

# HAMILTONIAN $S^1$ -MANIFOLDS ARE UNIRULED

DUSA MCDUFF

ABSTRACT. The main result of this note is that every closed Hamiltonian  $S^1$  manifold is uniruled, i.e. it has a nonzero Gromov–Witten invariant one of whose constraints is a point. The proof uses the Seidel representation of  $\pi_1$  of the Hamiltonian group in the small quantum homology of  $M$  as well as the blow up technique recently introduced by Hu, Li and Ruan. It applies more generally to manifolds that have a loop of Hamiltonian symplectomorphisms with a nondegenerate fixed maximum. Some consequences for Hofer geometry are explored. An appendix discusses the structure of the quantum homology ring of uniruled manifolds.

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## 1. INTRODUCTION

A projective manifold is said to be **projectively uniruled** if there is a holomorphic rational curve through every point. Projective manifolds with this property form an

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important class of varieties in birational geometry since they do not have minimal models but rather give rise to Fano fiber spaces; see for example Kollar [18, Ch IV].

One way to translate this property into the symplectic world is to call a symplectic manifold  $(M, \omega)$  **(symplectically) uniruled** if there is a nonzero genus zero Gromov–Witten invariant of the form  $\langle pt, a_2, \dots, a_k \rangle_{k, \beta}^M$  where  $\beta \neq 0$ . Here  $k \geq 1$ ,  $0 \neq \beta \in H_2(M)$ , and  $a_i \in H_*(M)$ , and we consider the invariant where the  $k$  marked points are allowed to vary freely. Because Gromov–Witten invariants are preserved under symplectic deformation, a symplectically uniruled manifold  $(M, \omega)$  has a  $J$ -holomorphic rational curve through every point for every  $J$  that is tamed by some symplectic form deformation equivalent to  $\omega$ . Further justification for this definition is given in the foundational paper by Hu–Li–Ruan [15]. They show that  $(M, \omega)$  is symplectically uniruled whenever there is *any* nontrivial genus zero Gromov–Witten invariant  $\langle \tau_{i_1} pt, \tau_{i_2} a_2, \dots, \tau_{i_k} a_k \rangle_{k, \beta}^M$  with  $\beta \neq 0$ , where the  $i_j \geq 0$  denote the degrees of the descendent insertions.<sup>1</sup> They also show that the uniruled property is preserved under symplectic blowing up and down, and is therefore preserved by symplectic birational equivalences.

We will call a symplectic manifold  $(M, \omega)$  **strongly uniruled** if there is a nonzero invariant  $\langle pt, a_2, a_3 \rangle_{3, \beta}^M$ . (Since one can always add marked points with insertions given by divisors, this is the same as requiring there be some nonzero invariant with  $k \leq 3$ , but it is slightly different from Lu’s usage of the term in [25, Def 1.14].) Manifolds with this property may be detected by the special structure of their small quantum cohomology rings; see Lemma 2.1.

Kollar and Ruan showed in [18, 34] (see also [15, Thm. 4.2]) that every projectively uniruled manifold is strongly uniruled; i.e. if there is a holomorphic  $\mathbb{P}^1$  through every point there is a nonzero invariant  $\langle pt, a_2, a_3 \rangle_{\beta}^M$ . It is not clear that the same is true in the symplectic category. Two questions are included here. Firstly, there is a question about the behavior of Gromov–Witten invariants: if there is some nonzero  $k$ -point invariant with a point insertion must there be a similar nonzero 3-point invariant? Secondly, there is a more geometric question. Suppose that  $M$  is covered by  $J$ -holomorphic 2-spheres either for one  $\omega$ -tame  $J$  or for a significant class of  $J$ . Must  $M$  then be uniruled? Hamiltonian  $S^1$ -manifolds are a good test case here since, when  $J$  is  $\omega$ -compatible and  $S^1$ -invariant, there is an  $S^1$ -invariant  $J$  sphere through every point (given by the orbit of a gradient flow trajectory of the moment map with respect to the associated metric  $g_J$ .)

In this note we show that every Hamiltonian  $S^1$ -manifold is uniruled. This is obvious when  $n = 1$  and is well known for  $n = 2$  since, by Audin [2] and Karshon [17], the only 4-dimensional Hamiltonian  $S^1$ -manifolds are blow ups of rational or ruled surfaces. Our proof in higher dimensions relies heavily on the approach used by Hu–Li–Ruan [15] to analyse the Gromov–Witten invariants of a blow up.

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<sup>1</sup>Descendent insertions  $\tau_i$  are defined in the discussion following equation (3.2).

The first step is to argue as follows. By Lemma 2.18 we may blow  $M$  up along its maximal and minimal fixed point sets  $F_{\max}, F_{\min}$  until these two are divisors. Then consider the gradient flow of the (normalized) moment map  $K$  with respect to an  $S^1$ -invariant metric  $g_J$  constructed from an  $\omega$ -compatible invariant almost complex structure  $J$ . The  $S^1$ -orbit of any gradient flow line is  $J$ -holomorphic. If  $\alpha$  is the class of the  $S^1$ -orbit of a flow line from  $F_{\max}$  to  $F_{\min}$  then  $c_1(\alpha) = 2$ , and because there is just one of these spheres through a generic point of  $M$  one might naively think that the Gromov–Witten invariant  $\langle pt, F_{\min}, F_{\max} \rangle_{\alpha}^M$  is 1. However, there could be other curves in class  $\alpha$  that cancel this one. Almost the only case in which one can be sure this does not happen is when the  $S^1$  action is semifree, i.e. no point in  $M$  has finite stabilizer: see Proposition 4.3. If the order of the stabilizers is at most 2, then one can also show that  $(M, \omega)$  must be strongly uniruled: see Proposition 4.2. However it is not clear whether  $(M, \omega)$  must be strongly uniruled when the isotropy has higher order.

To deal with the general case, we use the Seidel representation of  $\pi_1(\text{Ham}(M, \omega))$  in the group of multiplicative units of the quantum homology ring  $QH_*(M)$ .<sup>2</sup> To make the argument work, we blow up once more at a point in  $F_{\max}$ , obtaining an  $S^1$ -manifold called  $(\widetilde{M}, \widetilde{\omega})$ . By Proposition 2.14 the Seidel element  $\mathcal{S}(\widetilde{\gamma}) \in QH_{2n}(\widetilde{M})$  of the resulting  $S^1$  action  $\widetilde{\gamma}$  on  $\widetilde{M}$  involves the exceptional divisor  $E$  on  $\widetilde{M}$ . Although we know very little about the structure of  $QH_*(\widetilde{M})$ , the fact  $(M, \omega)$  is not uniruled implies by Proposition 2.5 that the part of  $QH_*(\widetilde{M})$  that does not involve  $E$  forms an ideal. Moreover, the quotient of  $QH_*(\widetilde{M})$  by this ideal has an understandable structure. Using this, we show that the invertibility of  $\mathcal{S}(\widetilde{\gamma})$  implies that certain terms in the inverse element  $\mathcal{S}(\widetilde{\gamma}^{-1})$  cannot vanish. This tells us that certain section invariants<sup>3</sup> of the fibration  $\widetilde{P}' \rightarrow S^2$  defined by the loop  $\widetilde{\gamma}^{-1}$  cannot vanish. The homological constraints here involve  $E$ . The final step is to show that these invariants can be nonzero only if  $(M, \omega)$  is uniruled. Hence the original manifold is as well, by the blow down result of Hu–Li–Ruan [15, Thm 1.1]. Their blowing down argument does not give control on the number of insertions; a 3-point invariant might blow down to an invariant with more insertions. Hence we cannot conclude that  $(M, \omega)$  is strongly uniruled.

This argument does not use the properties of  $F_{\min}$  nor does it use much about the circle action. We do need to assume that the loop  $\gamma = \{\phi_t\}$  has a **fixed maximal submanifold**; i.e. that there is a nonempty (but possibly disconnected) submanifold  $F_{\max}$  such that at each time  $t$  the generating Hamiltonian  $K_t$  for  $\gamma$  takes its maximum on  $F_{\max}$  in the strict sense that if  $x_{\max} \in F_{\max}$  then  $K_t(x) \leq K_t(x_{\max})$  for all  $x \in M$  with equality iff  $x \in F_{\max}$ . Further we need  $\gamma$  to restrict to an  $S^1$  action near  $F_{\max}$ ; i.e. in appropriate coordinates  $(z_1, \dots, z_k)$  normal to  $F_{\max}$  we assume that near  $F_{\max}$  the

<sup>2</sup>Although we describe this here in terms of genus zero Gromov–Witten invariants, another way to think of it is as the transformation induced on Hamiltonian Floer homology by continuation along noncontractible Hamiltonian loops.

<sup>3</sup>Suppose that  $P \rightarrow S^2$  is a bundle with fiber  $(M, \omega)$  with Hamiltonian structure group. Then the fiberwise symplectic form  $\omega$  extends to a symplectic form  $\Omega$  on  $P$ . Gromov–Witten invariants of  $P$  in classes  $\beta \in H_2(P; \mathbb{Z})$  that project to the positive generator of  $H_2(S^2; \mathbb{Z})$  are called *section invariants*, while those whose class lies in the image of  $H_2(M; \mathbb{Z})$  are called *fiber invariants*.

generating Hamiltonian  $K_t$  has the form  $\text{const} - \sum m_i |z_i^2|$  for some positive integers  $m_i$ . In these circumstances, we shall say that  $\gamma$  is a **circle action near its maximum**. Note that this action is effective (i.e. no element other than the identity acts trivially) iff  $\gcd(m_1, \dots, m_k) = 1$ .

Our main result is the following.

**Theorem 1.1.** *Suppose that  $\text{Ham}(M)$  contains a loop  $\gamma$  with a fixed maximum near which  $\gamma$  is an effective circle action. Then  $(M, \omega)$  is uniruled.*

**Corollary 1.2.** *Every Hamiltonian  $S^1$ -manifold is uniruled.*

**Remark 1.3.** Since our proofs involve the blow up of  $M$  they work only when  $n := \frac{1}{2} \dim M \geq 2$ . However Theorem 1.1 is elementary when  $n = 1$ . For then, if  $M \neq S^2$ , the group  $\text{Ham}M$  is contractible, as is each component of the group  $G$  of elements in  $\text{Ham}M$  that fix  $x_0$ . Therefore the homomorphism  $\pi_1(G) \rightarrow \pi_1(\text{Sp}(2, \mathbb{R}))$  given by taking the derivative at  $x_0$  is trivial. This implies that there is no loop  $\gamma$  in  $\text{Ham}M$  that is a nonconstant circle action near  $x_0$ .

In fact, in this case there is also no nonconstant loop that is the identity near a fixed maximum. For  $\omega$  is exact on  $M \setminus \{x_0\}$  so that the Calabi homomorphism  $\text{Cal} : \text{Ham}^c(M \setminus \{x_0\}) \rightarrow \mathbb{R}$  is well defined on the group  $\text{Ham}^c$  of compactly supported Hamiltonian symplectomorphisms of  $M \setminus \{x_0\}$ ; cf. [31, Ch. 10.3]. In particular

$$\text{Cal}(\phi) := \int_0^1 \left( \int_M (H_t(x) - H_t(x_0)) \omega \right) dt,$$

is independent of the choice of path  $\phi_t^H$  in  $\text{Ham}^c$  with time 1 map  $\phi$ . But if  $\gamma$  were a nonconstant loop in  $\text{Ham}^c$  with fixed maximum at  $x_0$  we could arrive at  $\phi = \text{id}$  by a path for which this integral is negative.

If all we know is that  $\gamma$  has a fixed maximum  $F_{\max}$  then for each  $x \in F_{\max}$  the linearized flow  $A_t(x)$  in the  $2k$ -dimensional normal space  $N_x^F := T_x M / T_x(F_{\max})$  is generated by a family of nonpositive quadratic forms. We shall say that  $F_{\max}$  is **nondegenerate** if these quadratic forms are everywhere negative definite. In this case, the linearized flow is a so-called positive loop in the symplectic group  $\text{Sp}(2k; \mathbb{R})$ , i.e. a loop generated by a family of negative definite quadratic forms; cf. Lalonde–McDuff [20]. If  $2k \leq 4$  Slimowitz [37, Thm 4.1] shows that any two positive loops that are homotopic in  $\text{Sp}(2k; \mathbb{R})$  are in fact homotopic through a family of positive loops. (In principle this should hold in all dimensions, but the details have been worked out only in low dimensions.) By Lemma 2.22 below, in the 4-dimensional case one can then homotop  $\gamma$  so that it is an effective local circle action near its maximum. Thus we find:

**Proposition 1.4.** *Suppose that  $\dim M \leq 4$  and the loop  $\gamma$  has a nondegenerate maximum at  $x_{\max}$ . Then  $\gamma$  can be homotoped so that it is a nonconstant circle action near its maximum, that when  $\dim M = 4$  can be assumed effective. Thus  $(M, \omega)$  is uniruled.*

Slimowitz's result also implies that every  $S^1$  action on  $\mathbb{C}^k$  with strictly positive weights  $(m_1, \dots, m_k)$  is homotopic through positive loops to an action with positive weights  $(m'_1, m'_2, m_3, \dots, m_k)$  where  $m'_1, m'_2$  are mutually prime. (Here we do not change

the action on the last  $(k - 2)$  coordinates.) Since actions are effective iff their weights at any fixed point are mutually prime we deduce:

**Corollary 1.5.** *Theorem 1.1 holds for any loop that is a circle action near its maximum.*

These results have implications for Hofer geometry. It is so far unknown which elements in  $\pi_1(\text{Ham}(M))$  can be represented by loops that minimize the Hofer length. By Lalonde–McDuff [19, Prop. 2.1] such a loop has to have a fixed maximum and minimum  $x_{\max}, x_{\min}$ , i.e. points such that the generating Hamiltonian  $K_t$  satisfies the inequalities  $K_t(x_{\min}) \leq K_t(x) \leq K_t(x_{\max})$  for all  $x \in M$  and  $t \in [0, 1]$ . However, these extrema could be degenerate and need not form the same manifold for all  $t$ .

If fixed extrema exist, then the  $S^1$ -orbit of a path from  $x_{\max}$  to  $x_{\min}$  forms a 2-sphere, and it is easy to check that the integral of  $\omega$  over this 2-sphere is just  $\|K\| := \int (K_t(x_{\max}) - K_t(x_{\min})) dt$ ; cf. [32, Ex.9.1.11]. Hence such a loop cannot exist on a symplectically aspherical manifold  $(M, \omega)$ . However, so far no examples are known of symplectically aspherical manifolds with nonzero  $\pi_1(\text{Ham}(M))$ .

It is easy to construct nonzero elements in  $\pi_1(\text{Ham}(M))$  for manifolds that are not uniruled. For example, by [29, Prop 1.10] one could take  $M$  to be the two point blow up of any symplectic 4-manifold. Hence, if one could better understand the situation when the fixed points are degenerate, it might be possible to exhibit manifolds for which  $\pi_1(\text{Ham})$  is nontrivial but where no nonzero element in this group has a length minimizing representative. This question is the subject of ongoing research.

**Remark 1.6.** (i) It is not clear whether a Hamiltonian  $S^1$ -manifold  $(M, \omega)$  always has a blow up that is strongly uniruled. However, as we show in Proposition A.4 there is not much difference between the uniruled and the strongly uniruled conditions. Moreover by Corollary 4.5 they coincide in the case when  $H^*(M; \mathbb{Q})$  is generated by  $H^2(M; \mathbb{Q})$ .

(ii) It is essential in Corollary 1.2 that the loop is Hamiltonian; it is not enough that it has fixed points. For example, the semifree symplectic circle action constructed in [26] has fixed points but these all have index and coindex equal to 2. Hence the first Chern class  $c_1$  vanishes on  $\pi_2(M)$  and  $M$  cannot be uniruled for dimensional reasons.

(iii) One might wonder why we work with the blow up  $\widetilde{M}$  rather than with  $M$  itself. One answer is seen in results such as Lemma 2.8 and Propositions 2.14 and 2.19. These make clear that the exceptional divisor in  $\widetilde{M}$  forms a visible marker that allows us to show that certain Gromov–Witten invariants of  $M$  do not vanish. The point is that even though we cannot calculate  $QH_*(\widetilde{M})$  in general, the results of §2.1 show that we can calculate the quotient  $QH_*(\widetilde{M})/\mathcal{I}$  provided that  $(M, \omega)$  is *not* uniruled.

As always, the problem is that it is very hard to calculate Gromov–Witten invariants for a general symplectic manifold; if one has no global information about the manifold, the contribution to an invariant of a known  $J$ -holomorphic curve could very well be cancelled by some other unseen curve. The Seidel representation is one of the very few general tools that can be used to show that certain invariants of an arbitrary symplectic manifold cannot vanish and hence it has found several interesting applications; cf.

Seidel [36] and Lalonde–McDuff [21]. It is made by counting section invariants of Hamiltonian bundles  $P$  with fiber  $M$  and base  $S^2$ . It is fairly clear that the existence of fiber invariants of the blow up bundle  $\tilde{P}$  that involve the exceptional divisor  $E$  should imply that  $(M, \omega)$  is uniruled. A crucial step in our argument is Lemma 2.20 which states that the existence of certain section invariants of  $P$  also implies that  $(M, \omega)$  is uniruled.

Another way of getting global information on Gromov–Witten invariants is to use symplectic field theory. This approach was recently taken by J. He [9] who shows that subcritical manifolds are strongly uniruled, thus proving a conjecture of Biran and Cieliebak in [3, §9]. This case is somewhat different than the one considered here; for example  $\mathbb{P}^n$  is subcritical while  $\mathbb{P}^1 \times \mathbb{P}^1$  is not.

The main proofs are given in §2. They use some properties of absolute and relative Gromov–Witten invariants established in §3. Conditions that imply  $(M, \omega)$  is strongly uniruled are discussed in §4. The appendix treats general questions about the structure of the quantum homology ring.

**Remark 1.7** (Technical underpinnings of the proof). The main technical tool used in this paper is the decomposition rule for relative genus zero Gromov–Witten invariants. Since we use this with descendent absolute insertions, it is useful to have this in a strong form in which the virtual moduli cycles in question are at least  $C^1$ . To date the most relevant references for this result are Li–Ruan [23, Thm 5.7] and Hu–Li–Ruan [15, §3]. However, relative Gromov–Witten theory is a part of the symplectic field theory package (cf. [6, 4]). Therefore the work of Hofer–Wysocki–Zehnder [10, 11, 12, 13] on polyfolds should eventually lead to an alternative proof. In §3 we explain the relative moduli spaces, but our proofs assume the basic results about them. Note that we cannot get anywhere by restricting to the semi-positive case (a favorite way of avoiding virtual cycles) since we need to consider almost complex structures on  $M$  that are  $S^1$ -invariant and these are almost never regular unless the action is semifree.

Another important ingredient of the proof is the Seidel representation of  $\pi_1$  of the Hamiltonian group  $\text{Ham}(M, \omega)$ , which is proved for general symplectic manifolds in [27, §1]. Finally, we use the identities (3.9) and (3.10) for genus zero Gromov–Witten invariants. The former is fairly well known and is not hard to deduce from the corresponding result for genus zero stable curves, while the latter was first proved by Lee–Pandharipande [22, Thm 1] in the algebraic case. For Theorem 1.1 we only need a very special case of this identity whose proof we explain in Lemma 3.14. The full force of (3.10) is used to prove Proposition 4.4, but this result is not central to the main argument.

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## 2. THE MAIN ARGUMENT

Because the main argument is somewhat involved, we shall begin by outlining it. In §2.1 we explain the structure of the quantum homology ring  $QH_*(\widetilde{M})$  of the one point blow up of a manifold  $(M, \omega)$  that is not uniruled. By Proposition 2.5 this condition on  $(M, \omega)$  implies that the subspace of  $QH_*(\widetilde{M})$  spanned by the homology classes in  $H_*(M \setminus pt)$  forms an ideal  $\mathcal{I}$ . Moreover, the quotient  $\mathcal{R} := QH_*(\widetilde{M})/\mathcal{I}$  decomposes as the sum of two fields (Lemma 2.7), and we can work out an explicit formula for the inverse  $u^{-1}$  of any unit  $u$  in  $\mathcal{R}$ .

The unit in question is the Seidel element  $\mathcal{S}(\widetilde{\gamma}) \in QH_*(\widetilde{M})$  of the Hamiltonian loop  $\widetilde{\gamma}$  on  $\widetilde{M}$ . Usually it is very hard to calculate  $\mathcal{S}(\widetilde{\gamma})$ . However, when the maximum fixed point set  $\widetilde{F}_{\max}$  of  $\widetilde{\gamma}$  is a divisor, one can calculate its leading term (Corollary 2.12). Moreover, because the final blow up is at a point of  $F_{\max}$  we show in Proposition 2.14 that this leading term has nontrivial image in  $\mathcal{R}$ . We next use the explicit formula for  $u^{-1} \in \mathcal{R}$  to deduce the nonvanishing of certain terms in the Seidel element  $\mathcal{S}(\widetilde{\gamma}')$  for the inverse loop  $\widetilde{\gamma}' := \widetilde{\gamma}^{-1}$ . But  $\mathcal{S}(\widetilde{\gamma}')$  is calculated from the Gromov–Witten invariants of the fibration  $\widetilde{P}' \rightarrow \mathbb{P}^1$  determined by  $\widetilde{\gamma}'$ . Hence we deduce in Corollary 2.15 that some of these invariants do not vanish.

Finally we show that these particular invariants can be nonzero only if  $M$  is uniruled, so that our initial assumption that  $(M, \omega)$  is not uniruled is untenable. The proof here involves two steps. First, by blowing down the bundle  $\widetilde{P}'$  to  $P'$ , the bundle determined by  $\gamma' := \gamma^{-1}$ , we deduce the nonvanishing of certain Gromov–Witten invariants in  $P'$  (Proposition 2.19). Second, we show in Lemma 2.20 that these invariants can be nonzero only if  $(M, \omega)$  is uniruled.

In this section we prove essentially all results that do not involve comparing the Gromov–Witten invariants of a manifold and its blow up. (Those proofs are deferred to §3.) We assume throughout that  $(M, \omega)$  is a closed symplectic manifold of dimension  $2n$  where  $n \geq 2$ .

**2.1. Uniruled manifolds and their pointwise blowups.** Consider the small quantum homology  $QM_*(M) := H_*(M) \otimes \Lambda$  of  $M$ . Here we use the Novikov ring  $\Lambda := \Lambda_\omega[q, q^{-1}]$  where  $q$  is a polynomial variable of degree 2 and  $\Lambda_\omega$  denotes the generalized Laurent series ring with elements  $\sum_{i \geq 1} r_i t^{-\kappa_i}$ , where  $r_i \in \mathbb{Q}$  and  $\kappa_i$  is a strictly increasing sequence that tends to  $\infty$  and lies in the period subgroup  $\mathcal{P}_\omega$  of  $[\omega]$ , i.e. the image of the homomorphism  $I_\omega : \pi_2(M) \rightarrow \mathbb{R}$  given by integrating  $\omega$ . We assume that  $[\omega]$  is chosen so that  $I_\omega$  is injective. We write the elements of  $QH_*(M)$  as infinite sums  $\sum_{i \geq 1} a_i \otimes q^{d_i} t^{-\kappa_i}$ , where  $a_i \in H_*(M; \mathbb{Q}) =: H_*(M)$ , the  $|d_i|$  are bounded and  $\kappa_i$  is as before. The term  $a \otimes q^{d} t^{\kappa}$  has degree  $2d + \deg a$ .

The quantum product  $a * b$  of the elements  $a, b \in H_*(M) \subset QH_*(M)$  is defined as follows. Let  $\xi_i, i \in I$ , be a basis for  $H_*(M)$  and write  $\xi_i^*, i \in I$ , for the basis of  $H_*(M)$  that is dual with respect to the intersection pairing, that is  $\xi_j^* \cdot \xi_i = \delta_{ij}$ . Then

$$(2.1) \quad a * b := \sum_{i, \beta \in H_2(M; \mathbb{Z})} \langle a, b, \xi_i \rangle_{\beta}^M \xi_i^* \otimes q^{-c_1(\beta)} t^{-\omega(\beta)},$$

where  $\langle a, b, \xi_i \rangle_{\beta}^M$  denotes the Gromov–Witten invariant in  $M$  that counts curves in class  $\beta$  through the homological constraints  $a, b, \xi_i$ . Note that if  $(a * b)_{\beta} := \sum_i \langle a, b, \xi_i \rangle_{\beta}^M \xi_i^*$ , then  $(a * b)_{\beta} \cdot c = \langle a, b, c \rangle_{\beta}^M$ . Further,  $\deg(a * b) = \deg a + \deg b - 2n$ , and the identity element is  $\mathbb{1} := [M]$ .

**Lemma 2.1.** *The following conditions are equivalent:*

- (i)  $(M, \omega)$  is not strongly uniruled;
- (ii)  $\mathcal{Q}_- := \bigoplus_{i < 2n} H_i(M) \otimes \Lambda$  is an ideal in the small quantum homology  $QH_*(M)$ ;
- (iii)  $pt * a = 0$  for all  $a \in \mathcal{Q}_-$ , where  $*$  is the quantum product.

Moreover, if these conditions hold every unit  $u \in QH_*(M)^{\times}$  in  $QH_*(M)$  has the form  $u = \mathbb{1} \otimes \lambda + x$ , where  $x \in \mathcal{Q}_-$  and  $\lambda$  is a unit in  $\Lambda$ .

