If $b = 0$ then the equation becomes

$$1 + a\overline{w} + \overline{a}w = 0,$$

which is a line. If $b \neq 0$, then use the fact $\overline{b} = b$ to get

$$\frac{1}{b} + \frac{a}{b}\overline{w} + (\overline{\frac{a}{b}})w + w\overline{w} = 0$$

which is a circle.

5a. Let A be a 2x2 compelx matrix defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Denote the map $z \rightsquigarrow \frac{az+b}{cz+d}$ by the notation f_A. Now, f_A is non-constant iff $det(A) \neq 0$ i.e. A is non-singular. So every linear fractional transformation (LFT) written as f_A, where A is a 2x2 non-singular complex matrix, and vice-versa. If f_A and f_B are two LFT's, do the computation to find that, $f_A \circ f_B = f_{AB}$. If $det(A) \neq 0$ and $det(B) \neq 0$ then $det(AB) \neq 0$, and hence the result.

The above observation trivially shows that $(f_A)^{-1} = f_{A^{-1}}$.

Let $f(z) = \frac{az+b}{cz+d}$. If $c = 0$, then the proof is trivial. If $c \neq 0$, define,

$$h_1(z) = \frac{a}{c} + (b - \frac{a}{c}d)z$$

$$h_2(z) = \frac{1}{z}$$

$$h_3(z) = cz + d$$

Then observe that,

$$f = h_1 \circ h_2 \circ h_3$$

.

5b. Follows from Problem 3.

5c. Observe that, if $f(H) = H$,then $f(\partial H) = \partial H$. And $\partial H = \mathbb{R}$. Then for $t \in \mathbb{R}$, we must have,

$$f(t) = \frac{at + b}{ct + d} \in \mathbb{R}$$

. Put $t = 0$ to get that $\frac{b}{d} \in \mathbb{R}$. Since ∞ can be seen a limit of a real sequence, and LFT's are continuous at ∞, we have, $f(\infty) \in \mathbb{R}$. Hence $\frac{a}{c} \in \mathbb{R}$. Put $t = 1$ and the fact that $ad - bc \neq 0$ to get that $\frac{c}{d} \in \mathbb{R}$. Hence $a, b, c, d \in \mathbb{R}$. Now,

$$f(i) = \frac{(a + bd) + i(ad - bc)}{c^2 + d^2}$$

. Hence $ad - bc > 0$. Therefore all such LFT can be represented as f_A where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $det(A) > 0$. It is easy to check that any such matrix would give an LFT which preserves the upper-half plane.