J-curves and the classification of rational and ruled symplectic 4-manifolds

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In this paper, we present a complete classification of rational and ruled symplectic 4-manifolds up to symplectomorphism as well as describing ways of recognizing these manifolds. The proofs are essentially self-contained.

1 Introduction

The classification of symplectic structures on $\mathbf{CP}^2$ has followed a quite simple story: Gromov showed in [2] that the existence of an embedded sympectic 2-sphere in the homology class $[\mathbf{CP}^1]$ of a line implies that the symplectic structure is diffeomorphic to the standard Kähler structure on $\mathbf{CP}^2$, and Taubes showed in [37] that such a sphere always exists if the space is diffeomorphic to $\mathbf{CP}^2$. Both proofs are due to spectacular advances in the application of elliptic PDEs methods to symplectic 4-manifolds.

The classification of rational ruled symplectic 4-manifolds, that is the classification of symplectic S^2-fibrations over a 2-sphere, was established by Gromov [2] in a rather special case, and was extended to all cases by McDuff [14]. Her

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proof was based on two new ideas: the realization that in 4-dimensions cusp-curves do not affect the behaviour of the evaluation map for generic J, and the construction of symplectic sections of a rational ruled 4-manifold with a method suggested by Eliashberg.

However, the classification of symplectic structures on spaces diffeomorphic to S^2-bundles over Riemann surfaces of genus > 0 is more complicated. There have been several attempts to solve this problem: partial results (including a complete solution of the problem for ruled surfaces over the torus) were obtained by McDuff in [14, 21, 23] and by Lalonde in [5]. Actually the main difficulty, as we will see below, is not to derive the existence of an embedded symplectic 2-sphere. It is to classify the symplectic S^2-fibrations on a given S^2-bundle over a Riemann surface of genus > 0. This required more work, both on the *symplectic* and *elliptic* aspects of the problem. The complete classification finally appeared in our recent note [7]. It is based on a simpler and more general geometric argument which reduces the classification of the irrational ruled manifolds to the rational ones, up to the computation of some Gromov invariant on the given irrational ruled manifold. The first step of this reduction – the "cutting and pasting" argument – appeared in McDuff [14], and the second step of the reduction – the transformation of a symplectic deformation into a genuine isotopy via a one-parameter family of symplectic submanifolds – was worked out by Lalonde in [5]. The computation of the relevant Gromov invariant is based on the equivalence of the Gromov and Seiberg-Witten invariants, and is easily reduced to the calculation of a wall-crossing number at the reducible solutions of the Seiberg-Witten equations.

The second theme of this article is the characterization of rational and ruled symplectic manifolds. While developing the theory of J-holomorphic spheres, McDuff discovered that any symplectic 4-manifold which contains a symplectically embedded 2-sphere C of nonnegative self-intersection is the blow-up of a rational or ruled manifold. Taubes's work now applies to show that this remains true if we just assume that C is smoothly embedded. Another beautiful new result was recently proved by Liu [10]. He showed that Gompf's conjecture holds: namely, any minimal symplectic 4-manifold whose canonical class K is such that $K^2 < 0$ is irrational ruled. As a corollary, it follows that any minimal symplectic 4-manifold which admits a metric of positive scalar curvature is rational or ruled: see Liu [10] and Ohta–Ono [31].

2 Statement of the classification theorems

We assume that the reader is familiar with blowing-up and blowing-down in the symplectic category: see for instance [27] or [5], §II, for a convenient summary.

Recall that there are only two S^2-bundles over any given Riemann surface B, the trivial bundle $\pi : B \times S^2 \to B$ and the nontrivial bundle $\pi : M_B \to B$. By analogy with the complex case, such a bundle is said to be a **ruled symplectic manifold** if the total space is equipped with a symplectic form ω which is nondegenerate on each fiber. In this case, we also say that ω is **compatible** with the given ruling π. Similarly, we say that a symplectic 4-manifold (M, ω) is **rational** if it can be obtained from the standard $\mathbf{CP}^2$ by a sequence of blow-ups and blow-downs. (This is slightly unfortunate terminology, since in other contexts the word rational is used to denote a rationality condition on some cohomology class, for instance on the class $[\omega]$ or on some Maslov class.)