*Proof.* Choose the basis  $\xi_i$  so that  $\xi_0 = pt$ ,  $\xi_M = [M] =: \mathbb{1}$  and all other  $\xi_i$  lie in  $H_j(M)$  for  $1 \leq j < 2n$ . Then  $\xi_0^* = \xi_M$ , and the coefficient of  $\mathbb{1} \otimes q^{-d} t^{-\kappa}$  in  $a * b$  is

$$\sum_{\beta: \omega(\beta) = \kappa, c_1(\beta) = d} \langle a, b, pt \rangle_{\beta}.$$

Since we assume that  $I_{\omega} : \pi_2(M) \rightarrow \Gamma_{\omega}$  is an isomorphism and  $\kappa \in \Gamma_{\omega}$ , there is precisely one class  $\beta$  such that  $\omega(\beta) = \kappa$ . If  $\beta = 0$  this invariant is the usual intersection product  $a \cap b \cap pt$  and so always vanishes for  $a, b \in H_{< 2n}(M)$ . Therefore, this coefficient is nonzero for some such  $a, b$  iff  $(M, \omega)$  is strongly uniruled. But this coefficient is nonzero iff  $\mathcal{Q}_-$  is not a subring. Note finally that since  $\mathcal{Q}_-$  has codimension 1, it is an ideal iff it is a subring.

This proves the equivalence of (i) and (ii). The other statements are proved by similar arguments.  $\square$

**Remark 2.2.** It seems very likely that  $(M, \omega)$  is strongly uniruled iff  $\mathcal{Q}_-$  contains no units. We prove this in the appendix, as well as a generalization to uniruled manifolds, in the case when the odd Betti numbers of  $M$  vanish.

Now let  $(\widetilde{M}, \widetilde{\omega})$  be the 1-point blow up of  $M$  with exceptional divisor  $E$ . Put  $\varepsilon := E^{n-1}$ , the class of a line in  $E$ . The blow up parameter is  $\delta := \widetilde{\omega}(\varepsilon)$ , which we assume to be  $\mathbb{Q}$ -linearly independent from  $\Gamma_{\omega}$ . Later it will be convenient to write  $\Lambda_{\widetilde{\omega}} =: \Lambda_{\omega, \delta}$ .

**Lemma 2.3.** *Invariants of the form*

$$\langle E^i, E^j, E^k \rangle_{p\varepsilon}^{\widetilde{M}}, \quad 0 < i, j, k < n, p > 0$$

are nonzero only if  $p = 1$  and  $i + j + k = 2n - 1$ , and in this case the invariant is  $-1$ .

*Proof.* Since  $c_1(\varepsilon) = n - 1$  the invariant  $\langle E^i, E^j, E^k \rangle_{p\varepsilon}^{\widetilde{M}}$  is nonzero only if  $n + p(n - 1) = i + j + k$ . Since  $i + j + k \leq 3n - 3$  we must have  $p = 1$ . The result now follows from the fact that if  $J$  is standard near  $E$  then the only  $\varepsilon$  curves are lines in  $E$ .  $\square$

**Corollary 2.4.** *Let  $0 \leq i, j < n$ . Then*

$$\begin{aligned} E^i * E^j &= E^i \cap E^j = E^{i+j} && \text{if } i + j < n, \\ E^i * E^{n-i} &= -pt + E \otimes q^{-n+1}t^{-\delta}, && \text{and} \\ E^i * E^j &= E^{i+j-n+1} \otimes q^{-n+1}t^{-\delta} && \text{if } n < i + j < 2n - 1. \end{aligned}$$

We prove the next result in §3 by the technique of Hu–Li–Ruan [15].

**Proposition 2.5.** *Let  $(\widetilde{M}, \widetilde{\omega})$  be the one point blow up of  $M$  with exceptional divisor  $E$ , and let  $a, b \in H_{<2n}(M)$ . If any invariant of the form*

$$\langle a, b, E^k \rangle_{\widetilde{\beta}}^{\widetilde{M}}, \quad \langle a, E^i, E^j \rangle_{\widetilde{\beta}}^{\widetilde{M}}, \quad \langle E^i, E^j, E^k \rangle_{\widetilde{\beta}}^{\widetilde{M}}, \quad \text{for } 0 \neq \widetilde{\beta} \neq p\varepsilon, \quad i, j, k \geq 1$$

*is nonzero, then  $(M, \omega)$  is uniruled.*

Decompose  $QH_*(\widetilde{M})$  additively as  $\mathcal{I} \oplus \mathcal{E}$  where  $\mathcal{I}$  is generated as a  $\Lambda_{\widetilde{\omega}}$ -module by the image of  $H_{<2n}(M)$  in  $H_*(\widetilde{M})$  and  $\mathcal{E}$  is spanned by  $\mathbb{1}, E, \dots, E^{n-1} = \varepsilon$ . Proposition 2.5 implies:

**Corollary 2.6.** *If  $(M, \omega)$  is not uniruled,  $\mathcal{I}$  is an ideal in  $QH_*(\widetilde{M})$ .*

*Proof.* Let  $a, b \in H_{<2n}(M)$ . Then  $a * b \in \mathcal{I}$  iff  $(a * b)_{\widetilde{\beta}} \cdot E^k = \langle a, b, E^k \rangle_{\widetilde{\beta}}^{\widetilde{M}} = 0$  for all  $\widetilde{\beta} \in H_2(\widetilde{M})$  and all  $k \geq 1$ . Therefore the hypotheses imply that  $\mathcal{I}$  is a subring. It is an ideal because  $a * E^k = 0$  for all  $k \geq 1$ .  $\square$

Let  $\mathcal{E}_{2n}$  denote the degree 2n part of  $\mathcal{E}$ . This is not a subring of  $QH_*(\widetilde{M})$  since for example

$$(E \otimes q) * (\varepsilon \otimes q^{n-1}) = -pt \otimes q^n + E \otimes qt^{-\delta}.$$

Nevertheless if  $(M, \omega)$  is not uniruled,  $\mathcal{E}_{2n}$  can be given the ring structure of the quotient  $QH_{2n}(\widetilde{M})/\mathcal{I}$ . By Corollary 2.4 and Proposition 2.5, there is a ring isomorphism

$$\mathcal{E}_{2n} \rightarrow \Lambda_{\omega, \delta}[s]/(s^n = st^{-\delta}) \quad \text{given by } E \otimes q \mapsto s.$$

Composing with the quotient map  $QH_{2n}(\widetilde{M}) \rightarrow \mathcal{E}_{2n} \cong QH_{2n}(\widetilde{M})/\mathcal{I}$  we get a homomorphism

$$\Phi_E : QH_{2n}(\widetilde{M}) \rightarrow \mathcal{R} := \Lambda_{\omega, \delta}[s]/(s^n = st^{-\delta}).$$

**Lemma 2.7.** *The ring  $\mathcal{R}$  decomposes as the direct sum of two fields  $\mathcal{R}_1 \oplus \mathcal{R}_2$ .*

*Proof.* Denote  $e_1 := 1 - s^{n-1}t^\delta$  and  $e_2 := s^{n-1}t^\delta$ . Because  $se_1 = 0, se_2 = s$  we find  $e_i^2 = e_i$  and  $e_1e_2 = 0$ . Moreover  $\mathcal{R}_1 := e_1\mathcal{R}$  is isomorphic to the field  $e_1\Lambda_{\omega, \delta}$ . To see that  $\mathcal{R}_2 := e_2\mathcal{R}$  is a field, consider the homomorphism

$$F : \mathcal{R} \rightarrow \Lambda_{\omega, \delta/(n-1)} \quad \text{given by } s \mapsto t^{-\delta/(n-1)}, t \mapsto t.$$

Then  $F(e_2) = 1$  and the kernel of  $F$  is  $e_1\mathcal{R}$ . Hence  $F$  gives an isomorphism between  $e_2\mathcal{R}$  and the field  $\Lambda_{\omega,\delta/(n-1)}$ .  $\square$

Denote

$$\mathcal{X} := \left\{ x \in \mathcal{R} : x = \sum_{i \geq 0} r_i s^{d_i} t^{-\kappa_i}, r_i \in \mathbb{Q}, 0 \leq d_i \leq n, \kappa_i > 0 \right\}.$$

Then for all  $x \in \mathcal{X}$  the element  $1 + x$  is invertible with inverse  $1 - x + x^2 - \dots$ .

**Lemma 2.8.** *Let  $u \in \mathcal{R}$  be a unit of the form  $1 + rst^{\kappa_0}(1 + x)$  where  $x \in \mathcal{X}$ ,  $r \neq 0$  and  $\kappa_0 \in \Lambda_{\omega}$  is  $> \delta$ .*

(i) *If  $n \geq 3$  then*

$$u^{-1} = 1 - s^{n-1}t^{\delta} + \frac{1}{r}s^{n-2}t^{\delta-\kappa_0}(1 + y), \quad \text{for some } y \in \mathcal{X}.$$

(ii) *If  $n = 2$  then  $u^{-1} = 1 - st^{\delta} + \frac{1}{r}st^{2\delta-\kappa_0}(1 + y)$  for some  $y \in \mathcal{X}$ .*

*Proof.* Since  $e_2u = e_2(e_2 + rst^{\kappa_0}(1 + x))$  we may write

$$\begin{aligned} u = e_1u + e_2u &= e_1 + e_2 \left( s^{n-1}t^{\delta} + rst^{\kappa_0}(1 + x) \right) \\ &= e_1 + e_2rst^{\kappa_0}(1 + x') \quad \text{where } x' := x + \frac{1}{r}s^{n-2}t^{\delta-\kappa_0}. \end{aligned}$$

If  $n > 2$  define  $v := e_1 + \frac{1}{r}e_2s^{n-2}t^{\delta-\kappa_0}(1 + x')^{-1}$ . Then  $e_1uv = e_1$  while  $e_2uv = e_2s^{n-1}t^{\delta} = e_2$ . Hence  $u^{-1} = v$ . Since  $e_2s^{n-2} = s^{n-2}$  when  $n > 2$ , this has the form required in (i). When  $n = 2$  we take  $v = e_1 + \frac{1}{r}e_2st^{2\delta-\kappa_0}(1 + x')^{-1}$ .  $\square$

**Remark 2.9.** Here we restricted the coefficients of  $QH_*(M)$  to  $\Lambda_{\omega}[q, q^{-1}]$  to make the structure of  $\mathcal{R}$  as simple as possible. However, in order to define the Seidel representation we will need to use the larger coefficient ring  $\Lambda^{\text{univ}}[q, q^{-1}]$  where  $\Lambda^{\text{univ}}$  consists of all formal series  $\sum_{i \geq 1} r_i t^{-\kappa_i}$ , where  $\kappa_i \in \mathbb{R}$  is any increasing sequence that tends to  $\infty$ . Since  $\mathcal{R}$  injects into  $\mathcal{R}' := \Lambda^{\text{univ}}[s]/(s^n = st^{-\delta})$  the statement in Lemma 2.8 remains valid when we think of  $u$  as an element of  $\mathcal{R}'$ .

**2.2. The Seidel representation.** This is a homomorphism  $\mathcal{S}$  from  $\pi_1(\text{Ham}(M, \omega))$  to the degree  $2n$  multiplicative units  $QH_{2n}(M)^{\times}$  of the small quantum homology ring first considered by Seidel in [35]. To define it, observe that each loop  $\gamma = \{\phi_t\}$  in  $\text{Ham}(M)$  gives rise to an  $M$ -bundle  $P_{\gamma} \rightarrow \mathbb{P}^1$  defined by the clutching function  $\gamma$ :

$$P_{\gamma} := M \times D_+ \cup M \times D_- / \sim \quad \text{where } (\phi_t(x), e^{2\pi it})_+ \sim (x, e^{2\pi it})_-.$$

Because the loop  $\gamma$  is Hamiltonian, the fiberwise symplectic form  $\omega$  extends to a closed form  $\Omega$  on  $P$ , that we can arrange to be symplectic by adding to it the pullback of a suitable form on the base  $\mathbb{P}^1$ ; see the proof of Proposition 2.11 below.

In the case of a circle action with normalized moment map  $K : M \rightarrow \mathbb{R}$  we may simply take  $(P_{\gamma}, \Omega)$  to be the quotient  $(M \times_{S^1} S^3, \Omega_c)$ , where  $S^1$  acts diagonally on  $S^3$  and  $\Omega_c$  pulls back to  $\omega + d((c - K)\alpha)$ . Here  $\alpha$  is the standard contact form on  $S^3$  normalized so that it descends to an area form on  $S^2$  with total area 1, and  $c$  is any

constant larger than the maximum  $K_{\max}$  of  $K$ . Points  $x_{\max}, x_{\min}$  in the fixed point sets  $F_{\max}$  and  $F_{\min}$  give rise to sections  $s_{\max} := x_{\max} \times \mathbb{P}^1$  and  $s_{\min} := x_{\min} \times \mathbb{P}^1$ . Note that our orientation conventions are chosen so that the integral of  $\Omega$  over the section  $s_{\min}$  is *larger* than that over  $s_{\max}$ . For example, if  $M = S^2$  and  $\gamma$  is a full rotation,  $P_\gamma$  can be identified with the one point blow up of  $\mathbb{P}^2$ , and  $s_{\max}$  is the exceptional divisor. In the following we denote particular sections as  $s_{\max}$  or  $s_{\min}$ , while writing  $\sigma_{\max}, \sigma_{\min}$  for the homology classes they represent.

The bundle  $P_\gamma \rightarrow \mathbb{P}^1$  carries two canonical cohomology classes, the first Chern class  $c_1^{\text{Vert}}$  of the vertical tangent bundle and the coupling class  $u_\gamma$ , the unique class that extends the fiberwise symplectic class  $[\omega]$  and is such that  $u_\gamma^{n+1} = 0$ . Then

$$(2.2) \quad \mathcal{S}(\gamma) := \sum_{\sigma} a_{\sigma} \otimes q^{-c_1^{\text{Vert}}(\sigma)} t^{-u_{\gamma}(\sigma)},$$

where  $\sigma \in H_2(P; \mathbb{Z})$  runs over all section classes (i.e. classes that cover the positive generator of  $H_2(\mathbb{P}^1; \mathbb{Z})$ ) and  $a_{\sigma} \in H_*(M)$  is defined by the requirement that

$$(2.3) \quad a_{\sigma} \cdot_M c = \langle c \rangle_{\sigma}^P, \quad \text{for all } c \in H_*(M).$$

(Cf. [32, Def. 11.4.1]. For this to make sense we must use  $\Lambda^{\text{univ}}$  instead of  $\Lambda_{\omega}$  as explained in Remark 2.9. Note that we write  $\cdot_M$  for the intersection product in  $M$ .) Further for all  $b, c \in H_*(M)$

$$(2.4) \quad \mathcal{S}(\gamma) * b = \sum_{\sigma} b_{\sigma} \otimes q^{-c_1^{\text{Vert}}(\sigma)} t^{-u_{\gamma}(\sigma)}, \quad \text{where } b_{\sigma} \cdot_M c := \langle b, c \rangle_{\sigma}^P.$$

**Remark 2.10.** Lemma 2.1 shows that if  $(M, \omega)$  is not strongly uniruled then  $\mathcal{S}(\gamma) = \mathbb{1} \otimes \lambda + x$  where  $x \in \mathcal{Q}_-(M)$ . Therefore  $\mathcal{S}(\gamma) * pt = pt \otimes \lambda$ , which in turn means that all 2-point invariants  $\langle pt, c \rangle_{\sigma}^P$  with  $c \in H_{<2n}(M)$  must vanish. In other words, a section invariant in  $P_\gamma$  with more than a single point constraint must vanish. We shall expand on this theme later in Lemmas 2.20 and 2.21.

In general it is very hard to calculate  $\mathcal{S}(\gamma)$ . The following result is essentially due to Seidel [35, §11]; see also McDuff–Tolman [Thm 1.10][33]. We shall say that  $K_t$  is a normalized generating Hamiltonian for  $\gamma = \{\phi_t\}$  iff

$$\omega(\dot{\phi}_t, \cdot) = -dK_t, \quad \text{and} \quad \int_M K_t \omega^n = 0.$$

Further we define

$$(2.5) \quad K_{\max} := \int_0^1 \max_{x \in M} K_t(x) dt.$$

**Proposition 2.11.** *Suppose that  $\gamma$  has a nonempty maximal submanifold  $F_{\max}$  and is generated by the normalized Hamiltonian  $K_t$ . Then*

$$\mathcal{S}(\gamma) := a_{\max} \otimes q^{m_{\max}} t^{K_{\max}} + \sum_{\beta \in H_2(M; \mathbb{Z}), \omega(\beta) > 0} a_{\beta} \otimes q^{m_{\max} - c_1(\beta)} t^{K_{\max} - \omega(\beta)},$$

where  $m_{\max} := -c_1^{\text{Vert}}(\sigma_{\max})$ . Moreover  $a_{\max} \cdot_M c = \langle c \rangle_{\sigma_{\max}}^P$ .

*Proof.* Because  $\gamma$  is assumed to have a fixed maximum, we may write the section classes  $\sigma$  appearing in  $\mathcal{S}(\gamma)$  as  $\sigma_{\max} + \beta$  where  $\beta \in H_2(M)$ , and then denote  $a_\beta := a_\sigma$ . Therefore by equation (2.2) the result will follow if we show that:

- (a) *the only class with  $\omega(\beta) \leq 0$  that contributes to the sum is the class  $\beta = 0$ , and*
- (b)  $-u_\gamma(\sigma_{\max}) = K_{\max}$ .

Consider the renormalized polar coordinates  $(r, t)$  on the unit disc  $D^2$ , where  $t := \theta/2\pi$ . Define  $\Omega_- := \omega + \varepsilon d(r^2) \wedge dt$  on  $M \times D_-$  for some small constant  $\varepsilon > 0$ , and set

$$\Omega_+ := \omega + \left( \kappa(r^2, t)d(r^2) - d(\rho(r^2)K_t) \right) \wedge dt, \quad \text{on } M \times D_+,$$

where  $\rho(r^2)$  is a nondecreasing function that equals 0 near 0 and 1 near 1. If  $\kappa(r^2, t) = \varepsilon$  near  $r = 1$ , these two forms fit together to give a closed form  $\Omega$  on  $P$ . Moreover,  $\Omega$  is symplectic iff  $\kappa(r^2, t) - \rho'(r^2)K_t(x) > 0$  for all  $r, t$  and  $x \in M$ . Hence we may suppose that

$$(2.6) \quad \nu := \int_{s_{\max}} \Omega = \pi\varepsilon + \int_{D_+} (\kappa(r^2, t) - \rho'(r^2) \max_{x \in M} K_t(x)) 2r dr dt$$

is as close to zero as we like.

A class  $\beta$  contributes to  $\mathcal{S}(\gamma)$  iff some invariant  $\langle c \rangle_{\sigma_{\max} + \beta}^P \neq 0$ . In particular, we must have  $\int_{\sigma_{\max} + \beta} \Omega = \nu + \omega(\beta) > 0$  for all symplectic forms on  $P$ . Hence we need  $\omega(\beta) \geq 0$ .

It remains to investigate the case  $\omega(\beta) = 0$ . Note that  $\Omega$  defines a connection on  $P \rightarrow \mathbb{P}^1$  whose horizontal spaces are the  $\Omega$ -orthogonals to the vertical tangent spaces. Further, this connection does not depend on the choice of function  $\kappa$ . Choose an  $\Omega$ -compatible almost complex structure  $J$  on  $P$ . We may assume that both the fibers of  $P \rightarrow \mathbb{P}^1$  and the horizontal subspaces are  $J$ -invariant. Thus  $J$  induces a compact family  $J_z, z \in \mathbb{P}^1$ , of  $\omega$ -tame almost complex structures on the fiber  $M$ . Let  $\hbar$  be the minimum of the energies of all nonconstant  $J_z$ -holomorphic spheres in  $M$  for  $z \in \mathbb{P}^1$ . Standard compactness results imply that  $\hbar > 0$ . Note that if we change  $\kappa$  (keeping  $\Omega$  symplectic)  $J$  remains  $\Omega$ -compatible. Hence we may suppose that  $\nu := \int_{s_{\max}} \Omega < \hbar$ .

We now claim that the only  $J$ -holomorphic sections of  $P$  with energy  $\leq \nu$  are the constant sections  $x \times \mathbb{P}^1$  for  $x \in F_{\max}$ . To see this, decompose tangent vectors to  $P$  as  $v + h$  where  $v$  is tangent to the fiber and  $h$  is horizontal. Then, because of our choice of  $J$ , at a point  $(x, z)$  over the fiber at  $z = (r, t) \in D_+$ ,

$$\begin{aligned} \Omega(v + h, Jv + Jh) &= \omega(v, Jv) + \Omega(h, Jh) \geq \Omega(h, Jh) \\ &\geq 2r(\kappa(r^2, t) - \rho'(r^2) \max_{x \in M} K_t(x)) dr \wedge dt(h, Jh). \end{aligned}$$

Here the first inequality is an equality only along a section that is everywhere horizontal, while the second is an equality only if the section is contained in  $F_{\max} \times \mathbb{P}^1$ . (Recall that by assumption  $K_t(x) < K_{\max}$  for  $x \notin F_{\max}$ .) Similarly, the corresponding integral over  $D_-$  is  $\geq \varepsilon$  with equality only if the section is constant. Comparing with equation (2.6) we see that the energy of a section is  $\geq \nu$  with equality only if the section is constant. This proves the claim.

Now suppose that the class  $\beta$  with  $\omega(\beta) = 0$  contributes to  $\mathcal{S}(\gamma)$ . Then the class  $\sigma_{\max} + \beta$  of energy  $\nu$  must be represented by a  $J$ -holomorphic stable map consisting of a section, possibly with some other bubble components in the fibers. Since each bubble has energy  $\geq \hbar > \nu$ , the stable map has no bubbles and hence by the preceding paragraph must consist of a single constant section in the class  $\sigma_{\max}$ . Thus  $\beta = 0$ . This completes the proof of (a).

To prove (b) it suffices to check that the coupling class  $u_\gamma$  on  $P_\gamma$  is represented by the form  $\Omega_0$  that equals  $\omega$  on  $M \times D_-$  and

$$\omega - d(\rho(r^2)K_t) \wedge dt, \quad \text{on } M \times D_+.$$

For this to hold we need that  $\int_P \Omega_0^{n+1} = 0$ . But this follows easily from the fact that  $K_t$  is normalized.  $\square$

**Corollary 2.12.** *Suppose that the fixed maximum  $F_{\max}$  of  $\gamma$  is a divisor and that  $\gamma$  is an effective circle action near  $F_{\max}$ . Then*

$$(2.7) \quad \mathcal{S}(\gamma) = F_{\max} \otimes q t^{K_{\max}} + \sum_{\beta \in H_2(M; \mathbb{Z}), \omega(\beta) > 0} a_\beta \otimes q^{1-c_1(\beta)} t^{K_{\max} - \omega(\beta)}.$$

*Proof.* By Proposition 2.11, it remains to check that the contribution  $a_{\max}$  of the sections in class  $\sigma_{\max}$  to  $\mathcal{S}(\gamma)$  is  $F_{\max}$ . We saw there that the moduli space of (unparametrized) holomorphic sections in class  $\sigma_{\max}$  can be identified with  $F_{\max}$ . In particular, it is compact. Because  $S^1$  acts with normal weight  $-1$ ,  $c_1^{\text{vert}}(\sigma_{\max}) = -m_{\max} = -1$  and one can easily see that these sections are regular. (A proof is given in [33, Lemma 3.2].) Hence  $a_{\max} \cdot a := \langle a \rangle_{\sigma_{\max}}^P = F_{\max} \cdot a$  for all  $a \in H_*(M)$ .  $\square$

If  $(M, \omega)$  is not strongly uniruled then, by Lemma 2.1, the unit  $\mathcal{S}(\gamma)$  must be of the form  $\mathbb{1} \otimes \lambda + x$  where  $x \in H_{<2n}(M)$  and  $\lambda \in \Lambda$  is also a unit. The previous lemma shows that  $\lambda = r t^{K_{\max} - \kappa_0} + \text{l.o.t.}$  where  $r \in \mathbb{Q}$  is nonzero,  $\kappa_0 = \omega(\beta) > 0$ , and l.o.t. denotes lower order terms. We need to estimate the size of  $\kappa_0$  for the blow up  $(\widetilde{M}, \widetilde{\omega})$ . In the course of the argument we shall also need to consider the bundles  $P' \rightarrow \mathbb{P}^1$  and  $\widetilde{P}' \rightarrow \mathbb{P}^1$  defined by  $\gamma' := \gamma^{-1}$  and its blow up  $(\widetilde{\gamma})^{-1}$ . We will represent the inverse loop  $\gamma^{-1}$  by the path  $\{\phi_{1-t}\}$ , so that it is generated by the Hamiltonian  $K'_t := -K_{1-t}$ . If for any normalized Hamiltonian  $K_t$  we set  $K_{\min} := \int \min_{x \in M} K_t(x) dt$ , then

$$K'_{\max} = -K_{\min}, \quad \text{and} \quad K'_{\min} = -K_{\max}.$$

Similarly, if  $\widetilde{K}_t$  and  $\widetilde{K}'_t$  generate the blow up loop  $\widetilde{\gamma}$  and  $\widetilde{\gamma}' := \widetilde{\gamma}^{-1}$  then

$$\widetilde{K}'_{\max} = -\widetilde{K}_{\min}, \quad \text{and} \quad \widetilde{K}'_{\min} = -\widetilde{K}_{\max}.$$

The above remarks imply that the coefficients  $\lambda, \lambda'$  of  $\mathbb{1}$  in the formulas for  $\mathcal{S}(\gamma)$ ,  $\mathcal{S}(\gamma')$  have the form

$$\lambda = r_0 t^{K_{\max} - \kappa_0} + \text{l.o.t.}, \quad \lambda' = r'_0 t^{K'_{\max} - \kappa'_0} + \text{l.o.t.}$$

for some nonzero  $r_0, r'_0 \in \mathbb{Q}$ . Similarly, we write the coefficients  $\tilde{\lambda}, \tilde{\lambda}'$  of  $\mathbb{1}$  in the formulas for  $\mathcal{S}(\tilde{\gamma}), \mathcal{S}(\tilde{\gamma}')$  as

$$\tilde{\lambda} = \tilde{r}_0 t^{\tilde{K}_{\max} - \tilde{\kappa}_0} + \text{l.o.t.}, \quad \tilde{\lambda}' = \tilde{r}'_0 t^{\tilde{K}'_{\max} - \tilde{\kappa}'_0} + \text{l.o.t.}$$

where  $\tilde{r}_0, \tilde{r}'_0$  are nonzero. Corollary 2.12 implies that if  $[\omega]$  is assumed to be integral then  $\kappa_0 \in \mathbb{Z}$  because it is a value  $\omega(\beta)$  of  $[\omega]$  on an integral class  $\beta \in H_2(M; \mathbb{Z})$ . Similarly  $\tilde{\kappa}_0$  is a value of  $\tilde{\omega}$  on  $H_2(\tilde{M}; \mathbb{Z})$ . On the other hand, because the loop  $\gamma'$  need not have a fixed maximum  $\kappa'_0$  need not be a value of  $\omega$  on  $H_2(\tilde{M}; \mathbb{Z})$ , although  $-K'_{\max} - \kappa'_0$  is a value of the coupling class  $u_{\gamma'}$  on  $H_2(P'; \mathbb{Z})$ .