An **exceptional sphere** in a symplectic 4-manifold is a symplectically embedded 2-sphere of self-intersection -1.[1] It was shown in [17, 14] that these spheres behave very much like exceptional curves in complex surfaces. In particular, they can be blown down and replaced by a ball of appropriate size. (In fact, if the *weight* $\omega(E)$ of the exceptional curve E is πr^2 the ball should have radius r.) If we define a **minimal** symplectic 4-manifold to be one which contains no exceptional spheres, one can reduce every symplectic 4-manifold to a minimal one by blowing down a maximal collection of disjoint exceptional spheres. Moreover, just as in the complex case, this minimal reduction is unique unless it it is rational or ruled: see [18].

It is also convenient to consider manifolds M which are such that the complement $M - C$ of a symplectic 2-submanifold C contains no exceptional spheres. (In this situation (M, C) is sometimes called a **minimal pair**.) Observe that given any such C in M one can obtain a minimal pair by blowing down a maximal collection of disjoint exceptional spheres in $M - C$.

Here are the main results.

Theorem 2.1 (Gromov–McDuff) *Let (M, ω) be a closed symplectic 4-manifold which contains an embedded symplectic 2-sphere C of self-intersection ≥ 0. Then (M, ω) is symplectomorphic to a blow-up either of $\mathbf{CP}^2$ with its standard Kahler structure (which is unique up to scaling by a constant) or of a ruled symplectic manifold. In particular, if $M - C$ is minimal then (M, ω) is either symplectomorphic to $\mathbf{CP}^2$ or is ruled.*

Remarks. (1) In the above theorem, the natural hypothesis would be to assume M minimal. But then one does not recover the case of the topologically nontrivial S^2-bundle over S^2, which coincides with the blow-up of $\mathbf{CP}^2$ at one point. In order to include this case, we must assume only the minimality of $M - C$.
(2) The symplectomorphism of the theorem can be chosen so that the curve C corresponds either to a line or quadric in $\mathbf{CP}^2$ or to a fiber of the S^2-bundle, or to a section of the bundle when the base is the sphere.

[1] To say that a 2-dimensional submanifold S of M is symplectically embedded is equivalent to saying that the restriction $\omega|_S$ never vanishes.

(3) This theorem implies that the rational manifolds are actually blow-ups of the standard $\mathbf{CP}^2$ or of the ruled $S^2 \times S^2$ (which by Theorem 2.3 below must also be standard, ie a product). In other words one does not need to take a sequence of blow-ups and blow-downs to arrive at an arbitrary rational symplectic manifold.

Theorem 2.2 *(i)* **(Taubes)** *If (M, ω) is a symplectic manifold and M is smoothly diffeomorphic to $\mathbf{CP}^2$, then ω is diffeomorphic to a standard Kahler form.*
(ii) **(Li–Liu)** *Let $\pi : M \to \Sigma$ be an smooth S^2-bundle over a closed orientable surface. Then any symplectic form ω on M is diffeomorphic to a form which is compatible with the given ruling. Moreover, we can assume that this diffeomorphism acts trivially on homology.*

Theorem 2.3 (McDuff) *Let (M, ω) be a ruled symplectic manifold over a 2-sphere. Then ω is isotopic to a standard Kahler form on M.*

Theorem 2.4 (Lalonde–McDuff) *Let (M, ω) be a ruled symplectic manifold over a closed orientable surface. Then ω is isotopic to a standard Kahler form on M.*

In the last two theorems, the statement is more precisely: any two cohomologous symplectic forms compatible with the ruling of a given smooth bundle $S^2 \hookrightarrow M \to \Sigma$ are isotopic (and therefore isotopic to the standard Kahler form in the given cohomology class).

Remarks. (1) It follows from the last three theorems that symplectic forms on smooth S^2-bundles are determined up to diffeomorphism by the classes $[\omega] \in H^2(M, \mathbf{R})$.
(2) Taubes's work also leads to some general recognition theorems for rational and ruled manifolds. One of the nicest of these is due to Liu, who showed in [10] that Gompf's conjecture holds. Namely every minimal symplectic 4-manifold (M, ω) with $K^2 < 0$ is a ruled surface over a base of genus > 1. Here $K = -c_1(\omega)$ is the canonical class. It follows that any symplectic 4-manifold which admits a symplectic submanifold C with $c_1(C) > 0$ is a blow-up of a rational or ruled manifold, which generalizes a well-known theorem in complex geometry.

As we said above, the classification of symplectic structures on spaces diffeomorphic to $\mathbf{CP}^2$ and the classification of a restricted kind of symplectic structure on $S^2 \times S^2$ was proved by Gromov [2]. Theorem 2.1 is McDuff's classification theorem from [14], which was proved by further developing the pseudoholomorphic techniques introduced in Gromov's seminal work [2]. The proof has two steps. One first shows that when $M - C$ is minimal M contains a symplectically embedded sphere of self-intersection either 0 or 1, and then analyses the diffeomorphism type of M in these two cases.

The second theorem is one of the consequences of Taubes's recent beautiful work on the equivalence of the Gromov and the Seiberg-Witten invariants. He proved part (i) of the theorem in [37] by finding a symplectic sphere of self-intersection 1. Part (ii) was proved by Li and Liu [8, 9] using a direct extension of Taubes's argument. We sketch the outlines of these proofs in §6.3 below.

The third theorem was proved by McDuff in [14]. Its proof is based on an extension of Gromov's techniques and on some new ideas that are discussed further below.

The proof of the fourth theorem involves an extended foray into symplectic geometry. Recall that two symplectic forms are said to be *deformation equivalent* if they can be joined by a path of not necessarily cohomologous symplectic forms, while they are *isotopic* if the path ω_t consists of cohomologous forms. In the latter case, Moser stability implies that there is a path of diffeomorphisms ϕ_t of M starting at the identity such that $\phi_t^*(\omega_t) = \omega_0$. Deformation equivalent forms, on the other hand, while they have the same Gromov invariants may not be geometrically equivalent. More precisely, there are examples of cohomologous and deformation equivalent symplectic forms which are not even diffeomorphic (see [13]), though no such examples are known in dimansion 4.

The first step in the proof of Theorem 2.4 is a cutting and pasting argument which shows that all symplectic forms on a ruled manifold compatible with a given ruling are deformation equivalent. This was proved in McDuff's paper [14] by enlarging the base to make enough room in which to cut open the ruled surface over a set of loops in the base in such a way that the monodromies over each such loop are trivial. This yields a ruled surface over a cell in $\mathbf{R}^2$ with a symplectic form which is standard near the boundary of the cell. One can then complete this to a ruled surface over S^2 and invoke uniqueness for ruled surfaces over S^2. The second step consists in showing that, if ω, ω' are two cohomologous and deformation equivalent symplectic forms compatible with a given ruling, then they are isotopic. A rather complicated geometric way of doing this was proposed in [14] and used in [21, 23] to prove the theorem in special cases, for example when the base is a torus. This procedure was greatly simplified by Lalonde in [5], Lemma I.5. His idea is to transform the one-parameter family of symplectic forms $\omega_{t\in[0,1]}$ by adding at each time an appropriate multiple of the Thom class of a ω_t-symplectic surface Z_t of M and a multiple of the Thom class of a ω_t-symplectic S^2-fiber. This will change the family ω_t into a family ω_t' of constant cohomology class provided that the homology class of the surfaces Z_t is correctly chosen. (This choice depends on the ratios of the slopes of the vectors $[\omega_t]$ and $[\omega_t']$ in $H^2(M, \mathbf{R}) = \mathbf{R}^2$.) This reduces the proof of the fourth classification theorem to the existence of the symplectic surfaces Z_t, which in turn is established by the computation of a Gromov invariant with the help of the equivalence between Seiberg-Witten and Gromov invariants.

Let us mention finally that it is easy to see that all symplectic forms on the k-fold blow-up of a rational or ruled manifold are deformation equivalent. However, it is nontrivial to decide whether two such forms are isotopic whenever they are cohomologous. Some progress with this uniqueness question has been made: see McDuff [17, 20] and Lalonde [5].