In the next lemma we assume that  $(\tilde{M}, \tilde{\omega})$  is not strongly uniruled. Hu [14, Thm. 1.2] showed that  $(M, \omega)$  is strongly uniruled only if  $(\tilde{M}, \tilde{\omega})$  is also (cf. Lemma 3.3). Hence our hypothesis implies that  $(M, \omega)$  also is not strongly uniruled.

**Lemma 2.13.** *Let  $(\tilde{M}, \tilde{\omega})$  be the blow up of  $(M, \omega)$  at a point of  $F_{\max}$ . Then:*

- (i)  $K_{\max} - K_{\min} = \tilde{K}_{\max} - \tilde{K}_{\min}$ .
- (ii) *If  $(\tilde{M}, \tilde{\omega})$  is not strongly uniruled then  $\kappa_0 + \kappa'_0 = \tilde{\kappa}_0 + \tilde{\kappa}'_0 = K_{\max} - K_{\min}$ .*
- (iii) *If  $(\tilde{M}, \tilde{\omega})$  is not strongly uniruled, then  $\tilde{\kappa}_0 = \kappa_0$  and  $\tilde{\kappa}'_0 = \kappa'_0$ .*

*Proof.* Observe that the  $S^1$  action near the point  $x_{\max} \in F_{\max}$  in  $M$  extends to a local toric structure in some neighborhood  $U$  of  $x_{\max}$  that preserves the submanifold  $U \cap F_{\max}$ . Hence we can form  $(\tilde{M}, \tilde{\omega})$  by cutting off a neighborhood of the vertex of the local moment polytope corresponding to  $x_{\max}$ . Since we do not cut off the whole of  $F_{\max}$  this does not change the length  $\max_{x \in U} K_t(x) - \min_{x \in U} K_t(x)$  of the image of the normalized local  $S^1$ -moment map at each  $t$ . This proves (i).

The identity

$$\mathbb{1} = \mathcal{S}(\gamma) * \mathcal{S}(\gamma') = (\mathbb{1} \otimes \lambda + x)(\mathbb{1} \otimes \lambda' + x')$$

implies that  $\lambda' = \lambda^{-1}$ . By Lemma 2.1 this is possible only if  $K_{\max} - \kappa_0 + K'_{\max} - \kappa'_0 = 0$ . Therefore  $\kappa_0 + \kappa'_0 = K_{\max} - K_{\min}$ . (Here we use the fact that  $K'_{\max} = -K_{\min}$ .) A similar argument shows that  $\tilde{\kappa}_0 + \tilde{\kappa}'_0 = \tilde{K}_{\max} - \tilde{K}_{\min}$ . In view of (i), this proves (ii).

The proof of (iii) is deferred to the end of this section (after Corollary 2.15).  $\square$

We now prove the key result of this section that ties the current ideas to the previously developed algebra.

**Proposition 2.14.** *Suppose that  $(M, \omega)$  is not uniruled, and that  $[\omega] \in H_*(M; \mathbb{Z})$ . Suppose further that  $\gamma$  has a maximal fixed point set  $F_{\max}$  that is a divisor, and that  $\gamma$  acts near  $F_{\max}$  as an effective circle action. Then there is a unit  $u \in \mathcal{E}_{2n} := QH_{2n}(\tilde{M})/\mathcal{I}$  of the form*

$$u = rE \otimes qt^{\kappa_0} + \mathbb{1} + x, \quad r \neq 0,$$

where  $x = \sum_{\kappa_i \geq \delta} E^{j_i} \otimes q^{j_i} t^{\kappa_0 - \kappa_i}$  for some  $0 < j_i < n$  and  $\kappa_0 \in \Gamma_\omega$  is positive.

*Proof.* Let  $(\tilde{M}, \tilde{\omega})$  be the one point blow up of  $(M, \omega)$  with blow up parameter  $\delta$  as before. Then  $[\tilde{F}_{\max}] = [F_{\max}] - E$  where  $E$  is (the class of) the exceptional divisor

and we consider  $[F_{\max}] \in H_{2n-2}(M) \equiv H_{2n-2}(\widetilde{M} \setminus E)$ . Denote by  $\widetilde{K}_t$  the normalized generating Hamiltonian for the lift  $\widetilde{\gamma}$  of  $\gamma$  to  $\widetilde{M}$ . By Corollary 2.12 the Seidel element of  $\widetilde{\gamma}$  has the form

$$(2.8) \quad \begin{aligned} \mathcal{S}(\widetilde{\gamma}) &= \widetilde{F}_{\max} \otimes q t^{\widetilde{K}_{\max}} + \sum_{\widetilde{\beta} \in H_2(\widetilde{M}; \mathbb{Z}), \widetilde{\omega}(\widetilde{\beta}) > 0} a_{\beta} \otimes q^{1-c_1(\widetilde{\beta})} t^{\widetilde{K}_{\max} - \widetilde{\omega}(\widetilde{\beta})} \\ &= -E \otimes q t^{\widetilde{K}_{\max}} + \sum_i E^{j_i} \otimes q^{j_i} t^{\widetilde{K}_{\max} - \widetilde{\kappa}_i} \pmod{\mathcal{I}}. \end{aligned}$$

Denote the coefficient of  $\mathbb{1} = E^0$  in this formula by  $\widetilde{\lambda} \in \Lambda^{\text{univ}}$ . Since  $(\widetilde{M}, \widetilde{\omega})$  is the blow up of  $(M, \omega)$  it is not uniruled by Hu–Li–Ruan [15, Thm 1.1]. Therefore the fact that  $\mathcal{S}(\widetilde{\gamma})$  is a unit implies by Lemma 2.1 that  $\widetilde{\lambda}$  is also a unit. Write  $\widetilde{\lambda} = t^{\widetilde{K}_{\max} - \widetilde{\kappa}_0}(\widetilde{r}_0 + y)$  where  $\widetilde{r}_0 \neq 0$  and  $y$  is a sum of negative powers of  $t$ . Lemma 2.13(iii) implies that  $\widetilde{\kappa}_0 = \kappa_0 = \omega(\beta) > 0$  for some  $\beta \in H_2(M)$ . Now let  $u = \mathcal{S}(\widetilde{\gamma}) \otimes \widetilde{\lambda}^{-1} \pmod{\mathcal{I}}$ . This has the required form.  $\square$

In the next corollary we denote the exceptional divisor in the fiber  $\widetilde{M}$  of  $\widetilde{P}' \rightarrow \mathbb{P}^1$  by  $E$ , and write  $E^j$  for its  $j$ -fold intersection product in  $\widetilde{M}$  considered (where appropriate) as a class in  $\widetilde{P}'$ .

**Corollary 2.15.** (i) *Let  $n \geq 3$ . Then under the conditions of Proposition 2.14 there is  $\sigma \in H_2(P'; \mathbb{Z})$  such that  $\langle E^2 \rangle_{\sigma - \varepsilon}^{\widetilde{P}'} \neq 0$ .*

(ii) *Similarly, if  $n = 2$  there is  $\sigma \in H_2(P'; \mathbb{Z})$  such that  $\langle E \rangle_{\sigma - 2\varepsilon}^{\widetilde{P}'} \neq 0$ .*

*Proof.* (i) By Lemma 2.8  $u^{-1}$  has the form  $1 - s^{n-1}t^\delta + \frac{1}{r}s^{n-2}t^{\delta - \kappa_0}(1 + \text{l.o.t.})$ . But  $u^{-1} = \mathcal{S}(\widetilde{\gamma}') \otimes \widetilde{\lambda} \pmod{\mathcal{I}}$ . Therefore

$$\mathcal{S}(\widetilde{\gamma}') = \widetilde{\lambda}^{-1} \left( 1 - s^{n-1}t^\delta + \frac{1}{r}s^{n-2}t^{\delta - \kappa_0}(1 + \text{l.o.t.}) \right) \pmod{\mathcal{I}}.$$

By Lemma 2.13 (iii)  $\widetilde{\lambda}^{-1}$  is a series in  $t$  with highest order term  $t^{-\widetilde{K}_{\max} + \kappa_0}$ . Therefore the largest  $\kappa$  such that the coefficient of  $s^{n-2}t^\kappa$  in  $\mathcal{S}(\widetilde{\gamma}')$  is nonzero is

$$\kappa = -\widetilde{K}_{\max} + \kappa_0 + \delta - \kappa_0 = -\widetilde{K}_{\max} + \delta.$$

Since the coefficient of  $t^{\widetilde{K}'_{\max} - \kappa'_0}$  in  $\mathcal{S}(\gamma')$  is nonzero, it follows from the definition of  $\mathcal{S}$  in equation (2.7) that there is a section  $\sigma'_0$  of  $P'$  such that  $-u_{\gamma'}(\sigma'_0) = K'_{\max} - \kappa'_0$ . We will take this as our reference section, writing all other sections in  $P'$  in the form  $\sigma'_0 + \beta$  where  $\beta \in H_2(M; \mathbb{Z})$  and those of  $\widetilde{P}'$  as  $\sigma'_0 + \widetilde{\beta}$  where  $\widetilde{\beta} \in H_2(\widetilde{M}; \mathbb{Z})$ .

Observe that

$$-u_{\gamma'}(\sigma'_0) = \widetilde{K}'_{\max} - \widetilde{\kappa}'_0 = \widetilde{K}'_{\max} - \kappa'_0 = -\widetilde{K}_{\max} + \kappa_0,$$

where the second equality holds by Lemma 2.13 (iii), and the third by Lemma 2.13 (ii) and the identity  $\widetilde{K}'_{\max} = -\widetilde{K}_{\min}$ . Since the coefficient of  $t^\kappa$  in  $\mathcal{S}(\widetilde{\gamma}')$  is nonzero by

hypothesis, there is  $\tilde{\beta} \in H_2(\tilde{M}; \mathbb{Z})$  such that

$$\kappa = -u_{\tilde{\gamma}'}(\sigma'_0 + \tilde{\beta}) = -u_{\tilde{\gamma}'}(\sigma'_0) - \tilde{\omega}(\tilde{\beta}).$$

Therefore

$$\tilde{\omega}(\tilde{\beta}) = -u_{\tilde{\gamma}'}(\sigma'_0) - \kappa = -\tilde{K}_{\max} + \kappa_0 + \tilde{K}_{\max} - \delta = \kappa_0 - \delta.$$

Since  $I_{\tilde{\omega}}$  is injective  $\tilde{\beta}$  must have the form  $\beta - \varepsilon$  for some  $\beta \in H_2(M)$ . Hence we may write the class  $\sigma'_0 + \tilde{\beta}$  as  $\sigma'_0 + \beta - \varepsilon = \sigma - \varepsilon$  for some  $\sigma = \sigma'_0 + \beta \in H_2(P')$ . Therefore the coefficient of  $s^{n-2}t^\kappa$  in  $\mathcal{S}(\tilde{\gamma}')$  arises from a nonzero invariant of the form

$$\langle E^2 \rangle_{\sigma - \varepsilon}^{\tilde{P}'}$$

where  $\sigma \in H_2(P')$ . (To check the power of  $E$  here, note that  $E^2 \cdot_M E^{n-2} = -1$ .) This proves the first statement.

The proof of (ii) is similar and is left to the reader.  $\square$

**Remark 2.16.** (i) By construction  $-u_{\gamma'}(\sigma) = -K_{\min} - \kappa'_0 - \kappa_0 = -K_{\max} = -u_{\gamma}(s'_{\min})$ , where  $s'_{\min}$  is the section of  $P'$  that is blown up to get  $\tilde{P}'$ . Therefore the classes  $\sigma$  and  $\sigma'_{\min}$  are equal.

(ii) The reader might wonder why we ignored the coefficient of  $s^{n-1}t^\delta$  in  $\mathcal{S}(\tilde{\lambda}')$ . But, reasoning as above, one can check that this term gives rise to a nonzero invariant of the form

$$\langle E \rangle_{\sigma - \varepsilon}^{\tilde{P}'}$$

where  $\sigma = \sigma'_0$ . When one blows  $\tilde{P}'$  down to  $P'$  this corresponds to the invariant  $\langle pt \rangle_\sigma^P$ , which we already know is nonzero. Hence this term gives no new information.

We finish this section by proving Lemma 2.13 (iii). For this we need the following preliminary result.

**Lemma 2.17.** *Let  $P, \tilde{P}, P', \tilde{P}'$  be as above. Then:*

(i) *For any section class  $\sigma \in H_2(P)$*

$$\langle pt \rangle_\sigma^P = \langle pt \rangle_\sigma^{\tilde{P}},$$

*where on RHS we consider  $\sigma \in H_2(\tilde{P})$ .*

(ii) *Similarly, for any section class  $\sigma \in H_2(P')$*

$$\langle pt \rangle_\sigma^{P'} = \langle pt \rangle_\sigma^{\tilde{P}'},$$

*where on RHS we consider  $\sigma \in H_2(\tilde{P}')$ .*

*Proof.* Hu showed in [14, Thm. 1.5] that if  $\tilde{Q}$  is the blow up of  $Q$  along an embedded 2-sphere  $C$  in a class with  $c_1(C) \geq 0$  then the Gromov-Witten invariants in classes  $\sigma \in H_2(Q)$  with homological constraints in  $H_*(Q \setminus C)$  remain unchanged. (The proof is similar to that of Proposition 2.19 below.) The result now follows because  $\tilde{P}$  is the blow up of  $P$  along the section  $s_{\max}$  with  $c_1(s_{\max}) = 1$ , while  $\tilde{P}'$  is the blow up of  $P'$

along a section  $s'_{\min}$  with  $c_1(s'_{\min}) = 3$ . (Cf. §3.3 for more detail on the construction of  $\tilde{P}'$ .)  $\square$

**Proof of Lemma 2.13 (iii).** We need to show that  $\kappa_0 = \tilde{\kappa}_0$  and  $\tilde{\kappa}'_0 = \kappa'_0$ . Since  $\kappa_0 + \kappa'_0 = \tilde{\kappa}_0 + \tilde{\kappa}'_0$  by part (ii) of Lemma 2.13, it suffices to show that  $\tilde{\kappa}_0 \leq \kappa_0$  and  $\tilde{\kappa}'_0 \leq \kappa'_0$ .

But by equations (2.2) and (2.3),  $\kappa_0$  is the minimum of  $\omega(\beta)$  over all classes  $\sigma = \sigma_{\max} + \beta$  such that  $\langle pt \rangle_{\sigma}^P \neq 0$ . Similarly,  $\tilde{\kappa}_0$  is the minimum of  $\tilde{\omega}(\tilde{\beta})$  over all classes  $\tilde{\sigma} = \tilde{\sigma}_{\max} + \tilde{\beta}$  such that  $\langle pt \rangle_{\tilde{\sigma}}^P \neq 0$ . But  $\sigma_{\max} = \tilde{\sigma}_{\max}$ , and by Lemma 2.17(i) every class  $\sigma$  with nonzero invariant in  $P$  also has nonzero invariant in  $\tilde{P}$ . Hence we must have  $\tilde{\kappa}_0 \leq \kappa_0$ .

Now consider the fibrations  $P' \rightarrow \mathbb{P}^1, \tilde{P}' \rightarrow \mathbb{P}^1$ . To compare  $\kappa'_0$  with  $\tilde{\kappa}'_0$  we need to use suitable reference sections that do not change under blow up. Instead of the maximal sections  $s_{\max}, \tilde{s}_{\max}$  we use the corresponding minimal sections  $s'_{\min}$  and  $\tilde{s}'_{\min}$  that represent the classes  $\sigma'_{\min}$  and  $\tilde{\sigma}'_{\min}$ . Thus we write every section class  $\sigma'$  of  $P'$  as  $\sigma'_{\min} + \beta$ , where  $\beta \in H_2(M)$ . Note that  $\sigma'_{\min}$  is taken to  $\tilde{\sigma}'_{\min}$  under the natural inclusion  $H_2(P') \rightarrow H_2(\tilde{P}')$ .

The symplectomorphism from the fiber connect sum  $P\#P'$  to the trivial bundle takes the connect sum of the sections  $s_{\max}$  and  $s'_{\min}$  to the trivial section  $\mathbb{P}^1 \times pt$  and the connect sum of the coupling classes  $u_{\gamma}\#u_{\gamma'}$  to the coupling class  $pr_M^*(\omega)$  of the trivial bundle  $\mathbb{P}^1 \times M \xrightarrow{pr} \mathbb{P}^1$ . Hence

$$u_{\gamma}(s_{\max}) + u_{\gamma'}(s'_{\min}) = pr_M^*(\omega)(\mathbb{P}^1 \times pt) = 0.$$

Therefore  $u_{\gamma'}(s'_{\min}) = -u_{\gamma}(s_{\max}) = K_{\max}$ . A similar argument applied to the blowups shows that  $u_{\tilde{\gamma}'}(\tilde{s}'_{\min}) = \tilde{K}_{\max}$ .

By Proposition 2.11 we can interpret  $\kappa'_0$  as the minimum of  $K'_{\max} + u_{\gamma'}(\sigma')$  over all classes  $\sigma'$  such that  $\langle pt \rangle_{\sigma'}^{P'} \neq 0$ . Writing  $\sigma'$  as  $\sigma'_{\min} + \beta$  and using  $K'_{\max} = -K_{\min}$ , we find

$$K'_{\max} + u_{\gamma'}(\sigma'_{\min} + \beta) = K_{\max} - K_{\min} + \omega(\beta).$$

Hence  $\kappa'_0$  is the minimum of  $K_{\max} - K_{\min} + \omega(\beta)$  over the corresponding set  $\mathcal{B}$  of classes  $\beta$ . Similarly,  $\tilde{\kappa}'_0$  is the minimum of  $\tilde{K}_{\max} - \tilde{K}_{\min} + \tilde{\omega}(\tilde{\beta})$  over a set of classes  $\tilde{\beta}$  that, by Lemma 2.17 (ii) and the fact that  $\sigma'_{\min} = \tilde{\sigma}'_{\min}$ , includes  $\mathcal{B}$ . Since  $K_{\max} - K_{\min} = \tilde{K}_{\max} - \tilde{K}_{\min}$ , we find that  $\tilde{\kappa}'_0 \leq \kappa'_0$  as required.  $\square$

**2.3. Proof of the main results.** Suppose that  $\gamma$  is a loop of Hamiltonian symplectomorphisms of  $(M, \omega)$  with a fixed maximum near which it is an effective circle action. A standard Moser argument<sup>4</sup> implies that we can assume that  $I_{\omega}$  is injective. The next lemma shows that we can always arrange for the other conditions of Proposition 2.14 to hold.

<sup>4</sup>Perturb the given form to a suitable class and then average over the  $S^1$  action.

**Lemma 2.18.** *Suppose that the submanifold  $F_{\max}$  of  $M$  is a fixed maximum of the loop  $\gamma$  and that nearby  $\gamma$  is an effective  $S^1$  action. Then we can blow up along  $F_{\max}$  to achieve an action on some blow up of  $M$  for which the new  $F_{\max}$  is a divisor with normal weight  $m = 1$ .*

*Proof.* Suppose that the weights of the action normal to  $F_{\max}$  are  $(-m_1, \dots, -m_k)$  where  $0 < m_1 \leq m_2 \leq \dots \leq m_k$ . Let  $\widetilde{M}$  be the blow up of  $M$  along  $F_{\max}$ . Then the weights of the induced action normal to the new maximal fixed point set are  $-m_1, -(m_i - m_1), \dots, -(m_k - m_1)$ , where  $i$  is the smallest index such that  $m_i > m_1$ . Therefore by repeated blowing up along the maximal fixed point of the moment we can reduce to the case when there is just one normal weight, i.e.  $F_{\max}$  is a divisor. The normal weight is 1 since the action is effective.  $\square$

Corollary 2.15 summarizes the information given to us by the Seidel representation. To make use of it, we need one more result relating the Gromov–Witten invariants of  $\widetilde{P}'$  and  $P'$ . This is proved in §3.3.

**Proposition 2.19.** *Suppose that  $(M, \omega)$  is not uniruled.*

- (i) *If  $n \geq 3$  and  $\langle E^2 \rangle_{\sigma - \varepsilon}^{\widetilde{P}'} \neq 0$  for some section class  $\sigma \in H_2(P'; \mathbb{Z})$  then  $\langle \tau_1 pt \rangle_{\sigma}^{P'} \neq 0$ .*
- (ii) *If  $n = 2$  and  $\langle E \rangle_{\sigma - 2\varepsilon}^{\widetilde{P}'} \neq 0$  then for some section  $s$  in  $P'$  at least one of  $\langle pt, s \rangle_{\sigma}^{P'}$  and  $\langle \tau_1 pt \rangle_{\sigma}^{P'}$  is nonzero.*

We next show that the situation discussed in the above proposition cannot occur: in fact, if the conclusions hold  $(M, \omega)$  must be strongly uniruled.

**Lemma 2.20.** *If there is an element  $\gamma \in \pi_1(\text{Ham}(M))$  such that the descendent invariant*

$$\langle \tau_k pt \rangle_{\sigma}^P \neq 0$$

*for some  $k > 0$  and some section class  $\sigma$  in  $P := P_{\gamma}$  then  $(M, \omega)$  is strongly uniruled.*

*Proof.* As explained in §4 (see equation (3.9)) we may express this invariant as a sum

$$\langle \tau_k pt, M, M \rangle_{\sigma}^P = \sum_{i, \alpha_1 + \alpha_2 = \sigma} \langle \tau_{k-1} pt, \xi_i \rangle_{\alpha_1}^P \langle \xi_i^*, M, M \rangle_{\alpha_2}^P,$$

where  $\alpha_j \in H_2(P)$  and  $\xi_i$  runs over a basis for  $H_*(P)$  with dual basis  $\xi_i^*$ . Because the symplectic manifold  $(P, \Omega)$  supports an  $\Omega$ -tame almost complex structure  $J$  such that the projection  $P \rightarrow \mathbb{P}^1$  is holomorphic, the only classes in  $H_2(P)$  with nontrivial GW invariants project to nonnegative multiples of the generator of  $H_2(\mathbb{P}^1)$ . Thus one of the classes  $\alpha_j$  is a section class and the other is a fiber class, i.e. a class in  $H_2(M)$ . But if  $\alpha_2$  is a fiber class then it cannot meet two separate copies of  $M$ . Hence any nonzero term in this expansion must have section class  $\alpha_2$ . Then  $\alpha_1$  is a fiber class, and so by Lemma 3.8

$$\langle \tau_{k-1} pt, \xi_i \rangle_{\alpha_1}^P = \langle \tau_{k-1} pt, \xi_i \cap M \rangle_{\alpha_1}^M.$$

But now by repeated applications of equation (3.9) one finds that  $M$  is strongly uniruled; cf. the proof of Lemma 4.8.  $\square$

The following lemma is proved by a similar argument; see §3.4 at the end of §3.

**Lemma 2.21.** *Suppose that  $(M, \omega)$  is the blow up of a symplectic 4-manifold that is not rational or ruled. Then there is no  $\gamma \in \pi_1(\text{Ham}(M))$  such that the corresponding fibration  $P \rightarrow \mathbb{P}^1$  has a section  $s$  and a section class  $\sigma$  with*

$$\langle pt, s \rangle_\sigma^P \neq 0.$$

**Proof of Theorem 1.1.** Suppose first that  $\dim M \geq 6$  and that  $(M, \omega)$  is not uniruled. By Lemma 2.18 we may assume that we are in the situation of Proposition 2.14. Then Corollary 2.15(i) together with Lemma 2.20 implies that  $(M, \omega)$  is strongly uniruled. This contradiction shows that the initial hypothesis must be wrong: in other words,  $(M, \omega)$  is uniruled.