Here is a summary of the paper. In §3, we present the needed preliminaries about J-holomorphic curves: compatible almost complex structures, moduli spaces, compactness, regularity, positivity of intersection, and the adjunction formula (see [6] for a more complete summary). In §4, we prove Theorem 2.1 on the structure of rational and ruled symplectic 4-manifolds (except the statement on the uniqueness of the symplectic structure on $\mathbf{CP}^2$ which is proved in §5). In §5, we prove Theorems 2.3 and 2.1 as well as giving the proofs of some of the other fundamental results of Gromov, for example the fact that the group of compactly supported symplectomorphisms of $\mathbf{R}^4$ is contractible. In section §6, we prove Theorem 2.4, explaining in detail the symplectic and holomorphic parts of the argument, and showing how the rest of the proof can be reduced to the computation of the general wall-crossing formula for the Seiberg-Witten equations on ruled surfaces. Finally we sketch the proof of Theorem 2.2.

For the convenience of the reader, here is a list of contents.

3 Preliminaries

3.1 Summary of basic facts on J-holomorphic curves

All manifolds and maps will be assumed to be C^∞-smooth unless specific mention is made to the contrary.

An **almost complex structure** J on a manifold M is an automorphism $J : TM \to TM$ of the tangent bundle such that $J^2 = -\mathrm{Id}$. J is said to be **tamed** by ω if

$$\omega(v, Jv) \quad > \quad 0, \quad \text{for all nonzero } \ v \in TM,$$

and to be **compatible** with ω if in addition

$$\omega(Jv, Jw) \quad = \quad \omega(v, w), \quad \text{for all } \ v, w \in TM.$$

As usual we write $\mathcal{J} = \mathcal{J}(M, \omega)$ for the set of all ω-compatible J (or the set of all ω-tame structures J, depending on the context). These spaces are both nonempty and contractible (because $\mathrm{Sp}(2n; \mathbf{R})$ retracts onto its maximal compact subgroup $U(n)$). (See [27, Chapter 2.5].) Given any $J \in \mathcal{J}(M, \omega)$, we write $c_1 \in H^2(M; \mathbf{Z})$ for the first Chern class of the complex vector bundle (TM, J). This is independent of the choice of $J \in \mathcal{J}(M, \omega)$ since the latter space is connected.

Definition 3.1 (Curves) A (parametrized) J-holomorphic curve in M is a map u from a Riemann surface (Σ, j) to M which satisfies the generalized (nonlinear elliptic) Cauchy-Riemann equation:

$$\bar{\partial}_J u = \tfrac{1}{2}(du \circ j - J \circ du) = 0.$$

The corresponding unparametrized curve $\mathrm{Im}\, u$ will often be denoted C. Thus C has real dimension 2 and complex dimension 1. When Σ is the Riemann sphere, the curve is often said to be **rational**. It is important to note that a J-holomorphic map $u : \Sigma \to M$ is either a **multiple cover**, i.e. it factors through a holomorphic map $\Sigma \to \Sigma'$ of degree > 1, or it is **somewhere injective** in the sense that there is a point $z \in \Sigma$ such that

$$du_z \neq 0, \quad u^{-1}(u(z)) = \{z\}.$$

This is proved in McDuff [13] with more details given in [26, 2.3].

3.2 Local properties of J-curves

3.2.1 Singularities

We begin with a theorem essentially due to McDuff ([19, 22]) and improved by Sikorav in [36]. In particular, his argument works under much weaker smoothness assumptions on J.

Theorem 3.2 (Isolated singularities) *Let (M, J) be an almost complex manifold and u, u' two J-holomorphic maps to M defined on closed Riemann surfaces Σ, Σ'. Then the points where $u(z)$ coincides with $u'(z)$ are isolated. Further, the points where $du(z)$ vanishes are also isolated.*

Definition 3.3 Let $p \in (M, J)$ be the intersection of two J-holomorphic curves $C = u(\Sigma), C' = u'(\Sigma')$. By the previous theorem, we can define the contribution of p to the intersection of the two curves as follows: take small enough neighbourhoods U, U' of p in C, C' and perturb the interior of U, keeping its boundary fixed, so that:

(i) the perturbation of U, denoted say $\tilde{U}$, is C^0-small and stays disjoint from $\partial U'$, and

(ii) $\tilde{U}$ intersects U' transversally (at points which are regular both for $\tilde{U}$ and U').

Then define the contribution k_p of p to the intersection number of C and C' to equal the sum of the intersections of $\tilde{U}$ and U', each counted with the appropriate sign according to the given orientations of C, C', M.