When  $\dim M = 4$  the argument is similar, except that we use Lemma 2.21 instead of Lemma 2.20.  $\square$

Finally note that Proposition 1.4 follows from the next lemma.

**Lemma 2.22.** *Suppose that the loop  $\gamma$  in  $\text{Ham}(M, \omega)$  has a nondegenerate fixed maximum at  $x_{\max}$ . Suppose also that the linearized flow  $A_t, t \in [0, 1]$ , at  $x_{\max}$  is homotopic through positive paths to a linear circle action. Then  $\gamma$  is homotopic through Hamiltonian loops with fixed maximum at  $x_{\max}$  to a loop  $\gamma'$  that is a circle action near  $x_{\max}$ .*

*Proof.* Identify a neighborhood of  $x_{\max}$  with a neighborhood of  $\{0\}$  in  $\mathbb{R}^{2n}$  with its standard symplectic form  $\omega_0$ .

**Step 1:** *The loop  $\gamma =: \gamma_0$  is homotopic through loops  $\gamma_s = \{\phi_{st}\}$  with fixed maximum at  $\{0\}$  to a loop  $\gamma_1$  that equals the linear flow  $A_t$  in some neighborhood of  $\{0\}$ .*

Let  $\gamma = \{\phi_t\}$  with generating Hamiltonian  $H_t$ . By assumption  $H_t = H_t^0 + O(\|x\|^3)$  where for each  $t$  there is a positive definite symmetric matrix  $Q_t$  such that  $H_t^0(x) = -x^T Q_t x$ . We shall choose  $\phi_{st}$  of the form  $\phi_t \circ g_{st} \circ h_{st}$  where

- for each  $s \in [0, 1]$ , the loop  $g_{st}, t \in [0, 1]$ , consists of diffeomorphisms with support in some small ball  $B_{2r_1} := B_{2r_1}(0)$  such that for all  $s, t$  we have

$$dg_{st}(0) = id, \quad g_{0t} = id, \quad g_{1t} = (\phi_t)^{-1} \circ A_t \text{ in } B_{r_1};$$

- $h_{st}$  has support in the ball  $B_{3r_1}$  and is such that  $h_{st}^*(g_{st}^* \omega_0) = \omega_0$ . Moreover  $h_{1t} = id$  on  $B_{r_1}$ .

Choose  $g_{st}$  satisfying all conditions above and with support in  $\text{int } B_{2r_1}$ . Since  $dg_{st}(0) = id$ , we may write  $g_{st}(x) = x + O(\|x^2\|)$ . Hence we may arrange that for some constants  $c_0, c_1$  we have  $\|g_{st}(x) - x\| \leq c_0 \|x\|^2$  and  $|g_{st}^* \omega_0 - \omega_0| \leq c_1 \|x\|$ . Note that we may assume that these constants  $c_i$ , as well as all subsequent ones, depend only on the initial path  $\phi_t$ , i.e. there are constants  $R > 0$  and  $c_i$  such that suitable  $g_{st}$  exist for all  $r_1 < R$ . Then, reducing  $R$  as necessary,  $c_1 \|x\|$  is arbitrarily small on  $B_{2r_1}$  so that we may assume that the forms  $\omega_{st} := g_{st}^* \omega_0$  are all nondegenerate. Let  $\rho_{st} := \frac{d}{dt}(g_{st}^* \omega_0)$ . Since  $g_{st} = id$  for all  $s$  near 0 by construction,  $\rho_{0t} = 0$ . Similarly,  $\rho_{1t} = 0$  in  $B_{r_1}$ . Further for some constant  $c_2$  as above  $|\rho_{st}| \leq c_2 \|x\|$ .

Now,  $\rho_{st} := d\beta_{st}$ , where for each  $s, t$

$$\beta_{st}(x) = \int_0^1 \rho_{st}(\partial_r, \cdot)(\lambda x) d\lambda, \quad x \in B_{2r_1},$$

and  $\partial_r$  is the radial vector field in  $\mathbb{R}^{2n}$ . Hence  $\beta_{0t} \equiv 0$ , and  $\beta_{1t} = 0$  in  $B_{r_1}$ . Moreover, each 1-form  $\beta_{st}(x)$  satisfies  $|\beta_{st}(x)| \leq c_3 \|x\|^2$  and, because  $\rho_{st}$  has support in  $\text{int } B_{2r_1}$ , is closed and independent of  $r := \|x\|$  near  $\partial B_{2r_1}$ . Therefore, near  $\partial B_{2r_1}$  we may write  $\beta_{st} = df_{st}$  where  $f_{st}$  is a function on  $S^{2n-1}$  of norm  $\leq c_4 r_1^3$ . Therefore we may extend  $\beta_{st}$  by  $d(\alpha(r)f_{st})$  for a suitable cut off function  $\alpha$  so that it has support in  $B_{3r_1}$  and still satisfies an estimate  $|\beta_{st}(x)| \leq c_5 \|x\|^2$ .

Now construct  $h_{st}, t \in [0, 1]$ , for each fixed  $s$  by the usual Moser homotopy method so that  $\omega_{st}(\frac{d}{dt}h_{st}, \cdot) = \beta_{st}$ . Then  $h_{st}$  satisfies the required conditions. Moreover  $\|\frac{d}{dt}h_{st}\| \leq c_6 \|x\|^2$ . Hence the Hamiltonian  $G_{st}$  that generates  $g_{st} \circ h_{st}$  satisfies

$$|G_{st}| \leq c_7 \|x\|^3 \quad \text{for } \|x\| \leq 3r_1.$$

The generating Hamiltonian for  $\phi_{st} := \phi_t \circ g_{st} \circ h_{st}$  is

$$H_{st}(x) := H_t(x) + G_{st}(\phi_t^{-1}(x)) = H_t^0(x) + O(\|x\|^3).$$

Since  $c_7$  is independent of  $r_1$ , we can now choose  $r_1$  so small that  $x = 0$  is still the global maximum of  $H_{st}$ . (The size of  $r_1$  will depend on the smallest eigenvalues of the matrices  $Q_t$  and also on the second derivatives of the  $\phi_t$ , i.e. the cubic term in  $H_t$ .) This completes Step 1.

By assumption there is a homotopy of the path  $\{A_t\} := \{A_{1t}\}$  to a circle action  $\{A_{2t}\}$  through positive paths  $\{A_{st}\}_{t \in [0,1]}$ . Denote by  $Q_{st}$  the corresponding family of positive definite matrices. Reparametrize this homotopy with respect to  $s$  to stretch it out over the interval  $s \in [1, N]$  for some large  $N$  to be chosen later. Note that  $\phi_{1t} = A_{1t}$  on  $B_{r_1}$ .

**Step 2:** *There is a sequence  $r_k, k \geq 2$ , satisfying  $0 < 2r_{k+1} < r_k$  for all  $k$  and a finite sequence of homotopies  $\gamma_s = \{\phi_{st}\}_{t \in [k, k+1]}$ ,  $k = 1, \dots, N-1$ , of Hamiltonian loops with support in  $B_{r_k}$  so that:*

- for all  $k \in [1, N-1]$   $\phi_{k+1t} = A_{k+1t}$  on  $B_{r_{k+1}}$  and  $\phi_{k+1t} = A_{kt}$  on  $B_{r_k} \setminus B_{2r_{k+1}}$ .
- for each  $s$  the Hamiltonian loop  $\phi_{st}, t \in [0, 1]$ , has fixed maximum at  $\{0\}$ .

If  $r_k$  and  $\phi_{kt}$  are given, an obvious modification of the procedure described in Step 1 gives suitable  $r_{k+1}$  and  $\phi_{k+1t}$  provided that the derivative  $\frac{d}{ds}A_{st}, s \in [k, k+1]$ , is not too large. As before, the idea is first to use the linear homotopy  $A_{st}$  for  $s \in [k, k+1]$  to construct a smooth interpolation  $g_{st}$  between  $A_{k+1t}$  on  $B_{r_{k+1}}$  and  $A_{kt}$  on a neighborhood of  $\partial B_{r_k}$ , and then correct using a Moser homotopy. Then  $\|g_{st}(x) - A_{kt}(x)\| \leq c \|x\|^2$ , where  $c$  depends only on  $\frac{d}{ds}A_{st}$ . (Note that  $c$  does not depend on  $r_k$  because the linear maps  $A_{st}$  are invariant under homotheties.) But we can arrange that  $\frac{d}{ds}A_{st}$  is as small as we like by choosing  $N$  sufficiently large. In order that the resulting loops  $\phi_{st}$  have fixed maximum at  $\{0\}$ , the permissible size for  $c$  (and hence  $N$ ) will depend on the smallest eigenvalue of the compact family of matrices  $Q_{st}$ .

Step 2 completes the proof, since the loop  $\phi' := \phi_{Nt}$  satisfies the requirements of the lemma.  $\square$

### 3. GROMOV–WITTEN INVARIANTS AND BLOWING UP

We now prove the results on Gromov–Witten invariants needed above, i.e. Propositions 2.5 and 2.19. In [15, Theorem 5.15] Hu–Li–Ruan establish a correspondence between the relative genus  $g$  invariants of the blow up  $(\widetilde{M}, E)$  and certain corresponding sets of absolute invariants of  $M$ . Proposition 2.5 could be proved using a special case of this general correspondence. However, instead of quoting their result we shall reprove the parts we need, since we do not need the full force of their results and also need some other related results.

We begin this section with a discussion of relative GW invariants since this will be our main tool. For more details see Li–Ruan [23, §4,5] and Hu–Li–Ruan [15, §3], and Bourgeois *et al.* [4, §10] for relevant compactness results. After stating the decomposition formula, we prove Proposition 3.5, a generalization of Proposition 2.5 that characterizes uniruled manifolds  $M$  in terms of properties of the one point blow up  $\widetilde{M}$ .

Finally we prove Proposition 2.19 by using the decomposition formula for section classes of the bundle  $P \rightarrow \mathbb{P}^1$ .

**3.1. Relative invariants of genus zero.** Consider a pair  $(X, D)$  where  $D$  is a divisor in  $X$ , i.e. a codimension 2 symplectic submanifold. The relative invariants count connected  $J$ -holomorphic curves in the  $k$ -fold prolongation  $X_k$  (defined below) of  $X$ . Here  $J$  is an  $\omega$ -tame almost complex structure on  $X$  satisfying certain normalization conditions along  $D$ . In particular  $D$  is  $J$ -holomorphic, i.e.  $J(TD) \subset TD$ . The invariants are defined by first constructing a compact moduli space  $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{\beta, \underline{d}}^{X, D}(J)$  of genus zero  $J$ -holomorphic curves  $C$  in class  $\beta \in H_2(X)$  as described below and then integrating the given constraints over the corresponding virtual cycle  $\overline{\mathcal{M}}^{[vir]}$ . Here  $\underline{d} = (d_1, \dots, d_r)$  is a partition of  $d := \beta \cdot D \geq 0$ .

To a first approximation the moduli space consists of curves in  $X$ , i.e. equivalence classes  $C = [\Sigma, u, \dots]$  of stable maps to  $X$ , that intersect the divisor  $D$  at  $r$  points with multiplicities  $d_i$ . More precisely, each such curve has  $(k+1)$  levels  $C_i$  for some  $k \geq 0$ , the principal level  $C_0$  in  $X \setminus D$  and the higher levels  $C_i$  (sometimes called bubbles) in the  $\mathbb{C}^*$ -bundle  $L_D^* \setminus D$  where  $L_D^* \rightarrow D$  is the dual of the normal bundle to  $D$ . The whole curve  $C$  therefore lies in a space  $X_k$  called the  $k$ th prolongation of  $X$ , which is defined as follows. Identify  $L_D^* \setminus D$  with the complement of the sections  $D_0, D_\infty$  of the ruled manifold

$$\pi_Q : Q := \mathbb{P}(\mathbb{C} \oplus L_D^*) \rightarrow D,$$

where the zero section  $D_0 := \mathbb{P}(\mathbb{C} \oplus \{0\})$  has normal bundle  $L_D^*$  and the infinity section  $D_\infty := \mathbb{P}(\{0\} \oplus L_D^*)$  has normal bundle  $L_D$ . Think of the initial divisor  $D \subset X$  as the infinity section  $D_{0\infty}$  at level 0. Then the prolongation  $X_k$  is simply the disjoint union of  $X$  with  $k$  copies of  $Q$ , but it is useful to think that for each  $i \geq 1$  the zero section  $D_{i0}$  of the  $i$ th level is identified with the infinity section  $D_{i-1\infty}$  of the preceding level.

The most important divisor in  $X_k$  is  $D_{k\infty}$ , which carries the relative constraints and hence plays the role of the relative divisor.

For each  $i > 0$  we assume that the ends of the components of the  $i$ th level curve  $C_i$  along the zero section  $D_{i0}$  match with those of  $C_{i-1}$  along  $D_{i-1\infty}$ . The final level  $C_k$  carries the relative marked points which are mapped to  $D_{k\infty}$ . Assuming there are no absolute marked points, each level of  $C$  is an equivalence class of stable maps

$$[\Sigma_i, u_i, y_{10}, \dots, y_{r_{i0}0}, y_{1\infty}, \dots, y_{r_{i\infty}\infty}],$$

in some class  $\beta_i$ , where the internal relative marked points  $y_{10}, \dots, y_{r_{i0}0}$  are mapped to  $D_{i0}$  with multiplicities  $\underline{m}_{i0}$  that sum to  $\beta_i \cdot D_{i0}$  and the marked points  $y_{1\infty}, \dots, y_{r_{i\infty}\infty}$  are taken to  $D_{i\infty}$  with multiplicities  $\underline{m}_{i\infty}$  that sum to  $\beta_i \cdot D_{i\infty}$ . Note that the components  $C_i$  and  $C_{i+1}$  match along  $D_{i\infty} = D_{i+1,0}$  only if the multiplicities  $\underline{m}_{i\infty}$  and  $\underline{m}_{i+1,0}$  agree. In symplectic field theory such multilevel curves are called buildings: see [4, §7].

Each curve  $C$  in  $\overline{\mathcal{M}}$  might also have some absolute marked points; these could lie on any of the levels  $C_i$  but must be disjoint from the relative marked points. We require further that each component  $C_i$  be stable, i.e. have a finite group of automorphisms. For  $C_0$  this has the usual meaning. However, when  $i \geq 1$  we identify two level  $i$  curves  $C_i, C'_i$  if they lie in the same orbit of the fiberwise  $\mathbb{C}^*$  action; i.e. given representing maps  $u_i : (\Sigma_i, j) \rightarrow Q$  and  $u'_i : (\Sigma'_i, j') \rightarrow Q$ , the curves are identified if there is  $c \in \mathbb{C}^*$  and a holomorphic map  $h : (\Sigma_i, j) \rightarrow (\Sigma'_i, j')$  (preserving all marked points) such that  $u_i \circ h = c u'_i$ .<sup>5</sup> Thus  $L_D^* \setminus D$  should be thought of as a rubber space.

Note that although the whole curve is connected, the individual levels need not be but should fit together to form a genus zero curve. The homology class  $\beta$  of such a curve is defined to be the sum of the homology class of its principal component with the projections to  $D$  of the classes of its higher levels.<sup>6</sup>

When doing the analysis it is best to think that the domains and targets of the curves have cylindrical ends. However, their indices are the same as those of the corresponding compactified curves; see [23, Prop 5.3]. Thus the (complex) dimension of the moduli space  $\overline{\mathcal{M}}_{\beta, k, (d_1, \dots, d_r)}^{X, D}$  of genus zero curves in class  $\beta$  with  $k$  absolute marked points and  $r$  relative ones is

$$(3.1) \quad n + c_1^X(\beta) + k + r - 3 - \sum_{i=1}^r (d_i - 1) = n + c_1^X(\beta) + k + 2r - 3 - d.$$

Here  $k + r - 3$  is the contribution from moving the marked points and we subtract  $d_i - 1$  at each relative intersection point of multiplicity  $d_i$  since, as far as a dimensional count is concerned, what is happened at such a point is that  $d_i$  of the  $d := \beta \cdot D$  intersection points of the  $\beta$ -curve with  $D$  coincide.

<sup>5</sup>Hence a level  $C_i$  cannot consist only of multiply covered curves  $z \mapsto z^k$  in the fibers  $\mathbb{C}^*$  of  $L_D^* \setminus D$  because these are not stable.

<sup>6</sup>This is the class of the curve in  $X$  obtained by gluing all the levels together. When  $k = 0$  the curve only has one level and so the relative constraints lie along  $D \subset X$ . Ionel–Parker [16] work with the moduli space obtained by closing the space of 1-level curves, which in principle could give slightly different invariants from the ones considered here.

**Example 3.1.** Let  $X = \mathbb{P}^2 \# \overline{\mathbb{P}^2}$ , the one point blow up of  $\mathbb{P}^2$  and set  $D := E$ , the exceptional curve. Denote by  $\pi : X \rightarrow \mathbb{P}^1$  the projection, and fix another section  $H$  of  $\pi$  that is disjoint from  $E$ . Let  $J$  be the usual complex structure and  $\overline{\mathcal{M}}_\lambda^{X,E}(J; p)$  be the moduli space of holomorphic lines through some point  $p \in H$ , where  $\lambda = [H]$  is the class of a line. Since  $\lambda \cdot E = 0$  there are no relative constraints. Then  $\overline{\mathcal{M}}_\lambda^{X,E}(J; p)$  has complex dimension 1 and should be diffeomorphic to  $\mathbb{P}^1$ . The closure of the ordinary moduli space of lines in  $X$  though  $p$  contains all such lines together with one reducible curve consisting of the union of the exceptional divisor  $E$  with the fiber  $\pi^{-1}(\pi(p))$  through  $p$ . But the elements of  $\overline{\mathcal{M}}_\lambda^{X,E}(J; p)$  do not contain components in  $E$ . Instead, this component becomes a higher level curve lying in  $Q = \mathbb{P}(\mathbb{C} \oplus \mathcal{O}(1))$  that intersects  $E_0 = \mathbb{P}(\mathbb{C} \oplus \{0\})$  in the point  $E_0 \cap \pi^{-1}(\pi(p))$  and lies in the class  $\lambda_Q$  of the line in  $Q \cong X$ . Note that modulo the action of  $\mathbb{C}^*$  on  $Q$  there is a *unique* such bubble. Thus the corresponding two-level curve in  $\overline{\mathcal{M}}_\lambda^{X,E}(J; p)$  is a rigid object. Moreover, because  $\lambda_Q$  projects to the class  $\varepsilon \in H_2(E)$  of the exceptional divisor, its homology class  $(\lambda - \varepsilon) + pr(\lambda_Q)$  is  $(\lambda - \varepsilon) + \varepsilon = \lambda$ .

The constraints for the relative invariants consist of homology classes  $b_j$  in the divisor  $D$  (the relative insertions) together with absolute (possibly descendent) insertions  $\tau_{i_j} a_j$  where  $a_j \in H_*(X)$  and  $i_j \geq 0$ . We shall denote the (connected) relative genus zero invariants by:

$$(3.2) \quad \langle \tau_{i_1} a_1, \dots, \tau_{i_q} a_q \mid b_1, \dots, b_r \rangle_{\beta, (d_1, \dots, d_r)}^{X, D}$$

where  $a_i \in H_*(X), b_i \in H_*(D)$ , and  $d := \sum d_i = \beta \cdot D \geq 0$ . (If  $\beta \cdot D < 0$  then the moduli space is undefined and the corresponding invariants are set equal to zero.) This invariant counts isolated connected genus zero curves in class  $\beta$  that intersect  $D$  to order  $\underline{d} := (d_1, \dots, d_r)$  in the  $b_i$ , i.e the  $i$ th relative marked point intersects  $D$  to order  $d_i \geq 1$  at a point on some representing cycle for  $b_i$ . Moreover, the insertion  $\tau_i$  occurring at the  $j$ th absolute marked point  $z_j$  means that we add the constraint  $(c_j)^i$ , where  $c_j$  is the first Chern class of the cotangent bundle  $\mathcal{L}_j$  to the domain<sup>7</sup> at  $z_j$ . One can evaluate (3.2) (which in general is a rational number) by integrating an appropriate product of Chern classes over the virtual cycle corresponding to the moduli space of stable  $J$ -holomorphic maps that satisfy the given homological constraints and tangency conditions. This cut down virtual cycle has dimension equal to the index of the curves  $C$  satisfying these incidence conditions; if this dimension does not equal the total degree  $\sum_{j=1}^q i_j$  of the descendent classes the invariant is by definition set equal to 0. For details on how to construct this virtual cycle see for example Li–Ruan [23] or Hu–Li–Ruan [15]. (Also cf. Remark 1.7.)

We shall need the following information about specific genus zero relative invariants.

**Lemma 3.2.** *Let  $X = \mathbb{P}^n$ , and  $D = \mathbb{P}^{n-1}$  be the hyperplane. Denote by  $\lambda$  the class of a line. Then:*

<sup>7</sup>Thus, if  $z_j$  lies at the  $m$ th level, the fiber of  $\mathcal{L}_j$  at  $C$  is  $T_{z_j}^*(\Sigma_m)$ .

(i) if  $d > 0$  and  $n > 1$

$$\langle |b_1, \dots, b_r\rangle_{0, d\lambda, \underline{d}}^{\mathbb{P}^n, \mathbb{P}^{n-1}} = 0, \quad \text{for any } b_i \in H_*(\mathbb{P}^{n-1}).$$

(ii) (Hu [14, §3] and Gathmann [7, Lemma 2.2]) Let  $\widetilde{M}$  be a one point blow up with exceptional divisor  $E$  and suppose that  $\beta \in H_2(M) \subset H_2(\widetilde{M})$  so that  $\beta \cdot E = 0$ . Then for any  $a_i \in H_*(\widetilde{M} \setminus E)$ , the relative invariant  $\langle a_1, \dots, a_k | \rangle_{0, \beta}^{\widetilde{M}, E}$  equals the absolute invariant  $\langle a_1, \dots, a_k | \rangle_{0, \beta}^{\widetilde{M}}$ .

(iii) (Hu–Li–Ruan [15, Theorem 6.1]) The 2-point invariant

$$(3.3) \quad \langle \tau_k \text{pt} | D^j \rangle_{0, df, (d)}^{\mathbb{P}^n, \mathbb{P}^{n-1}}, \quad 1 \leq j \leq n, \quad d \geq 1,$$

is nonzero iff  $k = nd - j$ .

*Proof.* (i) holds for dimensional reasons. We shall show that under the given conditions on  $n$  and  $d$  it is impossible to choose  $\underline{d}$  and the  $b_i$  so that  $\overline{\mathcal{M}}_{d\lambda, \underline{d}}^{\mathbb{P}^n, \mathbb{P}^{n-1}}$  has formal dimension 0. Therefore the invariant vanishes by definition.

By equation (3.1) the formal (complex) dimension of the moduli space of genus zero stable maps through the relative constraints  $b_1, \dots, b_r$  is

$$n + d(n + 1) + r - 3 - (d - r) + \delta - rn,$$

where  $\delta$  is half the sum of the degrees of the  $b_i$ . Thus  $0 \leq \delta \leq rn$ . Since  $d - r \geq 0$  and  $r > 0$ , we therefore need  $(d - r)n + n + 2r + \delta = 3$ , which implies  $d = r$  and  $n = r = 1$ . Since we assumed  $n > 1$ , this is impossible.