3.2.2 Positivity of intersections and the adjunction formula

The positivity of intersections of two algebraic curves (which states that each point of intersection contributes positively) is a cornerstone of the algebraic geometry of surfaces. The previous theorem gives one hope that this principle extends to the almost complex case. And indeed it does, although the proof is much harder than in the integrable case. It is obvious that the sign of a point p of intersection of two J-curves is positive (and counts for $+1$) when the two curves are regular at p, and positivity is still quite easy to prove when at least one of the two curves is regular at p. It is more difficult when both curves are singular at p.

There is also a notion of positivity of intersection of a single curve: it says that each singular point of a complex curve gives a positive contribution to the self-intersection number of the curve. Again, this still holds in the almost complex case. The proof is quite elementary in the cases where the 1-jet of the singularity has the form (z^k, z^ℓ) with k, ℓ relatively prime, but is harder in the general case. In any case, part (i) of the following theorem was conjectured by Gromov in [2]. The full result (except with a C^0 rather than C^1 ϕ) was proved by McDuff in [16, 19, 22] by a topological argument based on perturbations. An improved and more analytical proof of everything except the last statement was given by Micallef and White in [29], by a method which works under considerably weaker smoothness assumptions on J.

Theorem 3.4 (Positivity) *(i) Let C, C' be two closed J-holomorphic curves. Then the contribution k_p of each point of intersection of C and C' to the intersection number $C \cdot C'$ is a strictly positive integer. It is equal to* 1 *if and only if C and C' are both regular at p and meet transversally at that point.*

(ii) For each singularity p of a J-holomorphic curve C, there is a neighbourhood U of p in M and a C^1-diffeomorphism ϕ from (U,p) to $(B,0) \subset (\mathbf{C}^2,0)$ which sends $C \cap U$ to an isolated singularity at 0 of a complex curve in the ball B. Moreover, there is a C^0-perturbation of J to J' with compact support inside U and a C^0-small isotopy of C to C' (or more precisely of its parametrisation) with compact support inside U such that the curve C' is immersed and J'-holomorphic.

Exercise 3.5 Prove the *adjunction formula*. This says that, if (M^4, J) is an almost complex manifold and $u : \Sigma \to M$ is a J-holomorphic curve which does not factorize through another Riemann surface by a multiple branched covering (that is: u is not a multiple covering), then the virtual genus of $C = \mathrm{Im}(u)$ defined by

$$g_v(C) = 1 + \frac{1}{2}(C \cdot C - c_1(C))$$

is always greater or equal to the genus of Σ, and it is equal to the genus of Σ if and only if u is an embedding. Note that this means that the total weight of singularities of a J-curve in (M^4, J) is a topological invariant. In particular, any J-curve homologous to an embedded J-curve of the same genus is also embedded.
Hint: suppose first that the curve is immersed and decompose the first Chern class of the ambient tangent bundle along the curve in the tangent and normal directions. Compute the self-intersection of the curve in terms of the self-intersection of the curve *inside* its normal bundle and of the number of self-intersection points of the curve (remember that these self-intersections are all positive!) Then deduce the general case from the immersed case using the last assertion of the theorem. For more details see [16].

3.3 Global geometry: moduli spaces and Gromov's compactness theorem

We now describe the global behaviour of the space of all J-holomorphic curves in a given homology class. It turns out that this space, as any solution space of an elliptic system of PDEs, is finite dimensional with dimension given by the Atiyah-Singer index theorem, at least when J is generic. We will see that, when this space is not empty, it is either compact or can be compactified by addition of what Gromov calls *cusp-curves*, which are the analogue of reducible curves in algebraic geometry. Actually, the picture is again very similar to the one in the integrable case. But the proofs are more delicate and rely on Riemannian estimates like the isoperimetric inequality and on properties of elliptic operators.

3.3.1 Fredholm framework

Let (M, ω) be a smooth compact symplectic manifold. As above, let $\mathcal{J}(M)$ be the Fréchet manifold of all almost complex structures which are tamed by

ω. For each class $A \in H_2(M, \mathbf{Z})$ and each non-negative integer g, consider the space $\mathcal{M}_p(A, g, \mathcal{J})$ of all triples (u, j, J) such that

(i) j belongs to the Teichmuller space $\mathcal{T}_g$ of the closed real oriented surface Σ_g of genus g,

(ii) $J \in \mathcal{J}$,

(iii) $u : (\Sigma_g, j) \to (M, J)$ is (j, J)-holomorphic and is not a multiple covering, and

(iv) $[u] = A$ (or more precisely: $u_*([\Sigma_g]) = A$ where the orientation of Σ_g is induced by j).