Hu's proof of (ii) uses the decomposition formula stated below. (His proof can be reconstructed by arguing as in the proof of Proposition 3.5.) The proof by Gathmann is more elementary but applies only in the case of projective algebraic manifolds. Claim (iii) is much deeper. It is proved by Hu–Li–Ruan using localization techniques.  $\square$

**3.2. Applications of the decomposition formula.** Our main tool is the decomposition rule of Li–Ruan [23, Thm 5.7] and (in a slightly different version) of Ionel–Parker [16]. So suppose that the manifold  $M$  is the fiber sum of  $(X, D)$  with  $(Y, D^+)$ , where the divisors  $D := D^-$  and  $D^+$  are symplectomorphic with dual normal bundles. Since this is the only case we shall need, let us assume that the absolute constraints can be represented by cycles in  $M$  that do not intersect the inverse image of the divisor, i.e. that  $a_i \in H_*(X \setminus D)$ ,  $i \leq q$ , and  $a_i \in H_*(Y \setminus D^+)$ ,  $q < i \leq p$ . For simplicity we assume also that the map  $H_2(M) \rightarrow H_2(X \cup_D Y)$  is injective. (This hypothesis is satisfied whenever  $H_1(D) = 0$ .) Further, let  $b_i, i \in I$ , be a basis for  $H_*(D) = H_*(D; \mathbb{Q})$  with dual basis  $b_i^*$  for  $H_*(D)$ . Then the genus zero decomposition formula has the following shape:

$$(3.4) \quad \langle a_1, \dots, a_p | \rangle_{\beta}^M = \sum_{\Gamma, \underline{d}, (i_1, \dots, i_r)} n_{\Gamma, \underline{d}} \langle a_1, \dots, a_q | b_{i_1}, \dots, b_{i_r} \rangle_{\beta_1, \underline{d}}^{\Gamma_1, X, D} \langle a_{q+1}, \dots, a_p | b_{i_1}^*, \dots, b_{i_r}^* \rangle_{\beta_2, \underline{d}}^{\Gamma_2, Y, D^+}.$$

Here we sum with rational weights  $n_{\Gamma, \underline{d}}$  over all decompositions  $\underline{d}$  of  $d$ , all possible connected labelled trees  $\Gamma$ , and all possible sets  $i_1, \dots, i_r$  of relative constraints. Each  $\Gamma$  describes a possible combinatorial structure for a stable map that glues to give a  $\beta$ -curve. Thus  $\Gamma$  is a disjoint union  $\Gamma_1 \cup \Gamma_2$ , where the graph  $\Gamma_1$  (resp.  $\Gamma_2$ ) describes the part of the curve lying in some  $X_k$  (resp. some  $Y_\ell$ ). Also  $\beta_1$  (resp.  $\beta_2$ ) is the part of its label that describes the homology class; the pair  $(\beta_1, \beta_2)$  runs through all decompositions such that the result of gluing the two curves in the prolongations  $X_{k_1}$  and  $Y_{k_2}$  along their intersections with the relative divisors gives a curve in class  $\beta$ . Moreover, there is a bijection between the labels  $\{(d_i, b_i) \in \mathbb{N} \times H_*(D)\}$  of the relative constraints in  $\Gamma_1$  and those  $\{(d_i, b_i^*) \in \mathbb{N} \times H_*(D^+)\}$  in  $\Gamma_2$ . (These labels are called relative ‘‘tails’’ in [15].)  $\Gamma_i$  need not be connected; if it is not, we define  $\langle \dots | \dots \rangle^{\Gamma_i}$  to be the product of the invariants defined by its connected components.

Because the total curve has genus zero, each component of  $\Gamma_1$  has at most one relative tail in common with each component of  $\Gamma_2$ . In many cases we will be able to show that  $\Gamma_1$  is connected, and hence that  $\Gamma_2$  has  $r$  components, one for each relative constraint.

Most of the next result is known: part (i) follows from Theorem 1.2 in Hu [14], while part (ii) is very close to [15, Theorem 6.1].

**Lemma 3.3.** *Let  $(\widetilde{M}, \widetilde{\omega})$  be the one point blow up of  $(M, \omega)$  with exceptional divisor  $E$ .*

(i) *If  $(M, \omega)$  is strongly uniruled, then  $(\widetilde{M}, \widetilde{\omega})$  is also.*

(ii) *If  $\langle a_1, a_2, pt \rangle_{\beta}^{\widetilde{M}} \neq 0$  for some  $a_i \in H_*(M) = H_*(\widetilde{M} \setminus E)$  and some  $\beta \in H_2(M) \subset H_2(\widetilde{M})$  then  $(M, \omega)$  is strongly uniruled.*

*Proof.* Consider (i). By hypothesis there is a nonzero invariant  $\langle a_1, a_2, pt \rangle_{\beta}^M$ . Think of  $M$  as the fiber sum of  $(\widetilde{M}, E)$  with  $(\mathbb{P}^n, \mathbb{P}^{n-1})$  and evaluate this invariant by the decomposition formula, putting all constraints into  $\widetilde{M} \setminus E$ . It follows that there is a nonzero relative invariant

$$\langle a_1, a_2, pt | E^{j_1}, \dots, E^{j_s} \rangle_{\beta, \underline{\ell}}^{\Gamma_1, \widetilde{M}, E}$$

with  $s \geq 0$  that is paired with a nonzero invariant  $\langle | E^{n-j_1+1}, \dots, E^{n-j_s+1} \rangle_{\ell \lambda, \underline{\ell}}^{\Gamma_2, \mathbb{P}^n, \mathbb{P}^{n-1}}$ .

Note that  $\widetilde{\beta} = \beta - \ell \varepsilon$ , where  $\ell = \sum \ell_i$ . Since there are no absolute constraints in  $\mathbb{P}^n$ , Lemma 3.2 (i) implies that  $s = \ell = 0$  and  $\Gamma_2 = \emptyset$ . But then Lemma 3.2 (ii) states that the above relative invariant equals the absolute invariant  $\langle a_1, a_2, pt \rangle_{\beta}^{\widetilde{M}}$ . This proves (i).

Now consider (ii). By Lemma 3.2 (ii)  $\langle a_1, a_2, pt \rangle_{\beta}^{\widetilde{M}} = \langle a_1, a_2, pt | \rangle_{\beta}^{\widetilde{M}, E} \neq 0$ . Now use the decomposition formula to evaluate  $\langle a_1, a_2, pt \rangle_{\beta}^M$ , again putting all the absolute constraints in  $\widetilde{M} \setminus E$ . Because there are no absolute constraints in  $(\mathbb{P}^n, \mathbb{P}^{n-1})$ , it follows as before that there are no terms in this formula with  $\Gamma_2 \neq \emptyset$ . Hence

$$\langle a_1, a_2, pt \rangle_{\beta}^M = \langle a_1, a_2, pt | \rangle_{\beta}^{\widetilde{M}} \neq 0$$

as required.  $\square$

**Remark 3.4.** The proof of Proposition 3.5 given below can be adapted to show that the statement in Lemma 3.3 (ii) holds also in the case  $\tilde{\beta} \cdot E = 1$ . However, it is not clear whether it continues to hold when  $\tilde{\beta} \cdot E > 1$ . It is also not known whether the strongly uniruled property persists under blow ups along arbitrary submanifolds.

We now prove the following version of Proposition 2.5.

**Proposition 3.5.** *If there is a nonzero invariant*

$$(3.5) \quad \langle a_1, \dots, a_p, E^{j_1}, \dots, E^{j_q} \rangle_{\tilde{\beta}}^{\widetilde{M}},$$

with  $a_i \in H_*(\widetilde{M} \setminus E) \cong H_*(M)$ ,  $q \geq 1$  and  $\tilde{\beta} \cdot E > 0$  then  $(M, \omega)$  is uniruled.

We prove this by the method of Hu–Li–Ruan [15]. Thus we first think of  $\widetilde{M}$  as the fiber (or Gompf) sum of  $(\widetilde{M}, E)$  with  $(X, E^+)$ , where  $X := \mathbb{P}(\mathcal{O}(-1) \oplus \mathbb{C})$  is the projectivized normal bundle to  $E = \mathbb{P}^{n-1}$  and  $E^+ = \mathbb{P}(\mathcal{O}(-1) \oplus \{0\}) \cong \mathbb{P}^{n-1}$  is the section with positive normal bundle. Using this decomposition, we show that the existence of the nonzero absolute invariant (3.5) implies the existence of a nontrivial relative invariant (3.7) for the pair  $(\widetilde{M}, E)$ . Next we identify the blow down  $M$  as the fiber sum of  $(\widetilde{M}, E)$  with the pair  $(\mathbb{P}^n, \mathbb{P}^{n-1})$  and deduce from the nontriviality of (3.7) the nontriviality of a suitable absolute invariant for  $M$ .

In the following we denote by  $\varepsilon \in H_2(\widetilde{M})$  the class of the line in the exceptional divisor  $E$ , and write the class  $\tilde{\beta} \in H_2(\widetilde{M})$  as  $\beta - d\varepsilon$ , where  $d := \tilde{\beta} \cdot E$  and  $\beta \in H_2(\widetilde{M} \setminus E) = H_2(M)$ . The homology of the exceptional divisor  $E \cong \mathbb{P}^{n-1}$  is generated by the hyperplane class in  $H_{2n-4}(E)$ . As a homology class we identify  $E$  with the class in  $H_{2n-2}(M)$  it represents. Hence the generator of  $H_{2n-4}(E)$  is  $E^2$ , and  $\varepsilon = E^{n-1}$ . The relative constraints for  $(\widetilde{M}, E)$  have the form  $E^j$ ,  $j = 1, \dots, n$ . With our conventions (which are different from [15]) the constraint in  $E$  dual to  $E^j$  is  $-E^{n-j+1}$ , i.e.

$$(3.6) \quad E^j \cdot_E (-E^{n-j+1}) = pt.$$

**Lemma 3.6.** *If there is a nonzero absolute invariant of the form (3.5) for a given  $p \geq 0$  and  $q \geq 1$  then there is a nonzero (connected) relative invariant of the form*

$$(3.7) \quad \langle a'_1, \dots, a'_m \mid E^{i_1}, \dots, E^{i_r} \rangle_{\beta' - \ell\varepsilon, (\ell_1, \dots, \ell_r)}^{\widetilde{M}, E}, \quad i_j \geq 1, \quad \sum \ell_i = \ell > 0,$$

where  $0 \leq m \leq p$ ,  $\beta' \in H_2(M)$  and  $a_i \in H_*(M)$ .

*Proof.* Decompose  $\widetilde{M}$  as the fiber sum of  $\widetilde{M}$  with the ruled manifold  $(X, E^+)$  as above. Apply the decomposition formula to evaluate the nonzero invariant

$$\langle a_1, \dots, a_p, E^{j_1}, \dots, E^{j_q} \rangle_{\tilde{\beta}}^{\widetilde{M}}$$

putting all the  $a_1, \dots, a_p$  insertions into  $\widetilde{M} \setminus E$  and the insertions  $E^{j_1}, \dots, E^{j_q}$  into  $X \setminus E^+$ . Because  $\tilde{\beta} \notin H_2(E)$ , each term in the decomposition formula must correspond

to a splitting  $\widetilde{\beta} = \widetilde{\beta}' + \alpha$  where  $0 \neq \widetilde{\beta}' \in H_2(\widetilde{M})$ . Hence there is a nonzero relative invariant for  $(\widetilde{M}, E)$  in some class  $\widetilde{\beta}' \neq 0$  that goes through all the constraints  $a_1, \dots, a_p$ . It counts curves modelled on the possibly disconnected graph  $\Gamma_1$  and hence is a product of connected invariants, each of which has the form (3.7) for some subset of  $a_1, \dots, a_p$ . Note that each such connected invariant has nonempty intersection with  $E$  because the initial  $\widetilde{\beta}$ -curve in  $\widetilde{M}$  is connected and  $q \geq 1$ .  $\square$

**Lemma 3.7.** *If there is a nonzero relative invariant for  $(\widetilde{M}, E)$  of the form (3.7) for some  $m \geq 0$  and  $r \geq 1$  then there is a nonzero absolute invariant on  $M$*

$$\langle a_1, \dots, a_t, \tau_{k_1} pt, \dots, \tau_{k_s} pt \rangle_{\beta}^M$$

for some  $\beta \neq 0$ ,  $t \leq m$ ,  $1 \leq s \leq r$ ,  $k_j \geq 0$  and  $a_i \in H_*(M)$ .

*Proof.* Choose a class  $\beta$  of minimal energy  $\omega(\beta)$  such that there is a nonzero connected relative invariant in some class  $\widetilde{\beta} := \beta - \ell\varepsilon$ ,  $\ell > 0$ , of the form (3.7) with  $t \leq m$  absolute constraints. Denote by  $s$  the smallest  $r > 0$  such that there is a nonzero invariant (3.7) with  $m = t$ , this class  $\beta$ , and  $r$  relative constraints. Denote the corresponding relative constraints by  $E^{n-j_1+1}, \dots, E^{n-j_s+1}$  (cf. equation (3.6)) and the multiplicities by  $\underline{\ell} = (\ell_1, \dots, \ell_s)$ . Note that  $\sum \ell_j = \ell > 0$ . In particular  $s > 0$ .

Decompose  $M$  into the fiber sum of  $\widetilde{M}$  with  $Y = \mathbb{P}^n$  by identifying  $E \subset \widetilde{M}$  with the hyperplane  $E^+ = \mathbb{P}^{n-1}$  in  $Y$ . Apply the decomposition formula to evaluate

$$(3.8) \quad \langle a_1, \dots, a_t, \tau_{k_1} pt, \dots, \tau_{k_s} pt \rangle_{\beta}^M, \quad k_i := n\ell_i - j_i,$$

putting all the point insertions into  $Y$  and the others into  $\widetilde{M} \setminus E$ . We claim that (3.8) is nonzero.

Note first that by Lemma 3.2(iii) there is a nonzero term  $T$  in this decomposition formula given by taking the product of the nonzero relative invariant (3.7) with  $s$  terms of the form  $\langle \tau_{k_i} pt | E^{j_i} \rangle_{\ell_i \lambda, (\ell_i)}^{\mathbb{P}^n, \mathbb{P}^{n-1}}$ . We need to check that all other terms in this formula vanish.

Consider an arbitrary nonzero term  $T'$  in this formula. It is a product of a relative invariant for  $(\widetilde{M}, E)$  modelled on a graph  $\Gamma_1$  and in some class  $\beta' - d\varepsilon$  with a relative invariant for  $(\mathbb{P}^n, \mathbb{P}^{n-1})$  in class  $d\lambda$ . Since the classes  $\beta' - d\varepsilon$  and  $d\lambda$  combine to give  $\beta'$ , we must have  $\beta' = \beta$ . The minimality of  $\omega(\beta)$  implies that  $\Gamma_1$  is connected, since otherwise each of its components is in a class  $\beta_i - d_i\varepsilon$  with  $0 < \omega(\beta_i) < \omega(\beta)$ .

Now look at the other side. By Lemma 3.2 (i) each connected relative invariant in  $(\mathbb{P}^n, \mathbb{P}^{n-1})$  must go through some absolute constraint. Hence  $\Gamma_2$  has at most  $s$  components. But it cannot have fewer than  $s$  components, because if it did  $\Gamma_1$  would have fewer than  $s$  relative constraints (since the genus zero requirement means that  $\Gamma_1$  meets each component of  $\Gamma_2$  at most once), which contradicts the minimality of  $s$ . Therefore there are  $s$  components of  $\Gamma_2$  and each goes through precisely one absolute constraint. Thus each component is a nonzero 2-point invariant of the form  $\langle \tau_{k_i} pt | E^j \rangle_{df, (d)}^{\mathbb{P}^n, \mathbb{P}^{n-1}}$  for some  $i, d$  and  $1 \leq j \leq n$ . But  $k_i$  has a unique decomposition of the form  $n\ell_i - j_i$  where

$1 \leq j_i \leq n$ . Hence, by Lemma 3.2 (iii) the relative constraints in  $\Gamma_2$  are precisely the  $E^{j_i}, i = 1 \dots, s$ , with intersection multiplicities  $\underline{\ell}$ . Thus  $T' = T$ .  $\square$

**Proof of Proposition 3.5.** By Lemma 3.7 it suffices to show that  $M$  is uniruled iff an invariant of the form  $\langle \tau_{i_1} a_1, \dots, \tau_{i_{k-1}} a_{k-1}, \tau_{i_k} pt \rangle_{\beta}^M$  is nonzero. Hu–Li–Ruan [15, Theorem 4.9] prove this by an inductive argument (similar to the proof of Proposition 4.4 below) that is based on the identity (3.9).  $\square$

**3.3. Blowing down section invariants.** It remains to prove Proposition 2.19. We begin with a general remark about the invariants of pairs of spaces  $(X, D)$  that are fibered over  $\mathbb{P}^1$  with fiber  $(F, D_F)$ . It concerns invariants in the class  $\beta \in H_2(F)$  of a fiber. Denote by  $\iota : H_*(F) \rightarrow H_*(X)$  the map induced by inclusion.

**Lemma 3.8.** *Let  $\pi : X \rightarrow \mathbb{P}^1$  be a Hamiltonian fibration with fiber  $F$  that induces a fibration  $F_D \rightarrow D \rightarrow \mathbb{P}^1$  on the divisor  $D$ . If  $a_1 := \iota(a_1)$  is a fiber class and  $\beta \in H_2(F)$  then*

$$\langle \iota(a_1), \dots | b_1, \dots \rangle_{\beta, \underline{d}}^{X, D} = \langle a_1, a_2 \cap F, \dots | b_1 \cap D_F, \dots \rangle_{\beta, \underline{d}}^{F, D_F}.$$

Similarly, if  $b_1 := \iota(b_1)$  is a fiber class then

$$\langle a_1, \dots | \iota(b_1), \dots \rangle_{\beta, \underline{d}}^{X, D} = \langle a_1 \cap F, \dots | b_1, b_2 \cap D_F, \dots \rangle_{\beta, \underline{d}}^{F, D_F}.$$

In particular, any invariant with two fiber insertions must vanish. Corresponding results hold for absolute invariants.

*Proof.* This holds because one can define  $\overline{\mathcal{M}}_{\beta}^{[vir]}$  in such a way that each of its elements represents a curve lying in some fiber of  $\pi$ . Therefore, if we represent the class  $a_1$  by a cycle lying in the fiber  $F_z := \pi^{-1}(z)$  all the curves that contribute to the invariant lie entirely in this fiber and hence must intersect the other cycles in representatives for  $a_i \cap F, b_j \cap F$ . The details of the proof in the absolute case are spelled out in [27, Prop 1.2(ii)]. The relative case is similar.  $\square$

We saw earlier that  $\tilde{P}$  is the blow up of  $P$  along a section  $s_{\max}$  in  $F_{\max} \times \mathbb{P}^1$ . Similarly,  $\tilde{P}'$  is the blow up of  $P'$  along the corresponding section  $s'_{\min}$  in  $F'_{\min} \times \mathbb{P}^1$ . Note that  $s'_{\min}$  has normal bundle  $\mathcal{O}(1) \oplus \mathbb{C}^{n-1}$ , where  $\mathcal{O}(m) \rightarrow \mathbb{P}^1$  denotes the holomorphic line bundle of Chern class  $m$ . We denote by  $D$  the exceptional divisor in  $\tilde{P}'$ . Thus it is a  $\mathbb{P}^{n-1}$ -bundle

$$\pi_D : D := \mathbb{P}(\mathcal{O}(1) \oplus \mathbb{C}^{n-1}) \rightarrow \mathbb{P}^1.$$

Consider the section  $s_D := \mathbb{P}(\{0\} \oplus \dots \oplus \{0\} \oplus \mathbb{C})$  of  $D \rightarrow \mathbb{P}^1$ . It lies with trivial normal bundle in the product divisor  $V := \mathbb{P}(\{0\} \oplus \mathbb{C}^{n-1}) \cong s_D \times \mathbb{P}^{n-2}$  and so  $c_1(s_D) = 3$ . The exceptional divisor  $E$  of  $\tilde{M}$  can be identified with  $\pi_D^{-1}(pt)$ . The line  $\varepsilon$  in  $E$  intersects  $V$  once. Therefore classes  $s_D - m\varepsilon$  with  $m > 1$  have no holomorphic representatives in  $D$  since they have negative intersection with  $V$  and cannot be represented in  $V$  itself. On the other hand the section class  $s_Z := s_D - \varepsilon$  has the unique representative  $\mathbb{P}(\mathcal{O}(1) \oplus \{0\} \oplus \dots \oplus \{0\})$ .

The proof of Proposition 2.19 has two steps: we need to show that the absolute invariant in  $\tilde{P}'$  equals a suitable relative invariant for  $(\tilde{P}', D)$ , and then look at the blow down correspondence. The first of these steps turns out to be the hardest, requiring a detailed study of the invariants of the ruled manifold  $(Y, D^+)$ . Here  $Y$  is the projectivization  $\mathbb{P}(L \oplus \mathbb{C})$  of the normal bundle to  $D \subset \tilde{P}'$  and  $D^+ := \mathbb{P}(L \oplus \{0\})$  as above; see Fig. 3.1. Thus  $(Y, D^+)$  fibers over  $\mathbb{P}^1$  with fiber  $(X, E^+)$  equal to the pair consisting of the 1-point blow up of  $\mathbb{P}^n$  and hyperplane class  $E^+$ , and there is a commutative diagram

$$\begin{array}{ccccc} \mathbb{P}^1 & \rightarrow & (X, E^+) & \xrightarrow{\rho_X} & E \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \rightarrow & (Y, D^+) & \xrightarrow{\rho_Y} & D \\ & & \pi_Y \downarrow & & \pi_D \downarrow \\ & & \mathbb{P}^1 & = & \mathbb{P}^1. \end{array}$$

We denote by  $D_0$  the divisor  $\mathbb{P}(\{0\} \oplus \mathbb{C})$  in  $Y$  that is disjoint from  $D^+$ , and by  $s_{D_0}$  the section in  $D_0$  corresponding to  $s_D$ . Since  $D_0$  has the same normal bundle as  $D \subset \tilde{P}'$ ,  $c_1(s_{D_0}) = 3$ . Also  $D_0 \cap X$  can be identified with the exceptional divisor  $E$  in  $\tilde{M}$ . Thus  $c_1(\lambda_X) = n - 1$  where  $\lambda_X$  is the class of a line in  $X \cap D_0$ . Note that  $H_2(Y)$  is generated by  $s_{D_0}, \lambda_X$  and  $f$ . Moreover  $\rho_Y(s_{D_0} - \lambda_X) = s_Z$ . Thus we write  $s_{Z_0} := s_{D_0} - \lambda_X$ .

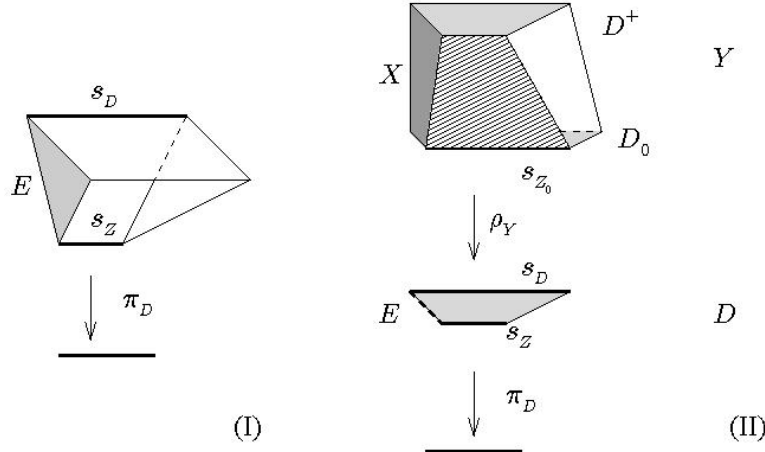


FIGURE 3.1. Diagram (I) is a 3D picture of the moment polytope of the toric manifold  $D$ , while (II) is a 3D picture of that for  $Y$  with  $D$  reduced to 2-dimensions

The next lemma contains some preliminary calculations. Here, and subsequently, we choose the relative constraints  $b_i^*$  from the elements of the following self-dual basis for  $H_*(D)$ :

$$D^i \in H_{2(n+1-i)}(D), \quad 1 \leq i \leq n, \quad E^j \in H_{2(n-j)}(D), \quad 1 \leq j \leq n.$$

As always  $E^j$  denotes the image in  $D$  of the class in  $\widetilde{M}$  represented by the  $j$ -fold intersection of  $E$  with itself in  $\widetilde{M}$ . Thus the  $E^j$  are fiber constraints for the projection  $\pi_D : D \rightarrow \mathbb{P}^1$ , while the  $D^i$  are nonfiber constraints. Also  $D^i \cdot E^j = 0$  unless  $i+j = n+1$ .

**Lemma 3.9.** (i) *If  $m \leq -2$ , all invariants of  $(Y, D^+)$  in the classes  $s_{D_0} + m\lambda_X + df$  vanish. They also vanish in the classes  $m\lambda_X + df$  with  $m \leq -1$ .*

(ii) *Let  $j = 1$  or  $2$ . Then  $\langle E^j | b_1^*, \dots, b_r^* \rangle_{s_{D_0} + df + m\lambda_X, \underline{d}}^{Y, D^+} = 0$  for all  $d$  such that  $0 \leq d \leq m+1$ .*

(iii) *Let  $d > 0$  and  $1 \leq j \leq 2$ . If  $\langle E^j | b_1^*, \dots, b_r^* \rangle_{df + m\lambda_X, \underline{d}}^{Y, D^+} \neq 0$  then  $b_i^*$  is a nonfiber constraint for all  $i$ . Moreover if  $n \geq 3$  and  $r = 1$  then  $m = 0$  and  $d = 1$ . The same holds if  $n = 2$  and  $j = r = 1$ .*

(iv) *If  $\langle | b_1^*, \dots, b_r^* \rangle_{df + m\lambda_X, (d)}^{Y, D^+} \neq 0$  for some  $m \geq 0$  and  $1 \leq r \leq 2$  then  $m < d$ .*

*Proof.* (i) Since  $\rho_Y : Y \rightarrow D$  is holomorphic every class with nonzero GW invariants in  $Y$  must map to a class with a holomorphic representative in  $D$ . Hence their image must have nonnegative intersection with the divisor  $V$  containing  $s_D$ . Since  $s_D \cdot V = 1$  and  $\varepsilon \cdot V = -1$  the result follows.

Now consider (ii). A dimensional calculation shows that when  $m+1 \geq d$  the invariant (ii) can be nonzero only if  $m = -1$ ,  $j = 2$  and  $d = r = 0$ . Hence it remains to consider the invariant  $\langle E^2 | \rangle_{s_{Z_0}}^{Y, D^+}$ , where  $s_{Z_0} := s_{D_0} - \lambda_X$ . Although this could be nonzero as far as dimensions are concerned, the geometry shows that it must be zero. To see this, consider the holomorphic fibration  $\pi_Y : Y \rightarrow \mathbb{P}^1$ . Since the normal bundle to  $s_Z$  in  $D$  is a sum of copies of  $\mathcal{O}(-1)$ , there is a unique curve in  $D$  in class  $s_Z$ . Since any holomorphic curve in  $Y$  in class  $s_{Z_0}$  projects to an  $s_Z$  curve in  $D$ , the  $s_{Z_0}$  curves in  $Y$  lie in the surface  $\rho_Y^{-1}(s_Z)$ ; cf. Figure 3.1 where this surface is crosshatched. This surface has 2-dimensions, while the constraint  $E^2$  has codimension 3 in  $Y$ . (Remember that  $E$  is the exceptional divisor in  $\widetilde{M}$  and  $E^2$  denotes its intersection in  $\widetilde{M}$ .) Hence there is a cycle in  $Y$  representing  $E^2$  that does not meet any  $s_{Z_0}$  curves. This proves (ii).

In case (iii) we are counting curves in a fiber class of the holomorphic projection  $\pi_Y : Y \rightarrow \mathbb{P}^1$ . Since  $E^j$  is a fiber constraint, the first claim is an immediate consequence of Lemma 3.8. To prove the second, note that by Lemma 3.8 the invariant reduces to one in  $(X, E^+)$ . A count of dimensions shows that  $m = 0$ . But then  $f$  is a fiber class for the projection  $\rho_X$  so that the invariant is nonzero only if  $\rho_X(b) \cap \rho_X(E^j) \neq 0$ . A further dimension count now implies that  $d = 1$  (or one can use the fact that one is counting  $d$ -fold covered spheres in  $\mathbb{P}^1$  with 2 homological constraints). The proof of the third claim is similar.

Consider (iv). Since the fibers  $X$  of the bundle  $Y \rightarrow \mathbb{P}^1$  can all be identified (as Kähler manifolds) with the 1-point blow up of  $\mathbb{P}^n$  (with  $X \cap D^+$  the hyperplane class), this invariant can be nonzero only if one of the relative constraints, say  $b_1^*$  is a fiber constraint while the others are not. Therefore  $\delta_b \geq r - 1$ . Now count dimensions.  $\square$

**Lemma 3.10.** *Suppose that  $(M, \omega)$  is not uniruled and that  $n \geq 3$ . Then*

$$\langle |E^2\rangle_{\sigma-\varepsilon}^{\tilde{P}', D} = \langle E^2\rangle_{\sigma-\varepsilon}^{\tilde{P}'}$$

*Proof.* As in the proof of Lemma 3.6 decompose  $\tilde{P}'$  as the connected sum of  $(\tilde{P}', D)$  with the ruled manifold  $(Y, D^+)$  considered above. Consider a typical term in the decomposition formula for  $\langle E^2\rangle_{\sigma-\varepsilon}^{\tilde{P}'}$  where we put the constraint  $E^2$  in  $D_0 \subset Y$ :

$$\langle |b_1, \dots, b_r\rangle_{\alpha_1 - \ell\varepsilon, \ell}^{\Gamma_1, \tilde{P}', D} \langle E^2 | b_1^*, \dots, b_r^* \rangle_{\alpha_2 + \ell f, \ell}^{\Gamma_2, Y, D^+}$$

where  $\alpha_1 \in H_2(P')$  and  $\alpha_2 \in H_2(D_0)$ . (Note that the intersections on each side of  $D$  match because  $\alpha_1 \cdot D = 0$  for all  $\alpha_i \in H_2(P')$ .) There is one term in this formula with  $\alpha_1 = \sigma, \ell = r = 1$  and  $b_1 = E^2$ . The first factor is then  $\langle |E^2\rangle_{\sigma-\varepsilon}^{\tilde{P}', D}$  while the second is  $\langle E^2 | D^{n-1} \rangle_f^{Y, D^+} = 1$ . Hence we must see that all other terms vanish.

The argument has several steps. Throughout the following discussion the words *fiber/section class* and *fiber/nonfiber constraint* apply to the fibration  $\pi$  over  $\mathbb{P}^1$ . Moreover a component of  $\Gamma_1$  or  $\Gamma_2$  is called a *fiber component* if it represents a class in the homology  $H_2(\tilde{M})$  of the fiber.

(a) *No component of  $\Gamma_1$  can lie in a fiber class  $\tilde{\beta}$  and go through a fiber constraint  $b$ .* For if there were such a component, Lemma 3.8 would imply that the corresponding invariant  $\langle |b, b_2, \dots\rangle_{\tilde{\beta}, k}^{\tilde{P}', D}$  equals the fiber invariant  $\langle |b, b_2 \cap \tilde{M}, \dots\rangle_{\tilde{\beta}, k}^{\tilde{M}, E}$ , and so  $(M, \omega)$  would be uniruled by Lemma 3.7.

(b) *Every fiber component of  $\Gamma_2$  has  $r \leq 2$ ; moreover a fiber component through  $E^j$  has  $r = 1$ .* If a component of  $\Gamma_2$  has two nonfiber relative constraints, say  $b_1^*, b_2^*$ , then the genus zero restriction implies that at least one of the dual fiber constraints  $b_1, b_2$  must lie in a fiber component of  $\Gamma_1$  (since  $\Gamma_1$  has at most one section component.) But this contradicts Step (a). Thus the second claim holds by the first part of Lemma 3.9(iii), while the first claim holds by Lemma 3.8 which says that only one of the relative constraints for each component can be a fiber class.

(c) *If  $\Gamma_2$  is a section class, its section component must go through the absolute constraint.* For otherwise  $\Gamma_2$  contains an invariant of the type considered in Lemma 3.9 (iii) whose relative constraint is dual to a fiber constraint. Hence  $\Gamma_1$  would have to contain a fiber component with a fiber constraint, contradicting (a).

(d)  *$\Gamma_2$  cannot be a section class.* If  $\Gamma_2$  is a section class then by (c) the fiber components in  $\Gamma_2$  have no absolute constraints. Hence by Lemma 3.9(i) and (iv) (which applies by (b)) the fiber classes have the form  $d_i f + m_i \lambda_X$  with  $0 \leq m_i < d_i$ . Suppose the section class is  $s_{D_0} + d_0 f + m_0 \lambda_X$ . Then these classes in  $\Gamma_2$  combine with  $\alpha_1 - d\varepsilon$  to make  $\sigma - \varepsilon$  where  $\sigma \in H_2(P')$ . But  $\lambda_X$  projects to the class  $\varepsilon$  in  $D$ . Hence

$$-d + \sum_{i \geq 0} m_i = \sum_{i \geq 0} (m_i - d_i) = -1.$$

We saw above that  $m_i - d_i \leq -1$  for all  $i > 0$ . Therefore either  $\Gamma_2$  is connected and  $m_0 - d_0 = -1$  or  $m_0 - d_0 \geq 0$ . Neither of these cases occur by Lemma 3.9(ii).

(e) *Completion of the proof.* As in step (d)  $d = \sum d_i$  and  $\sum_{i \geq 0} (m_i - d_i) = -1$ . The component of  $\Gamma_2$  through  $E^2$  has  $r = 1$  by (b) and hence  $m_0 = 0$  and  $d_0 = 1$  by Lemma 3.9(iii). Moreover all other components of  $\Gamma_2$  have  $m_i \geq 0$  by Lemma 3.9(i) and hence  $m_i < d_i$  by Lemma 3.9(iv). Therefore  $\Gamma_2$  is connected and lies in class  $f$ . Hence  $\Gamma_1$  is also connected and lies in class  $\sigma - \varepsilon$ . Therefore this is the term already considered.  $\square$

We now relate the relative invariant in  $\tilde{P}'$  to an absolute invariant in  $P'$ . To this end, consider the  $\mathbb{P}^n$ -bundle  $W := \mathbb{P}(\mathcal{O}(1) \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}) \rightarrow \mathbb{P}^1$ . It contains a copy  $D^+ := \mathbb{P}(\mathcal{O}(1) \oplus \mathbb{C}^{n-1} \oplus \{0\})$  of  $D$  with normal bundle  $L_D^*$  (where as usual  $L_D$  is the normal bundle of  $D$ ), and a disjoint section  $s_W := \mathbb{P}(\{0\} \oplus \{0\} \oplus \mathbb{C})$ . This can be identified with  $s_D$  and so  $c_1(s_W) = 3$ . We also denote by  $\lambda$  the line in the fiber of  $W \rightarrow \mathbb{P}^1$ .

**Lemma 3.11.** *Let  $n \geq 2$ . Then:*

- (i)  $\langle \tau_1 pt | b_1^*, \dots, b_r^* \rangle_{s_W + d\lambda, \underline{d}}^{W, D^+} = 0$  for all  $b_i^* \in H_*(D^+)$ .
- (ii)  $\langle \tau_1 pt | b_1^*, \dots, b_r^* \rangle_{d\lambda, \underline{d}}^{W, D^+} \neq 0$  only if  $d = r = 1$  and  $b_1^*$  is a multiple of  $(D^+)^{n-1}$ .
- (iii) If  $d > 0$ ,  $\langle |b_1^*, \dots, b_r^* \rangle_{d\lambda, \underline{d}}^{W, D^+} = 0$  for all  $b_i^*$ .

*Proof.* (i) Since  $c_1(s_W) = 3$  and  $c_1(\lambda) = n + 1$  this invariant is nonzero only if

$$n + 1 + 3 + d(n + 1) + r + 1 - 3 + \delta_b - 1 - (d - r) = (r + 1)(n + 1),$$

which reduces to  $(d - r)n + \delta_b + r = 0$ . Hence it cannot hold when  $d \geq r \geq 1$ .

In case (ii) the dimensional condition is the same except that LHS is reduced by  $3 = c_1(s_W)$ . If  $d > r$  then we need  $n = 2, r = 1, d = 2$  and  $\delta_b = 0$ . But then the invariant has two point constraints and hence vanishes by Lemma 3.8: it counts curves in a fiber class through points that could lie in different fibers. If  $d = r$ , we need  $\delta_b + r = 3$ . If  $r > 1$  then at least one of the  $b_i^*$  is a point constraint, and the invariant vanishes as before. Hence we must have  $r = 1, \delta_b = 2$ . Since  $b$  cannot be a fiber class, it reduces to  $\langle \tau_1 pt | b \cap F \rangle_{\mathbb{P}^n, \mathbb{P}^{n-1}}$  which is nonzero when  $b = (D^+)^{n-1}$  by Lemma 3.2(iii).

Claim (iii) holds by a dimension count (or by Lemma 3.2(i)).  $\square$

**Corollary 3.12.** *Suppose that  $(M, \omega)$  is not uniruled and that  $n \geq 2$ . Then there is  $c \neq 0$  such that  $\langle \tau_1 pt \rangle_{\sigma}^{P'} = c \langle |E^2 \rangle_{\sigma - \varepsilon}^{\tilde{P}', D}$ .*

*Proof.* Identify  $P'$  with the fiber sum of  $(\tilde{P}', D)$  with  $(W, D^+)$  and calculate  $\langle \tau_1 pt \rangle_{\sigma}^{P'}$  using the decomposition formula, putting the point into  $W \setminus D'$ . A typical term in the decomposition formula (3.4) has the form

$$\langle |b_1, \dots, b_r \rangle_{\alpha_1 - d\varepsilon, \underline{d}}^{\Gamma_1, \tilde{P}', D} \langle \tau_1 pt | b_1^*, \dots, b_r^* \rangle_{\alpha_2 + d\lambda, \underline{d}}^{\Gamma_2, W, D^+}$$

where  $\alpha_2 = 0$  or  $s_W$ , and  $\lambda$  is the class of a line in the fiber of  $W$ .

We saw in Lemma 3.11 that in a nonzero term  $\alpha_2 = 0$ . Further each nonzero component of  $\Gamma_2$  must have an absolute constraint and  $d = r = 1$ . Hence there is only one such component and it has  $b_1^* = (D^+)^{n-1}$ . Hence  $b_1 = E^2$  and  $\Gamma_1$  must consist of the single component  $\langle |E^2\rangle_{\sigma-\varepsilon, (1)}^{\tilde{P}', D}$ . Hence there is only one term in the decomposition formula. The result follows.  $\square$

**Proof of Proposition 2.19(i).**

This follows immediately from Corollary 3.12 and Lemma 3.10.  $\square$

We now turn to the proof Proposition 2.19(ii) which concerns the case  $n = 2$ .

**Lemma 3.13.** *Let  $n = 2$ . If  $\langle E \rangle_{\sigma-2\varepsilon}^{\tilde{P}'} \neq 0$  then  $\langle |E, D \rangle_{\sigma-2\varepsilon, (1,1)}^{\tilde{P}', D} \neq 0$ .*

*Proof.* As before, we calculate the nonzero invariant  $\langle E \rangle_{\sigma-2\varepsilon}^{\tilde{P}'}$  by considering  $\tilde{P}'$  as a connected sum of  $(\tilde{P}', D)$  with  $(Y, D^+)$ , putting  $E$  into  $Y$ . Steps (a), (b) and (c) of the proof of Lemma 3.10 go through as before. So consider (d) and suppose that  $\Gamma_2$  is a section class. The homology class count is now

$$\sum_{i \geq 0} (m_i - d_i) = -2.$$

By Lemma 3.9 (i),  $m_0 \geq -1$  and  $m_i \geq 0$ . Therefore by Lemma 3.9 (ii)  $m_0 - d_0 < -1$  and by Lemma 3.9 (iv)  $m_i - d_i \leq -1$  for  $i > 0$ . Therefore  $\Gamma_2$  must be connected and  $m_0 - d_0 = -2$ . Since  $d_0 \geq 0$  this gives  $m_0 \leq -2$ , which is impossible.

Since  $m_i - d_i \leq -1$  for fiber constraints, this argument also shows that  $\Gamma_2$  has at most two fiber components. One of these goes through the  $E$  constraint and has  $m = 0, d = 1$  by Lemma 3.9 (iii). Hence there is exactly one other that has  $0 \leq m' = d' - 1$  and  $r \leq 2$ . Since this is a fiber invariant it reduces to an invariant in  $(X, E^+)$  of the form

$$\langle |a_1, \dots, a_r \rangle_{d'f + (d'-1)\lambda_X, d'}^{X, E^+}.$$

Since  $n = 2$ , this is possible only if  $d' = 1$ . Hence both components of  $\Gamma_2$  lie in the class  $f$ , and the  $\Gamma_2$  factor must be

$$\langle E|D^2 \rangle_f^{Y, D^+} \langle |pt \rangle_f^{Y, D^+}.$$

Hence  $\langle |E, D \rangle_{\sigma-2\varepsilon, (1,1)}^{\tilde{P}', D} \neq 0$ .  $\square$

**Proof of Proposition 2.19(ii).**

Blow down  $D$  by summing with  $(W, D^+)$  as in Corollary 3.12. Consider the resulting decomposition formula for  $\langle pt, s_W \rangle_{\sigma}^{P'}$  where the absolute constraints are put into  $W$ .

(a) *There is no term in this formula with  $\Gamma_1 = \emptyset$ .* In such a term  $\sigma$  would be a section class in  $W$  with  $c_1^W(\sigma) = 3$ . Since the only section classes are  $s_W + m\lambda$  and  $c_1^W(\lambda) = 3$ , we must have  $\sigma = s_W$ . But the only holomorphic representatives of  $s_W$  that meet the product divisor  $Z = \mathbb{P}(\{0\} \oplus \mathbb{C}^2) \cong s_W \times \mathbb{P}^1$  are parallel copies of  $s_W$  lying entirely in

$Z$ . Hence one can place the constraints  $pt, s_W$  in  $Z$  in such a way that no holomorphic  $s_W$ -curve meets them both.

(b)  $\Gamma_2$  *cannot be a section class*. For if so, by (a), there would have to be a nonzero invariant

$$\langle a_1, \dots, a_m | b_1^*, \dots, b_r^* \rangle_{s_W + d\lambda, \underline{d}}^{W, D^+}, \quad d > 0,$$

where  $0 \leq m \leq 2$  and the  $a_i$  form a subset of  $\{pt, s_W\}$ . This is impossible for dimensional reasons.

(c)  $\Gamma_2$  *has at most 2 components. Each has  $d = r = 1$* . By (b) each component in  $\Gamma_2$  lies in some class  $df$ . Even if one puts all the constraints into one component, a dimension count shows that the invariant  $\langle pt, s_W | b_1, \dots, b_r \rangle_{d\lambda, \underline{d}}^{W, D^+}$  is nonzero only if  $d = 1$ . It follows that  $d = 1$  in all cases. Thus we are counting lines, and so each component must have two constraints and hence since  $r = 1$  each must have an absolute constraint.

Therefore the only possibilities for  $\Gamma_2$  are: one component of type

$$\langle pt, s_W | D \rangle_{\lambda, (1)}^{W, D^+} = \langle pt, pt | E^+ \rangle_{\lambda, (1)}^{F, E^+},$$

or the two terms in the product

$$\langle pt | s_D \rangle_{\lambda, (1)}^{W, D^+} \langle s_W | pt \rangle_{\lambda, (1)}^{W, D^+}.$$

The corresponding  $\Gamma_1$  terms are

$$\langle |pt \rangle_{\sigma - \varepsilon}^{\tilde{P}', D}, \quad \text{and} \quad \langle |D, E \rangle_{\sigma - 2\varepsilon}^{\tilde{P}', D}.$$

We saw in Lemma 3.13 that the second of these terms is nonzero. Therefore, if the first term vanishes,  $\langle pt, s_W \rangle_{\sigma}^{P'} \neq 0$ . However, if the first term does not vanish, we may apply Corollary 3.12 to conclude that  $\langle \tau_1 pt \rangle_{\sigma}^{P'} \neq 0$ . This completes the proof.  $\square$

**3.4. Identities for descendent classes.** To complete the proof of Theorem 1.1 we must establish Lemma 2.21. Its proof is based on some identities for genus zero Gromov–Witten invariants that we now explain. We shall denote by  $\psi_i$  the first Chern class of the cotangent bundle to the domain of a stable map at the  $i$ th marked point, and by  $\overline{\mathcal{W}}_{\beta}$  the configuration space of all (not necessarily holomorphic) stable maps in class  $\beta$ . If  $i \neq j$  consider the subset  $D_{i, \beta_1 | j, \beta_2}$  of  $\overline{\mathcal{W}}_{\beta}$  consisting of all stable maps with at least two components, one in class  $\beta_1$  containing  $z_i$  and the other in class  $\beta_2 := \beta - \beta_1$  containing  $z_j$ . Further, if  $i, j, k$  are all distinct consider the subset  $D_{i | jk}$  of  $\overline{\mathcal{W}}_{\beta}$  consisting of all stable maps with at least two components, one in some class  $\beta_1$  containing  $z_i$  and the other in class  $\beta - \beta_1$  containing  $z_j, z_k$ . The virtual moduli cycle  $\overline{\mathcal{M}}_{\beta}^{[vir]}$  has a natural map to  $\overline{\mathcal{W}}_{\beta}$  that may be chosen to be transverse to the above subsets. These therefore pullback to real codimension 2 sub(branched) manifolds in  $\overline{\mathcal{M}}_{\beta}^{[vir]}$  which we denote by the same names.

In [22, Thm 1], Lee–Pandharipande prove the following identities in the algebraic case:

$$(3.9) \quad \psi_i = D_{i|jk}$$

$$(3.10) \quad ev_i^*(L) = ev_j^*(L) + (\beta \cdot L) \psi_j - \sum_{\beta_1 + \beta_2 = \beta} (\beta_1 \cdot L) D_{i, \beta_1 | j, \beta_2},$$

where the two sides are considered as elements of an appropriate Picard group and  $L \in H_{2n-2}(M; \mathbb{Z}) \cong H^2(M)$  is any divisor. We might try to think of these equations as identities in the second cohomology group of  $\overline{\mathcal{M}}_\beta^{[vir]}$ . However,  $\overline{\mathcal{M}}_\beta^{[vir]}$  is not really a space in its own right since it is only well defined up to certain kinds of cobordism. Hence it does not have a well defined  $H^2$ . Therefore we shall interpret these equations as identities for Gromov–Witten invariants. For example, (3.9) states that for all  $k \geq 3$ , all  $i_1 > 0$ ,  $i_j \geq 0$  and all classes  $\beta \in H_2(M)$ ,  $a_i \in H_*(M)$ ,

$$\langle \tau_{i_1} a_1, \dots, \tau_{i_k} a_k \rangle_\beta^M = \sum_{j, S, \beta_1 + \beta_2 = \beta} \langle \tau_{i_1-1} a_1, \dots, \xi_j \rangle_{\beta_1}^M \langle \xi_j^*, \tau_{i_2} a_2, \tau_{i_3} a_3, \dots \rangle_{\beta_2}^M.$$

Here the sum is over the elements of a basis  $\xi_j$  of  $H_*(M)$  (with dual basis  $\xi_j^*$ ), all decompositions  $\beta_1 + \beta_2 = \beta$  of  $\beta$ , and all distributions  $S$  of the constraints  $\tau_{i_4} a_4, \dots, \tau_{i_k} a_k$  over the two factors, subject only to the restriction that if  $\beta_1 = 0$  then the first factor must include at least one of the  $\tau_{i_\ell} a_\ell$ ,  $\ell \geq 4$ , so that it is stable. The second identity has a similar interpretation; cf. Lemma 3.14.

Identity (3.9) is proved for the space of genus zero stable curves  $\overline{\mathcal{M}}_{0,k}$  by Getzler at the beginning of §4 in [8]. It holds for stable maps because the descendent class  $\psi_i$  differs from the pullback of the corresponding class on  $\overline{\mathcal{M}}_{0,k}$  precisely by the boundary class consisting of the elements in  $\overline{\mathcal{M}}_\beta^{[vir]}$  such that the component containing the  $i$ th marked point is unstable. Note that the proof of Proposition 3.5 and hence of Theorem 1.1 requires the full strength of this identity.

Lee–Pandharipande’s proof of (3.10) in [22, §1.1] adapts readily to the symplectic case provided that one has a good framework to work in: one needs a model for the virtual cycle  $\overline{\mathcal{M}}_\beta^{[vir]}$  in which the elements in  $H_2(\overline{\mathcal{M}}_\beta^{[vir]})$  have “nice” representatives. One could work with ad hoc methods as in [27] and interpret the idea of transversality using the normal cones to the strata of the moduli space provided by the gluing data, but it is cleaner to work in the category of polyfolds newly introduced by Hofer–Wysocki–Zehnder since this provides the moduli spaces with a smooth structure. Note that because one needs to use multivalued perturbations the moduli spaces are in general branched. (Here one can work with various essentially equivalent definitions, see [28, Def 3.2] and [13, Def 1.3].)

To prove Theorem 1.1 we only need the very special case of this identity relevant to Lemma 2.21. We will sketch the proof in this case to give the general idea. Note that here  $i = 1, j = 2$  and the term  $ev_2^*(L)$  does not contribute since  $L \cap pt = 0$ . When we prove Lemma 2.21 we will explain how to construct the moduli space  $B$  in this context;

since we are working in a fibered 6 dimensional manifold, it is not hard to obtain it by ad hoc methods.

**Lemma 3.14.** *Given a class  $\sigma \in H_2(M)$  and a divisor  $H \in H_{2n-2}(M)$ , denote by  $B := \overline{\mathcal{M}}_{\sigma,2}^{[vir]}(pt)$  the virtual moduli space of stable maps in class  $\sigma$  with two marked points, one through a fixed point  $x_0 \in M$  and the other through a cycle representing  $H$ . Suppose that  $B$  can be constructed as a smooth branched manifold of real dimension 2 that is transverse to the strata in  $\overline{\mathcal{W}}$ . Then, for any divisor  $L \in H_{2n-2}(M)$*

$$\langle LH, pt \rangle_{\sigma}^M = (\sigma \cdot L) \langle H, \tau_1 pt \rangle_{\sigma}^M - \sum_{j, \alpha_1 + \alpha_2 = \sigma} (\alpha_1 \cdot L) \langle H, \xi_j \rangle_{\alpha_1}^M \langle \xi_j^*, pt \rangle_{\alpha_2}^M.$$

*Proof.* For present purposes we may think of a branched 2-dimensional manifold as the realization of a rational singular 2-cycle formed by taking the union of a finite number of (positively) rationally weighted oriented 2-simplexes  $(\lambda_i, \Delta_i)$ , where  $\lambda_i \in \mathbb{Q}^{>0}$ , appropriately identified along their boundaries. (This might have singularities at the vertices, but since these are codimension 2 this does not matter. See [28, 13] for a more complete description.) Since  $B$  is transverse to the strata in  $\overline{\mathcal{W}}_{\beta}$ , there is a finite set  $\text{Sing}_B$  of points in  $B$  corresponding to stable maps whose domain has two components, and the other points have domain  $S^2$ . We assume that each  $b \in \text{Sing}_B$  lies in the interior of a 2-simplex and therefore has a weight  $\lambda_b$ .