Hence the set $\mathcal{M}_p(A, g, \mathcal{J})$ is a subset of the product of three Fréchet spaces $\mathcal{F}_{A,g} \times \mathcal{T}_g \times \mathcal{J}$ where $\mathcal{F}_{A,g}$ is the space of all maps of class C^∞ from Σ_g to M in class A which are not multiple coverings. It is actually a Fréchet submanifold, with the induced Fréchet structure (see [13]). The main fact is that the projection $P_A : \mathcal{M}_p(A, g, \mathcal{J}) \to \mathcal{J}$ is a smooth Fredholm operator of real index

$$\dim_{\mathbf{R}} \mathcal{T}_g + 2(c_1(A) + n(1 - g))$$

where n is half the real dimension of M, and $c_1 \in H^2(M, \mathbf{Z})$ is the first Chern class of the tangent bundle (T_*M, J). The inverse image of a point $\{J\}$ by this projection will be denoted $\mathcal{M}_p(A, g, J)$: this is the space of all parametrized J-curves of genus g in class A. When $g = 0$, we will often denote the moduli space by $\mathcal{M}_p(A, J)$.

A clever argument due to Taubes (and written down in [26, 3.4]) shows that the Sard-Smale implicit function theorem, which normally applies only to Fredholm maps between Banach spaces, extends to the projection operator $P_A : \mathcal{M}_p(A, g, \mathcal{J}) \to \mathcal{J}$. It follows that the moduli space $\mathcal{M}_p(A, g, J)$ is a smooth manifold of dimension $6g - 6 + 2(c_1(A) + n(1 - g))$ whenever the linearization dP_A of P_A is surjective for all $u \in \mathcal{M}_p(A, g, J)$, i.e. whenever J is a regular value of P_A. The elements J with this property are said to be **regular for (A, g)-curves**, and form a set $\mathcal{J} = \mathcal{J}_{\text{reg}}(A, g)$ of second category in $\mathcal{J}$.

Note that the Mobius group $\mathrm{G} = \mathrm{PSL}(2, \mathbf{C})$ of holomorphic diffeomorphisms of the sphere acts freely by reparametrization on $\mathcal{M}_p(A, J)$:

$$u \mapsto u \circ \phi, \quad \phi \in \mathrm{G},$$

and the quotient $\mathcal{M}(A, J) = \mathcal{M}_p(A, J)/\mathrm{G}$ is the space of unparametrized J-rational A-curves.

3.3.2 Regularity in dimension 4

Let (M, J) be a 4-dimensional almost complex manifold and let C be a closed immersed J-holomorphic curve of any genus. Gromov showed in [2] (but see [3] for more details) that the space of unparametrised J-curves near C can be identified with the zero set of a non-linear partial differential operator $\bar{\partial}_\nu$ of order 1, which is defined on a neighbourhood of the zerosection in $\Gamma(N)$ and takes values in $\Omega^{0,1}(N)$, where $N = T_C M/TC$ is the normal bundle of C. The linearization L_ν of this operator is elliptic and differs from the ordinary $\bar{\partial}$ associated with a complex connection by an operator of order 0. Hence its index is the same as in the integrable case, which is given by Riemann-Roch:

$$\mathrm{ind}_{\mathbf{R}}(\bar{\partial}_\nu) = 2(c_1(\nu) + 1 - g).$$

Using the decomposition $T_C M = TC + N$ as complex vector bundles, we have $c_1(C) = c_1(\nu) + 2 - 2g$ and so:

$$\mathrm{ind}_{\mathbf{R}}(\bar{\partial}_\nu) = 2(c_1(C) - 1 + g).$$

Hofer, Lizan and Sikorav proved in [3]:

Proposition 3.6 *The above linearised operator L_ν is onto when $c_1(C) > 0$. Thus the space of unparametrised J-holomorphic curves near C is a manifold of dimension $2(c_1(C) - 1 + g)$ for every $J \in \mathcal{J}$.*

When C is embedded, a similar argument shows regularity of the parametrized curves as well. Thus we get:

Proposition 3.7 *If the virtual genus equals the genus and if $c_1(A) > 0$, then all elements of $\mathcal{J}$ are regular values of the Fredholm projection $P_{A,g}$.*

3.3.3 Compactness

Let (M, J, ω) be a compact tame almost complex manifold, and let (Σ, j) be a fixed Riemann surface.