Let  $\pi : C \rightarrow B$  be the universal curve formed by the domains with evaluation map  $f : C \rightarrow M$ . The marked points define two disjoint sections  $s_1, s_2$  of  $C \rightarrow B$ , numbered so that  $f \circ s_1(b) \in H, f \circ s_2(b) = x_0$ . We may assume that these sections are transverse to the pullback divisor (codimension 2 cycle)  $f^*L$ . Note that  $C$  is the blow up of an oriented  $S^2$ -bundle  $P \rightarrow B$  at a finite number of points, one in each fiber over  $\text{Sing}_B$ . For each such  $b$  we choose the exceptional sphere  $E_b$  to be the component that does not contain  $s_2(b)$ . Since the marked points never lie at nodal points of the domain, the sections  $s_1, s_2$  blow down to disjoint sections  $s'_1, s'_2$  of the  $S^2$  bundle  $P \rightarrow B$ . Hence  $P$  can be considered as the projectivization  $\mathbb{P}(V \oplus \mathbb{C})$  where  $V \rightarrow B$  is a line bundle and  $s'_1 = \mathbb{P}(V \oplus 0), s'_2 = \mathbb{P}(0 \oplus \mathbb{C})$ . Note also that  $s_2 = s'_2$  while  $s_1$  is the blow up of  $s'_1$  over those points  $b \in \text{Sing}_B$  for which  $s_1 \cap E_b \neq \emptyset$ . For such  $b \in \text{Sing}_B$  set  $\delta_1(b) := \lambda_b$ , and otherwise set  $\delta_1(b) := 0$ .

If  $B$  were a manifold then, as in the proof of Theorem 1 in [22, §1.1], we would have

$$\langle H, \tau_1 pt \rangle_{\sigma}^M = -s_2 \cdot s_2 = - \int_B c_1(V).$$

In our situation we must take the weights on  $B$  into account: each point  $y$  in the intersection  $s_2 \cdot s_2$  should be given the (positive) weight  $\lambda(\pi(y)) \in \mathbb{Q}$  of the corresponding point  $\pi(y) \in B$  as well as the sign  $o(y) \in \{\pm 1\}$  of the intersection. Thus we find

$$\langle H, \tau_1 pt \rangle_{\sigma}^M = - \sum_{y \in s_1 \cdot s_1} o(y) \lambda(\pi(y)) =: - \int_B c_1(V).$$

We want to calculate

$$\begin{aligned}
\langle LH, pt \rangle_\sigma^M &= \langle LH, pt \rangle_\sigma^M - \langle H, Lpt \rangle_\sigma^M \\
&= \int_B \text{ev}_1^*(L) - \text{ev}_2^*(L) \\
&= \int_{s_1} f^*(L) - \int_{s_2} f^*(L) \\
&= \sum_{y \in s_1 \cdot f^*(L)} o(y) \Lambda(\pi(y)) - \sum_{y \in s_2 \cdot f^*(L)} o(y) \Lambda(\pi(y)).
\end{aligned}$$

This can be done just as in [22]. The divisor  $f^*(L)$  intersects a generic fiber  $F$  of  $C \rightarrow B$  with multiplicity  $\sigma \cdot L$ , and intersects the exceptional divisor  $E_b$  with multiplicity  $\alpha_1(b) \cdot L$ , where  $\alpha_1(b) = [f_*(E_b)]$ . Since  $H_2(C; \mathbb{Q})$  splits as the sum  $H_2(P; \mathbb{Q}) \oplus \sum_{b \in \text{Sing}_B} [E_b] \mathbb{Q}$ , we may consider the difference  $[s_1] - [s_2] \in H_2(C; \mathbb{Q})$  as the sum of  $[s'_1] - [s'_2]$  with  $-\sum_{b \in \text{Sing}_B} \delta_1(b) [E_b]$ . But  $[s'_1] - [s'_2] = k[F]$  where  $k := -\int_B c_1(V)F$  and  $[F]$  denotes the fiber class of  $P \rightarrow B$ . It follows that

$$\begin{aligned}
\int_{s_1} f^*(L) - \int_{s_2} f^*(L) &= -(\sigma \cdot L) \int_B c_1(V) - \sum_{b \in \text{Sing}_B} \delta_1(b) \alpha_1(b) \cdot L \\
&= (\sigma \cdot L) \langle H, \tau_1 pt \rangle_\sigma^M - \sum_{\alpha_1 + \alpha_2 = \sigma} (\alpha_1 \cdot L) N_{\alpha_1, \alpha_2}
\end{aligned}$$

where  $N_{\alpha_1, \alpha_2} = \sum_j \langle H, \xi_j \rangle_{\alpha_1}^M \langle \xi_j^*, pt \rangle_{\alpha_2}^M$  is the number of two-component curves, one in class  $\alpha_1$  through  $H$  and the other in class  $\alpha_2$  through the point  $x_0$ . This completes the proof.  $\square$

**Proof of Lemma 2.21** If a symplectic 4-manifold  $(M, \omega)$  is not the blow up of a rational or ruled manifold then it has a unique maximal collection  $\{\varepsilon_1, \dots, \varepsilon_k\}$  of disjoint exceptional classes (i.e. classes that may be represented by symplectically embedded spheres of self-intersection  $-1$ ). If we blow these down, the resulting manifold  $(\overline{M}, \overline{\omega})$  (called the minimal reduction of  $(M, \omega)$ ) has trivial genus zero Gromov–Witten invariants. We show that for these  $M$  it is impossible for an invariant of the type  $\langle pt, s \rangle_\sigma^P$  to be nonzero.

Since  $(M, \omega)$  is not strongly uniruled, it follows from Remark 2.10 that  $\langle pt, a \rangle_\sigma^P = 0$  for all  $a \in H_2(M)$ . Therefore  $\langle pt, s \rangle_\sigma^P$  is the same for all section classes  $s$ . Choose classes  $\overline{h}_1, \overline{h}_2 \in H^2(\overline{M}; \mathbb{Z})$  with  $\overline{h}_1 \overline{h}_2 \neq 0$ . Pull them back to  $M$  and then extend them to  $H^2(P; \mathbb{Q})$  (which is possible by [27, Thm 1.1] for example.) Multiplying them by a suitable constant we get integral classes  $h_1, h_2 \in H^2(P; \mathbb{Z})$  with Poincaré duals  $H_1, H_2$ . By construction  $s_0 := H_1 H_2$  is a section class and  $H_j \cdot \varepsilon_i = 0$  for all exceptional divisors  $\varepsilon_i$  in  $M$ .

We now claim that we can apply Lemma 3.14 to evaluate the nonzero invariant  $\langle H_1 H_2, pt \rangle_\sigma^P$ . For this, it suffices to show that the space of (regularized) stable maps in class  $\sigma$  and through the point  $x_0$  can be constructed as a branched 2-manifold  $B'$ .

To this end, consider an  $\Omega$ -tame almost complex structure  $J_P$  on  $(P, \Omega)$  for which the projection  $\pi : (P, J_P) \rightarrow (S^2, j)$  is holomorphic. Then  $J_P$  restricts on each fiber  $P_z := \pi^{-1}(z)$  to an  $\omega$ -tame almost complex structure on  $M$ . Every  $J_P$ -holomorphic stable map in class  $\sigma$  consists of a holomorphic section plus some fiberwise bubbles. Since the family of such stable maps through some fixed point  $x_0$  has real dimension 2, we can assume that each such bubble is a  $k$ -fold cover of an embedded regular curve in some class  $\beta$  with  $c_1(\beta) \geq 0$  and  $\omega(k\beta) \leq \kappa = \Omega(\sigma)$ , and that the sections through  $x_0$  with energy  $\leq \kappa$  are regular and so lie in classes with  $2 \leq c_1(\sigma') \leq 3$ . For any  $J_P$ , each exceptional class  $\varepsilon_i$  is represented by a unique embedded sphere. Since  $(M, \omega)$  is not rational or ruled, it follows from Liu [24] that these are the only  $J_P$ -holomorphic spheres in classes  $\beta$  with  $c_1(\beta) > 0$ . (For details of this argument see [30, Cor 1.5].)

Since regular sections through  $x_0$  in classes with  $c_1(\sigma') = 2$  are isolated, there can be a finite number of two component stable maps whose bubble is an exceptional sphere and there are no stable maps involving multiply covered exceptional classes. However there may be some with multiply covered bubbles in classes  $\beta$  with  $c_1(\beta) = 0$ . To deal with these, choose a suitable very small multivalued perturbation  $\nu$  over the moduli spaces of fiberwise curves with class  $k\beta$  for  $\omega(k\beta) \leq \kappa$  so that there are only isolated solutions of the corresponding perturbed equation. Since the sections through  $x_0$  are regular, they can meet one of these isolated bubbles only if they lie in a moduli space of real dimension 2 and for generic choices of  $J_P$  and  $\nu$  they will meet only one such bubble. Therefore for this choice of  $\nu$  the perturbed moduli space contains isolated two-component stable maps. It remains to extend the perturbation over a neighborhood of this stratum in  $\bar{W}_\sigma$  (tapering it off to zero outside this neighborhood), and to define  $B'$  as the solution space of the resulting perturbed Cauchy–Riemann equation.

Lemma 3.14 now implies that

$$\langle H_1 H_2, pt \rangle_\sigma^P = (\sigma \cdot H_1) \langle H_2, \tau_1 pt \rangle_\sigma^P - \sum_{j, \alpha_1 + \alpha_2 = \sigma} (\alpha_1 \cdot H_1) \langle H_2, \xi_j \rangle_{\alpha_1}^P \langle \xi_j^*, pt \rangle_{\alpha_2}^P,$$

where  $\xi_j$  runs over a basis for  $H_*(P)$  with dual basis  $\xi_j^*$ . Note that we may choose this basis so that exactly one of each pair  $\xi_j, \xi_j^*$  is a fiber class.

The first term must vanish, since otherwise  $(M, \omega)$  is strongly uniruled by Lemma 2.20. Therefore there must be some nonzero product. If  $\alpha_2$  is a fiber class then by Lemma 3.8  $\langle \xi_j^*, pt \rangle_{\alpha_2}^P = \langle \xi_j^* \cap M, pt \rangle_{\alpha_2}^M$  so that  $(M, \omega)$  is strongly uniruled by definition. Hence these product terms vanish. Further if  $\alpha_2$  is a section class and  $\xi_j^* \neq [M]$  is a fiber class then  $(M, \omega)$  is strongly uniruled by Remark 2.10. On the other hand if  $\xi_j^* = [M]$  then there is a nonzero invariant  $\langle H_2, s \rangle_{\alpha_1}^P$  for  $\alpha_1 \in H_2(M)$  and some section class  $s$  which implies that  $c_1^M(\alpha_1) = 1$ . Also because  $\alpha_1 \cdot H_2 \neq 0$ , the choice of  $H_2$  implies that  $\alpha_1$  is not one of the exceptional classes  $\varepsilon_i$ . But this is impossible since we saw above that the only bubbles with Chern class 1 are the exceptional spheres. Therefore there must be a nonzero term in which both  $\alpha_1$  and  $\xi_j$  are fiber classes. In this case,  $\langle H_2, \xi_j \rangle_{\alpha_1}^P = \langle H_2, \xi_j \cap M \rangle_{\alpha_1}^M$  is an invariant in  $M$ . But the only nonzero classes  $\alpha_1 \in H_2(M)$  with nontrivial 2-point

invariants are the exceptional divisors  $\varepsilon_i$  and  $\varepsilon_i \cdot H_1 = 0$  by construction. Therefore these terms must vanish as well. This completes the proof.  $\square$

#### 4. SPECIAL CASES

We now discuss some special  $S^1$ -actions for which it is possible to prove directly that  $(M, \omega)$  is strongly uniruled.

**Proposition 4.1.** *Suppose that  $(M, \omega)$  is a semifree Hamiltonian  $S^1$ -manifold. Then  $(M, \omega)$  is strongly uniruled.*

*Proof.* Denote by  $\gamma$  the element in  $\pi_1(\text{Ham}M)$  represented by the circle action and by  $\mathcal{S}(\gamma) \in QH_*(M)^\times$  its Seidel element. Since the action is semifree, [33, Thm 1.15] shows that

$$\mathcal{S}(\gamma) * pt = a \otimes q^{-d} t^k + \text{l.o.t.}$$

where  $a$  is a nonzero element of  $H_{2d}(M)$  with  $d > 0$ . But if  $(M, \omega)$  is not strongly uniruled,  $\mathcal{S}(\gamma) * pt = (\mathbb{1} \otimes \lambda + x) * pt = pt \otimes \lambda$  by Lemma 2.1. An alternative proof is given in Proposition 4.3.  $\square$

The ideas of [33] also work when the isotropy weights have absolute value  $\leq 2$ . (In this case, we say that the action has at most 2-fold isotropy.) This property is stable under blow up along the maximal or minimal fixed point sets: cf. Lemma 2.18.

**Proposition 4.2.** *Suppose that  $(M, \omega)$  is an effective Hamiltonian  $S^1$ -manifold with at most 2-fold isotropy and isotropy weights 1 along  $F_{\max}$ . Then  $(M, \omega)$  is strongly uniruled.*

*Proof.* By Proposition 2.11

$$\mathcal{S}(\gamma) = a_0 \otimes q^d t^{K_{\max}} + \sum_{\beta \in H_2(M; \mathbb{Z}), \omega(\beta) > 0} a_\beta \otimes q^{m-c_1(\beta)} t^{K_{\max} - \omega(\beta)},$$

where  $a_\beta \in H_*(M)$  and  $a_0$  is in the image of  $H_*(F_{\max})$  in  $H_*(M)$ . Suppose that  $(M, \omega)$  is not strongly uniruled. By Lemma 2.1 (iii), there is at least one term in  $\mathcal{S}(\gamma)$  with  $a_\beta = r\mathbb{1}$  where  $r \neq 0$ . Consider the term of this form with minimal  $\omega(\beta)$ . Let  $J$  be a generic  $\omega$ -tame and  $S^1$ -invariant almost complex structure on  $M$ , with corresponding metric  $g_J$ . Proposition 3.4 of [33] shows that in order for  $r \neq 0$  there must be, for every point  $y \in F_{\min}$ , an  $S^1$ -invariant  $J$ -holomorphic genus zero stable map in class  $\beta$  that intersects  $F_{\max}$  and  $y$ . Such an invariant element consists of a connected string of 2-spheres from  $F_{\max}$  to  $y$ , possibly with added bubbles. Components of the string (called *beads* in [33]) either lie in the fixed point set  $M^{S^1}$  or are formed by the orbits of the  $g_J$ -gradient trajectories of  $K$ . The energy  $\omega(\beta')$  of an invariant sphere in class  $\beta'$  that joins the two fixed point components  $F_1, F_2$  is at least  $|K(F_1) - K(F_2)|/q$ , where  $q$  is the order of the isotropy at a generic point of the sphere. Therefore, if the isotropy has order at most 2 the energy needed to get from  $F_{\max}$  to  $y \in F_{\min}$  is at least  $(K_{\max} - K_{\min})/2$ . In the case at hand, it is strictly larger than  $(K_{\max} - K_{\min})/2$  since

the first element of the string has trivial isotropy. Therefore there is  $r \neq 0, x \in \mathcal{Q}_-$  such that

$$\mathcal{S}(\gamma) = \mathbb{1} \otimes (rt^{K_{\max}-\kappa} + \text{l.o.t}) + x, \quad \kappa > (K_{\max} - K_{\min})/2.$$

Similarly, there is  $r' \neq 0, x' \in \mathcal{Q}_-$  such that

$$\mathcal{S}(\gamma^{-1}) = \mathbb{1} \otimes (r't^{-K_{\min}-\kappa'} + \text{l.o.t}) + x', \quad \kappa' > (K_{\max} - K_{\min})/2.$$

Since  $\mathcal{S}$  is a homomorphism, we know that  $\mathcal{S}(\gamma) * \mathcal{S}(\gamma^{-1}) = \mathbb{1}$ . Now assume that  $M$  is not strongly uniruled. Then by Lemma 2.1(ii) the above expressions imply that

$$\mathcal{S}(\gamma) * \mathcal{S}(\gamma^{-1}) = rr' \mathbb{1} \otimes (t^\delta + \text{l.o.t}), \quad \delta < 0,$$

a contradiction.  $\square$

The previous results give conditions under which  $(M, \omega)$  is strongly uniruled, but they do not claim that the specific invariant  $\langle pt \rangle_\alpha^M$  is nonzero, where  $\alpha$  is the orbit of a generic gradient flow line from  $F_{\max}$  to  $F_{\min}$ . There are two cases when we can prove this. Note that condition (ii) is not very general since  $F_{\max}$  is often obtained by blow up and any such manifold is uniruled.

**Proposition 4.3.** *Suppose that  $(M, \omega)$  is a Hamiltonian  $S^1$ -manifold whose maximal and minimal fixed point sets are divisors. Suppose further that at least one of the following conditions holds:*

- (i) *the action is semifree, or*
- (ii) *there is an  $\omega$ -tame almost complex structure  $J$  on  $F_{\max}$  such that the nonconstant  $J$ -holomorphic spheres in  $F_{\max}$  do not go through every point.*

*Then  $\langle pt \rangle_\alpha^M \neq 0$ .*

*Proof.* Suppose first that the action is semifree and let  $J$  be a generic  $S^1$ -invariant almost complex structure on  $M$ . Then by [33, Lemma 4.5] the gradient flow of the moment map with respect to the associated metric  $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$  is Morse–Smale. Hence for a generic point  $x_0$  of  $F_{\max}$  all the gradient flow lines that start at  $x_0$  end on  $F_{\min}$ . The union of these flow lines is an invariant  $J$ -holomorphic  $\alpha$ -sphere through  $x_0$ . Moreover there is no other invariant  $J$ -holomorphic stable map in class  $\alpha$  through  $x_0$ . For as in the previous proof this would have to consist of a sphere  $C$  through  $x_0$  in  $F_{\max}$  together with a string of 2-spheres from a point  $x$  in  $F_{\max}$  to a point  $y$  in  $F_{\min}$ , possibly with added bubbles. (The string has to reach  $F_{\min}$  since  $\alpha \cdot F_{\min} = 1$ .) But because the action is semifree the energy of such a string is at least  $\omega(\alpha)$ . Since  $\omega(C) > 0$  this is impossible. Therefore there is only one invariant  $J$ -holomorphic stable map in class  $\alpha$  through  $x_0$ . Since this is regular,  $\langle pt \rangle_\alpha^M = 1$ .

A similar argument works in case (ii). Choose  $J$  to be a generic  $S^1$ -invariant extension of the given almost complex structure on  $F_{\max}$ . The arguments of [33, §4.1] show that the set  $X$  of points in  $F_{\max}$  that flow down to some intermediate fixed point set of  $K$  is closed and of codimension at least 2. Moreover by perturbing  $J$  in  $M \setminus F_{\max}$

we may jiggle  $X$  so that there is a point  $x_0 \in F_{\max} \setminus X$  that does not lie on any  $J$ -holomorphic sphere in  $F_{\max}$ . Hence again there is only one invariant stable map in class  $\alpha$  through  $x_0$ . The result follows as before.  $\square$

We end by discussing the case when  $H^*(M; \mathbb{Q})$  is generated by  $H^2(M)$ . Our main result here is the following.

**Proposition 4.4.** *Assume that  $H^*(M; \mathbb{Q})$  is generated by  $H^2(M)$ . Then  $(M, \omega)$  is strongly uniruled iff it is uniruled.*

The proof is given below. By Theorem 1.1 we immediately obtain:

**Corollary 4.5.** *Suppose that  $(M, \omega)$  is a Hamiltonian  $S^1$  manifold such that  $H^*(M; \mathbb{Q})$  is generated by  $H^2(M)$ . Then  $(M, \omega)$  is strongly uniruled.*

**Remark 4.6.** (i) Observe that if  $M$  is a Hamiltonian  $S^1$ -manifold such that  $H^*(M; \mathbb{Q})$  is generated by  $H^2(M)$  then the same holds for the blow up of  $M$  along any of its fixed point submanifolds  $F$ . For because the moment map  $K$  is a perfect Morse function the inclusion  $F \rightarrow M$  induces an injection on homology. (Any class  $c$  in  $H_*(F)$  can be written as  $c^- \cap F^+$ , where  $c^-$ ,  $F^+$  are the canonical downward and upward extensions of  $c$ ,  $[F]$  defined for example in [33, §4.1].) Hence  $H^*(F)$  is generated by the restrictions to  $F$  of the classes in  $H^2(M)$ . Since the exceptional divisor  $E$  is a  $\mathbb{P}^k$  bundle over  $F$ ,  $H^*(E)$  is also generated by  $H^2(E)$ : in fact the generators are the pullbacks of the classes in  $H^2(F)$  plus the first Chern class  $c \in H^2(E)$  of the canonical line bundle over  $E$ . But  $c$  is the restriction to  $E$  of the class  $\tilde{c}$  in  $\widetilde{M}$  that is Poincaré dual to  $E$ . It follows easily (using the Mayer-Vietoris sequence) that  $H^*(\widetilde{M})$  is generated by the pullback of the classes in  $H^2(M)$  together with  $\tilde{c}$ ; see the discussion of the cohomology of a blow up given in [15, §5.1].

(ii) It is not clear which Hamiltonian  $S^1$  manifolds have the property that  $H^*(M)$  is generated by  $H^2(M)$ . It is not enough that the fixed points are isolated. For example, Sue Tolman<sup>8</sup> pointed out that the complex Grassmannian  $Gr(2, 4)$  of 2-planes in  $\mathbb{C}^4$  has  $H^2$  of dimension 1 and  $H^4$  of dimension 2 so that  $H^4 \neq (H^2)^2$ . It also has an  $S^1$  action with precisely 6 fixed points. However, it is enough to have isolated fixed points plus semifree action, since in this case Tolman–Weitsman [38] show that  $H^*(M)$  is isomorphic as a ring to the cohomology of a product of 2-spheres. Also  $H^*(M)$  is generated by  $H^2(M)$  in the toric case. However, these cases are uninteresting in the present context since we already know that these manifolds are strongly uniruled, in the former case by Proposition 4.3 and in the latter by the fact that toric manifolds are projectively uniruled.

The proof of Proposition 4.4 uses the identities (3.9) and (3.10). Note that each time we apply one of these formulas we must take special care with the zero class. The following lemma is well known: cf. [15, Lemma 4.7].

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<sup>8</sup>Private communication.

**Lemma 4.7.**  $\langle \tau_{k_1} a_1, \dots, \tau_{k_p} a_p \rangle_0^M \neq 0$  for some  $p \geq 3$  only if the intersection product of the classes  $a_i$  is nonzero (i.e. the (real) codimensions of the  $a_i$  sum to  $2n$ ) and  $\sum k_i = p - 3$ .

*Proof.* This is immediate from the definition if  $k_i = 0$  for all  $i$ . To prove the general case, one can either argue directly or can construct an inductive proof based on the identity (3.9). To understand why the invariant vanishes when  $\dim(a_1 \cap \dots \cap a_m) > 0$ , observe that in this case the moduli space  $\mathcal{M}_0$  of constant maps with *fixed* marked points and through the constraints can be identified with  $a_1 \cap \dots \cap a_m$  and so has dimension  $> 0$ . But the classes  $\psi_i$  are trivial on  $\mathcal{M}_0$  and hence the integral of any product of the  $\psi_i$  over the full moduli space (with varying marked points) must vanish.  $\square$

#### Proof of Proposition 4.4.

Since any strongly uniruled manifold is uniruled, it suffices to prove the converse. This in turn is an immediate consequence of the next lemma.

**Lemma 4.8.** *Suppose that  $H^*(M; \mathbb{Q})$  is generated by  $H^2(M)$  and that there is a nonzero invariant of the form*

$$(4.1) \quad \langle \tau_{k_1} p t, \tau_{k_2} a_2, \dots, \tau_{k_m} a_m \rangle_\beta^M, \quad a_i \in H_*(M), k_i \geq 0, \beta \neq 0.$$

*Then  $(M, \omega)$  is strongly uniruled.*

*Proof.* The first two steps in this argument apply to all  $M$  and are contained in the proof of [15, Thm 4.9]. We include them for completeness. Without loss of generality we consider a nonzero invariant (4.1) such that  $m$  is minimal and  $\omega(\beta)$  is minimal among all nonzero invariants (4.1) of length  $m$  with  $\beta \neq 0$ . We then order the indices  $k_i$  so that  $k_2 \leq \dots \leq k_m$  and suppose that the  $k_i$  for  $i > 1$  are minimal with respect to the lexicographic order for the given  $m, \omega(\beta)$ . Finally we choose a minimal  $k_1$  for the given  $m, \omega(\beta)$ , and  $k_i, i > 1$ .

**Step 1:** *We may assume that  $k_i = 0$  for  $i > 1$ .*

If not, let  $r$  be the minimal integer greater than one such that  $k_r \neq 0$ . Suppose first that  $m \geq 3$  and  $r > 2$  and apply (3.9) with  $i = r, j = 1$  and  $k = 2$ . Then the invariant (4.1) is a sum of products

$$\langle \tau_{k_r-1} a_r, \xi, \dots \rangle_{\beta_1}^M \langle \xi^*, \tau_{k_1} p t, a_2, \dots \rangle_{\beta-\beta_1}^M,$$

where  $\xi$  runs over a basis for  $H_*(M)$  with dual basis  $\{\xi^*\}$ , and the dots represent the other constraints  $\tau_{k_\ell} a_\ell$  (which may be distributed in any way.) There must be a nonzero product of this form.

We now show that this is impossible. Suppose first that there is such a product with  $\omega(\beta_1) > 0$ . Then the second factor is an invariant of type (4.1) in a class  $\beta'$  with  $\omega(\beta') < \omega(\beta)$  and at most  $m$  constraints. Since our assumptions imply that all such terms vanish, this is impossible. Hence any nonzero product must have  $\omega(\beta_1) = 0$  and hence  $\beta_1 = 0$ . But then the second factor has at least one fewer nonzero  $k_i$ , since it has the homological constraint  $\xi^*$  instead of  $\tau_{k_r} a_r$ . This contradicts the assumed minimality of  $k_2, \dots, k_m$ .

This completes the proof when  $r > 2$ . If  $m \geq 3$  but  $r = 2$  use the same argument but take  $k = 3$  instead of  $k = 2$ . If  $m < 3$  add divisorial constraints to get a nonzero invariant with 3 constraints. The reader can check that the previous argument still goes through because it does not use the minimality of  $m$  in any essential way.

**Step 2:**  $k_1 = 0$ .

If  $k_1 > 0$  apply (3.9) with  $i = 1, j = 2$  and  $k = 3$ . Again, there must be a nonzero product of the form

$$\langle \tau_{k_1-1} pt, \xi, \dots \rangle_{\beta_1}^M \langle \xi^*, a_2, a_3, \dots \rangle_{\beta-\beta_1}^M.$$

Since the first factor can have at most  $(m-1)$  constraints, the minimality of  $m$  implies that  $\beta_1 = 0$ . But then  $pt \cap \xi \neq 0$ , so that  $\xi = [M]$ . Hence  $\xi^* = pt$  and the second factor is an invariant of the required form with  $k_1 = 0$ .

**Step 3:** *Completion of the proof.*

By hypothesis on  $M$  and Step 2, there is a nonzero invariant  $\langle pt, H_2^{i_2}, \dots, H_m^{i_m} \rangle_{\beta}^M$  with  $m$  constraints, where  $H_j \in H_{2n-2}(M)$ . Moreover, we may assume that all invariants (4.1) in a class  $\beta'$  with  $\omega(\beta') < \omega(\beta)$  or  $m' < m$  vanish and that the set  $i_2 \leq \dots \leq i_m$  is minimal in the lexicographic ordering. Note that  $i_2 > 1$  since otherwise we can reduce  $m$  by using the divisor equation. We must show that  $m \leq 3$ .

Suppose not and apply (3.10) with  $i = 2$  and  $j = 1$ . Since  $H_2 \cap pt = 0$  the first term on the RHS vanishes. The second is a multiple of  $\langle \tau pt, H_2^{i_2-1}, H_3^{i_3}, \dots, H_m^{i_m} \rangle_{\beta}^M$ . Suppose it is nonzero and apply (3.9) to it, with  $i = 1$  and  $j, k = 2, 3$ . This gives a sum of terms  $\langle pt, \xi, \dots \rangle_{\beta_1}^M \langle H_2^{i_2-1}, H_3^{i_3}, \xi^*, \dots \rangle_{\beta-\beta_1}^M$ . Since the first factor has  $< m$  constraints, we must have  $\beta_1 = 0$ . But then  $\xi = [M]$  so that  $\xi^* = pt$ . Hence all the other constraints must lie in the second factor (since it must have at least  $m$  constraints). Therefore the first factor is a constant map with only two constraints, i.e. it is unstable. But this is not allowed. Therefore, the second term in (3.10) must vanish.

It remains to consider the product terms in (3.10), namely

$$(\beta_1 \cdot H_2) \langle H_2^{i_2-1}, \xi, \dots \rangle_{\beta_1}^M \langle pt, \xi^*, \dots \rangle_{\beta_2}^M,$$

where  $\beta_1 + \beta_2 = \beta$ . In any nonzero product of this form,  $\beta_1 \neq 0$ . Also the stability condition on the second factor implies that if  $\beta_2 = 0$  this term must have another constraint. Since this must have the form  $H_j^{i_j}$  with  $i_j > 0$ , this is not possible by Lemma 4.7. Therefore  $0 < \omega(\beta_2) < \omega(\beta)$ . But then the second factor vanishes by the minimality of  $\omega(\beta)$ .  $\square$

**Remark 4.9.** Suppose that  $(M, \omega)$  is uniruled with even constraints, i.e. there is a nonzero invariant of the form (4.1) in which all the  $a_i$  have even degree. Then the proof of Lemma 4.8 goes through if we assume only that the even part  $H^{ev}(M)$  of the cohomology ring is generated by  $H^2$ . For if the  $a_i$  have even degree, odd dimensional homology classes appear in the above proof only as elements  $\xi, \xi^*$ . Since these always appear as part of invariants where all the other insertions have even dimension, all terms

involving odd dimensional  $\xi, \xi^*$  must vanish. The appendix contains other results about such manifolds; cf. Propositions A.2 and A.4.

### APPENDIX A. THE STRUCTURE OF $QH_*(M)$ FOR UNIRULED $M$

In this appendix we explore the extent to which the uniruled property can be seen in quantum homology, completing the discussion begun in Lemma 2.1. To simplify we shall ignore contributions to the quantum product from the odd dimensional homology classes. Hence our results are not as general as they might be.

Let  $\mathbb{F} := \Lambda$  be the field  $\Lambda^{\text{univ}}$  of generalized Laurent series. Observe that the even quantum homology

$$QH_*^{ev}(M) := \bigoplus_{i=0}^n H_{2i}(M; \mathbb{R}) \otimes \Lambda[q, q^{-1}]$$

is a subring of  $QH_*(M)$  because when  $a, b \in H_*(M)$  have even degree  $\langle a, b, c \rangle_\beta^M = 0$  unless  $c$  also has even degree. We shall denote by  $\mathcal{A}$  its subring  $QH_{2n}(M) = QH_{2n}^{ev}(M)$  of elements of degree  $2n$ , regarded as a commutative algebra over  $\mathbb{F}$ . (Equivalently we can think of  $\mathcal{A}$  as the algebra obtained from  $QH_{ev}(M)$  by setting  $q = 1$ .)

Choose a basis  $\xi_i$  for  $H_{ev}(M; \mathbb{Q})$  with  $\xi_0 = pt$ ,  $\xi_N = \mathbb{1}$  and so that  $0 < \deg(\xi_i) < 2n$  for the other  $i$ . These elements form a finite basis for  $\mathcal{A}$  considered as a vector space over  $\mathbb{F}$ . Hence there is a well defined linear map  $f : \mathcal{A} \rightarrow \mathbb{F}$  given by

$$f\left(\sum_{i=0}^N \lambda_i \xi_i\right) = \lambda_0.$$

Since the corresponding pairing  $(a, b) := f(ab)$  on  $\mathcal{A}$  is nondegenerate,  $(\mathcal{A}, f)$  is a commutative Frobenius algebra.<sup>9</sup> It is easy to see that it satisfies the other conditions of the next lemma, with  $p = pt$  and  $\mathcal{M} = \text{span}\{\xi_i : i \neq 0, N\}$ . In particular  $f(pa) = 0$  when  $a \in \mathcal{Q}_- := p\mathbb{F} \oplus \mathcal{M}$  because all Gromov–Witten invariants of the form

$$\langle a, b, [M] \rangle_\beta^M, \quad \beta \neq 0, \quad a, b \in H_{<2n}(M)$$

vanish.

**Lemma A.1.** *Let  $(\mathcal{A}, f)$  be a finite dimensional commutative Frobenius algebra over a field  $\mathbb{F}$  that decomposes additively as  $\mathcal{A} = p\mathbb{F} \oplus \mathcal{M} \oplus \mathbb{1}\mathbb{F}$  where  $f(p) = 1$  and  $\ker f = \mathcal{M} \oplus \mathbb{1}\mathbb{F}$ . Suppose further that  $f(pa) = 0$  for all  $a \in \mathcal{Q}_- := p\mathbb{F} \oplus \mathcal{M}$ . Then  $pa = 0$  for all  $a \in \mathcal{Q}_-$  iff there are no units in  $\mathcal{Q}_-$ .*

*Proof.* One implication here is obvious: if  $a \in \mathcal{Q}_-$  is a unit then  $pa \neq 0$ . Conversely, suppose that  $pa \neq 0$  for some  $a \in \mathcal{Q}_-$ , but that there are no units in  $\mathcal{Q}_-$ . We shall show by a sequence of steps that there are no Frobenius algebras  $(\mathcal{A}, f)$  that have this property as well as satisfying the other conditions in the statement of the lemma.

<sup>9</sup>A pair  $(\mathcal{A}, f)$  consisting of a commutative finite dimensional unital algebra together with a linear functional  $f : \mathcal{A} \rightarrow \mathbb{F}$  satisfies the Frobenius nondegeneracy condition iff  $\ker f$  contains no nontrivial ideals.

Decompose  $\mathcal{A}$  as a sum  $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k$  of indecomposables and let  $e_1, \dots, e_k$  be the corresponding unitpotents. Thus for all  $i, j$

$$e_i e_j = \delta_{ij} e_i, \quad \text{and} \quad \mathcal{A}_i = \mathcal{A} e_i.$$

Each  $e_i$  may be written uniquely in the form  $\lambda_i \mathbb{1} + x_i$  where  $x_i \in \mathcal{Q}_-$  and  $\lambda_i \in \mathbb{F}$ . Order the  $e_i$  so that the nonzero  $\lambda_i$  are  $\lambda_1, \dots, \lambda_\ell$ . Since  $\sum e_i = \mathbb{1}$ , we must have  $\sum_{i=1}^\ell \lambda_i = 1$ . In particular,  $\ell \geq 1$ .

**Step 1:** *All units in  $\mathcal{A}_i, i > 1$ , lie in  $\mathcal{Q}_-$ . Hence  $\ell = 1$ .*

Suppose there is a unit in  $\mathcal{A}_i$  of the form  $\mu_i \mathbb{1} + x$ , where  $\mu_i \neq 0$  and  $i > 1$ . Choose nonzero  $\nu_j \in \Lambda$  for  $j \neq i, j \leq \ell$ , so that  $\sum \nu_j \lambda_j = \mu_i$  and set  $\nu_j := 1$  when  $j > \ell$ . Then  $u := \sum_{j \neq i} \nu_j e_j - u_i$  is a unit of  $\mathcal{A}$  since it is a sum of units, one from each factor. By construction, the coefficient  $\mathbb{1}$  in  $u$  vanishes. Hence  $u$  is a unit of  $\mathcal{A}$  lying in  $\mathcal{Q}_-$ , which, by hypothesis, is impossible.

**Step 2:** *All nilpotent elements in  $\mathcal{A}$  lie in  $\mathcal{Q}_-$ .*

If  $n = \mathbb{1} - x$  is nilpotent for some  $x \in \mathcal{Q}_-$ , then  $x = \mathbb{1} - n$  is a unit of  $\mathcal{A}$  lying in  $\mathcal{Q}_-$ .

**Step 3:** *For all  $i > 1$ , each  $\mathcal{A}_i \subset \mathcal{Q}_-$  and  $p e_i = 0$ .*

The standard theory of indecomposable finite dimensional algebras over a field implies that every nonzero element in  $\mathcal{A}_i$  either is a unit or is nilpotent; see Curtis–Reiner [5, pp.370-2]. The units in  $\mathcal{A}_i, i > 1$ , lie in  $\mathcal{Q}_-$  by Step 1, and the nilpotent elements do too by Step 2. This proves the first statement. Thus  $e_i x \in \mathcal{Q}_-$  for all  $x \in \mathcal{A}$  and  $i > 1$ , so that by our initial assumptions  $f(p e_i x) = 0$  for all  $x$ . But the restriction of  $f$  to each summand  $\mathcal{A}_i$  is nondegenerate. Hence this is possible only if  $p e_i = 0$ .

**Step 4:** *Completion of the argument.*

Let  $e = \sum_{i>1} e_i$ . By the above we may assume that  $p = p(\mathbb{1} - e) \in \mathcal{A}_1$ . Decompose  $\mathcal{A}_1$  additively as the direct sum  $\mathcal{N} \oplus \mathcal{U}$  where  $\mathcal{N}$  is the subspace formed by the nilpotent elements and  $\mathcal{U}$  is a complementary subspace. Step 2 implies that  $\mathcal{N} \subset \mathcal{A}_1 \cap \mathcal{Q}_-$ . On the other hand, as in Step 3, any nonzero element in  $\mathcal{A}_1 \cap \mathcal{Q}_-$  that is not nilpotent is a unit  $u$  in  $\mathcal{A}_1$ . If any such existed then  $u + e$  would be a unit of  $\mathcal{A}$  lying in  $\mathcal{Q}_-$ . Hence by hypothesis we must have  $\mathcal{N} = \mathcal{A}_1 \cap \mathcal{Q}_-$ . Thus  $p$  is nilpotent and  $\dim \mathcal{U} = 1$ , spanned by  $e_1$ . Further because  $\mathcal{Q}_-$  is spanned by  $\mathcal{N}$  and the  $\mathcal{A}_i, i > 1$ ,  $p$  does not annihilate all elements in  $\mathcal{N}$ .

We now need to use further information about the structure of  $\mathcal{A}_1$ . Recall that the socle  $\mathcal{S}$  of an algebra  $\mathcal{A}_1$  is the annihilator of  $\mathcal{N}$ . Therefore, our assumption on  $p$  implies that  $p \notin \mathcal{S}$ . But there always is  $w \in \mathcal{N}$  such that  $p w$  is a nonzero element of  $\mathcal{S}$ . To see this, choose a set  $x_1 = p, x_2, \dots, x_k$  of multiplicative generators for  $\mathcal{N}$  that includes  $p$ . There is  $N$  such that all products of the  $x_i$  of length  $> N$  must vanish. (Take  $N$  to be the sum of the orders of the  $x_i$ .) Therefore there is a nonzero product of maximal length that contains  $p$  and so can be written as  $p w$ . Since  $p w x_i = 0$  for all  $i$ ,  $p w \in \mathcal{S}$ . (A more precise version of this argument is given in Abrams [1, Prop.3.3].)

The argument is now quickly completed. For, by the Frobenius nondegeneracy condition there must be  $z \in \mathcal{A}$  such that  $(p w, z) = f(p w z) \neq 0$ . By Step 3 we may assume that  $z \in \mathcal{A}_1$  so that  $z = \lambda e_1 + n$  for some  $n \in \mathcal{N}$ . But then  $p w z = p w(\lambda e_1) = \lambda p w$  so that  $f(\lambda p w) = \lambda f(p w) \neq 0$ . But  $w \in \mathcal{N} \subset \mathcal{Q}_-$  by construction. Hence  $f(p w) = 0$  by

our initial assumptions. This contradiction shows that our assumption that  $\mathcal{Q}_-$  has no units must be wrong.  $\square$

We shall say that  $(M, \omega)$  is **uniruled with even constraints** if there is a nonzero Gromov–Witten invariant of the form  $\langle pt, a_2, \dots, a_k \rangle_\beta^M$  with  $\beta \neq 0$  and all  $a_i$  of even degree. Similarly,  $(M, \omega)$  is **strongly uniruled with even constraints** if there is a nonzero invariant of this kind with  $k = 3$ . For example, [15, Cor. 4.3] shows that any projective manifold that is uniruled is in fact strongly uniruled with even constraints.

**Proposition A.2.**  *$(M, \omega)$  is strongly uniruled with even constraints iff the even quantum homology ring  $QH_*^{ev}(M)$  has a unit in  $\mathcal{Q}_-$ .*

*Proof.* Note that  $QH_*^{ev}(M)$  has a unit iff its degree  $2n$  part  $\mathcal{A}$  has. Also the subring  $\mathcal{Q}_-(\mathcal{A})$  of  $\mathcal{A}$  considered above is just the intersection  $\mathcal{Q}_- \cap QH_{2n}^{ev}(M)$ . Therefore this is an immediate consequence of Lemma A.1.  $\square$

If  $(M, \omega)$  is uniruled rather than strongly uniruled we can still see some effect on quantum homology if instead of considering the small quantum product  $*$  we consider the whole family of products  $*_a$ ,  $a \in \mathcal{H}$ . Here we shall take  $\mathcal{H} := H_{ev}(M; \mathbb{C})$  and correspondingly allow the coefficients  $r_i$  of the elements  $\sum r_i t^{\kappa_i}$  in  $\Lambda$  to be in  $\mathbb{C}$ . Let  $\xi_i, i = 0, \dots, N$ , be a basis for  $\mathcal{H}$  with  $\xi_0 = pt$  as before, and identify  $\mathcal{H}$  with  $\mathbb{C}^{N+1}$  by thinking of this as the standard basis in  $\mathbb{C}^{N+1}$ . Denote by  $T_0, \dots, T_N$  the corresponding coordinate functions on  $\mathcal{H}$ , thought of as formal variables. If  $\alpha = (\alpha_1, \dots, \alpha_p)$  is a multi-index with  $0 \leq \alpha_i \leq N$  define

$$\xi^\alpha := (\xi_{\alpha_1}, \dots, \xi_{\alpha_p}) \in \mathcal{H}^p, \quad T^\alpha := \prod_{i=1}^p T_{\alpha_i}.$$

In this language, the (even) Gromov–Witten potential  $\Phi(t, T)$  is the formal power series in the variables  $t$  and  $T$  given by

$$\Phi(t, T) := \sum_{\alpha} \sum_{\beta} \frac{1}{|\alpha|!} \langle \xi^\alpha \rangle_{\beta} t^{-\omega(\beta)} T^\alpha.$$

Let us assume<sup>10</sup> that the following condition holds:

**Condition (\*):** *there is  $\delta > 0$  such that this series converges if we consider the  $T_i$  to be complex numbers such that  $|T_i| \leq \delta$ .*

Then we may think of  $\Phi$  as a function defined near  $0 \in \mathcal{H}$  with values in the field  $\Lambda = \Lambda_{\mathbb{C}}^{univ}$ . Further, as explained for example in [32, Ch 11.5], the structure constants of the associative product  $x *_a y$  are given by evaluating the third derivatives of  $\Phi(t, T)$  with

<sup>10</sup>We make this assumption to simplify our subsequent discussion. It is satisfied in the case of manifolds such as  $\mathbb{C}P^n$  (cf. [32, Ch 7.5]), but there is at present little understanding of when it is satisfied in general. Even if it were not satisfied, one could presumably adapt the results below in any particular case of interest.

respect to the variables  $T_i$  at the point  $T = a$ . In other words,<sup>11</sup>  $\xi_i *_a \xi_j = \sum_k c_{ij}^k(a) \xi_k^*$ , where

$$(A.1) \quad c_{ij}^k(a) = \left. \frac{\partial^3 \Phi(t, T)}{\partial T_i \partial T_j \partial T_k} \right|_{T=a} = \sum_m \frac{1}{m!} \langle \xi_i, \xi_j, \xi_k, a, \dots, a \rangle_{m+3, \beta}^M t^{-\beta}.$$

We denote the corresponding (ungraded) rings by  $QH_{ev}^a(M)$ . As before, they are Frobenius algebras over the field  $\mathbb{F} := \Lambda$ .

The main point for us is the following lemma:

**Lemma A.3.** *Assume that condition (\*) holds and that there is a nonzero invariant  $\langle pt, a_2, \dots, a_k \rangle_{\beta}^M$  where  $\beta \neq 0$  and the  $a_i$  have even degree. Then there are  $a, b \in \mathcal{H}$  with  $\deg b < 2n$  such that  $pt *_a b \neq 0$ .*

*Proof.* Let  $m+3$  be the minimal  $k$  for which some  $\langle pt, a_2, \dots, a_k \rangle_{k, \beta}^M$  does not vanish. If  $m \leq 0$  then we can take  $a = \mathbb{1}$  so that, by Lemma 4.7,  $*_a$  is the usual product. If  $m > 0$  then the hypothesis implies that some invariant of the form  $\langle pt, \xi_j, \xi_k, a, \dots, a \rangle_{m+3, \beta}^M$  is nonzero. (This holds because GW invariants are symmetric and multilinear functions of their arguments.) But by equation (A.1) the coefficients  $c_{ij}^k(a)$  are power series in the coordinates of  $a \in \mathcal{H}$ . Hence the fact that one coefficient of the power series for  $c_{0j}^k(a)$  does not vanish implies that by perturbing  $a$  slightly if necessary we can arrange that  $c_{0j}^k(a)$  itself is nonzero. It follows that  $pt *_a \xi_j \neq 0$ .  $\square$

A similar argument proves the analog of the other statements in Lemma 2.1. Moreover, if  $(\mathcal{A}^a, f)$  denotes the Frobenius algebra  $QH_{2n}^a(M)$ , then one can check as before that  $(\mathcal{A}^a, f)$  satisfies the conditions of Lemma A.1. Thus we deduce:

**Proposition A.4.** *Assume that condition (\*) holds. Then  $(M, \omega)$  is uniruled with even constraints iff there is  $a \in \mathcal{H}$  such that the even quantum ring  $QH_{ev}^a(M)$  has a unit in  $\mathcal{Q}_-$ .*

**Remark A.5.** (i) Suppose that  $(\mathcal{A}, f)$  is a Frobenius algebra over a field  $\mathbb{F}$  with the property that  $f(\mathbb{1}) = 0$ . Suppose further that there is  $p \in \mathcal{A}$  such that  $f(p) \neq 0$  while  $f(p^2) = 0$ . Since the functional  $\ker f \rightarrow \mathbb{F}$  given by  $x \mapsto f(px)$  does not vanish when  $x = \mathbb{1}$ , its kernel  $\mathcal{M}$  is a complement to  $\mathbb{1}\mathbb{F}$  in  $\ker f$ . Moreover the assumption  $f(p^2) = 0$  implies that the subspace of  $\mathcal{A}$  orthogonal to  $p$  is  $p\mathbb{F} \oplus \mathcal{M} =: \mathcal{Q}_-$ . Hence  $\mathcal{A}$  decomposes additively as  $p\mathbb{F} \oplus \mathcal{M} \oplus \mathbb{1}\mathbb{F}$  as in Lemma A.1. Therefore the conditions in this lemma are satisfied for some  $\mathcal{M}$  provided only that  $f(\mathbb{1}) = 0$  and there is  $p$  with  $f(p) \neq 0, f(p^2) = 0$ .

(ii) The fact that there is such a nice characterization of the uniruled property in terms of the structure of quantum homology leads immediately to speculations about rational

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<sup>11</sup>One needs to interpret these formulas with some care. When we set  $T = a$  we are thinking of  $T$  as the set of numbers  $(T_0, \dots, T_k)$  that are the coordinates of  $a = \sum T_\ell \xi_\ell$ . On the other hand, the arguments of a Gromov–Witten invariant are homology classes. Thus  $\langle a, \dots, a \rangle_{p, \beta}^M = \sum_\alpha \langle \xi^\alpha \rangle_{p, \beta}^M t^{-\omega(\beta)} T^\alpha |_{T=a}$ .

connectedness. A projective manifold is said to be **rationally connected** if there is a holomorphic  $\mathbb{P}^1$  through every generic pair of points: see Kollar [18]. This implies that there is a holomorphic  $\mathbb{P}^1$  through generic sets of  $k$  points, for any  $k$ , but it is not yet known whether these spheres are visible in quantum homology, e.g. it is not known whether there must be a nontrivial Gromov–Witten invariant with more than one point constraint. This would correspond to the point class  $p := pt$  in  $QH_{ev}^a(M)$  having nonzero square  $p^2$ . This raises many questions. Are there symplectic manifolds with  $p$  nilpotent but with  $p^2 \neq 0$ ? If  $p$  is not nilpotent is the quantum homology semi-simple? Abrams’ condition for semi-simplicity in [1] involves the quantum Euler class. What is its relation to the class  $p$ ? There are many possible choices for the coefficient ring  $\Lambda$ ; how do these affect the situation?

(iii) The purpose of Proposition A.4 is to show that there is not much conceptual difference between the usual quantum product and its deformations  $*_a$ . All the usual applications of quantum homology (such as the Seidel representation and spectral invariants) should have analogs for  $*_a$ . For example, given  $a \in H_{2d}(M; \mathbb{Q})$  let  $\mathcal{G}^a$  be the extension of  $\pi_1(\text{Ham}(M))$  whose elements are pairs  $(\gamma, \tilde{a})$  consisting of an element  $\gamma \in \pi_1(\text{Ham}(M))$  with a class  $\tilde{a} \in H_{2d+2}(P_\gamma)$  such that  $\tilde{a} \cap [M] = a$ ; cf. the discussion of the group  $\widehat{G}$  in [32, Ch 12.5]. Then (assuming that the appropriate version of condition (\*) holds) one can define a homomorphism

$$\mathcal{S}^a : \mathcal{G}^a \rightarrow (QH_{ev}^a(M))^\times$$

by setting

$$\mathcal{S}^a((\gamma, \tilde{a})) = \sum_{\sigma, m, i} \frac{1}{m!} \langle \xi_i, \tilde{a}, \dots, \tilde{a} \rangle_{m+1, \sigma} \xi_i^* \otimes t^{-u_\gamma(\sigma)},$$

as in equation (2.2). It is not hard to check that this satisfies the analog of (2.4), namely

$$\mathcal{S}^a((\gamma, \tilde{a})) *_a b = \sum_{\sigma, m, i} \frac{1}{m!} \langle b, \xi_i, \tilde{a}, \dots, \tilde{a} \rangle_{m+2, \sigma} \xi_i^* \otimes t^{-u_\gamma(\sigma)}.$$

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DEPARTMENT OF MATHEMATICS, BARNARD COLLEGE, COLUMBIA UNIVERSITY, 2990 BROADWAY,  
NEW YORK, NY 10027

*E-mail address:* `dusa@math.columbia.edu`