Theorem 3.8 (Gromov) *Let $u_n : \Sigma \to M, n \in \mathbf{N}$, be a sequence of (j, J)-holomorphic maps whose ω-areas are uniformly bounded above. Then if the sequence u_n has no subsequence converging to a (j, J)-holomorphic map $u_\infty : \Sigma \to M$, there is a subsequence of $u_n, n \in \mathbf{N}$, which* bubbles off.

Definition 3.9 A sequence $u_n : \Sigma \to M, n \in \mathbf{N}$, of (j, J)-holomorphic maps *bubbles off* if there is a non-constant J-holomorphic map $v : \mathbf{CP}^1 = \mathbf{C} \cup \{\infty\} \to M$ and a sequence of conformal embeddings $\phi_n : D(R_n)(\subset \mathbf{C}) \to \Sigma$ with $R_n \to \infty$ and ϕ_n converging to a point $p \in \Sigma$, such that the sequence $u_n \circ \phi_n$ converges to v on each compact subset of $\mathbf{C}$.

Remarks. (1) Recall that all C^r-topologies on $M_p(A, g, J)$ are equivalent for $r \geq 0$, so it does not matter which one we use above (see [36], [26] or [6]).

(2) The bubble formation in the limit can be quite complicated: there might be bubbling at different points $p \in \Sigma$ or even many bubbles concentrated near a single point $p \in \Sigma$. However in all cases the attaching pattern of the bubbles can

be described by a tree (see Parker–Wolfson [33]). Note also that some bubbles may be multiple coverings. This phenomenon is not special to the non-integrable case: any bubbling off pattern that can appear in the non-integrable case can also appear in the integrable case.

(3) The theorem is still true if the sequence u_n is (j, J_n)-holomorphic with respect to a sequence J_n converging to a limit J.

The proof of this theorem can be found in chapter V of [1] by Sikorav, or in the book [26] by McDuff and Salamon.

To apply this compactness theorem, we need more information on the limit in the presence of bubbles. Actually, when the surface Σ has genus $g > 0$ and the conformal structure j_n on Σ is not fixed, there is a second way by which the sequence can diverge: the sequence j_n might converge to a point on the boundary of Teichmuller space $\mathcal{T}_g$ thus giving rise to a degenerate curve in the limit. However, this degeneration of the j_n occurs only when there is a non-contractible loop on Σ whose images by u_n shrink to a point. The theorem below shows that these are the only phenomena that can occur in a sequence of J-holomorphic curves of fixed genus and with bounded area. Before stating this formally, it is convenient to make the following definition.

Definition 3.10 Let (M, J) be an almost complex manifold. A *cusp-curve* is a (j, J)-holomorphic map $v : (\Sigma, j) \to (M, J)$ defined on a closed but not necessarily connected Riemann surface, whose image by v is connected.

Then we have:

Theorem 3.11 (Gromov's Compactness theorem) *Let M be a compact manifold equipped with sequences ω_n of symplectic forms and J_n of ω_n-tame almost complex structures which converge to some ω, J. Further, let Σ be a closed oriented surface of genus g, j_n a sequence of conformal structures on Σ, and $u_n : (\Sigma, j_n) \to (M, J_n)$ a sequence of (j_n, J_n)-holomorphic maps. Suppose that the areas of u_n are uniformly bounded above. Then if no subsequence of the (u_n, j_n) converges to a (possibly multiply-covered) J-holomorphic curve, there is a subsequence (also called (u_n, j_n)) which converges weakly to a cusp-curve*

$$u_\infty : (\Sigma_\infty, j_\infty) \to (M, J)$$

either by bubbling or by degeneration of the j_n or perhaps by both. Moreover the homotopy class of this subsequence eventually becomes constant, and

$$\lim \operatorname{area}_{\omega_n}(u_n) = \operatorname{area}_\omega(u_\infty).$$

The precise definition of "weak convergence" is somewhat complicated but it implies convergence in the Hausdorff topology of sets. More details and proofs of various versions of the compactness theorem can be found in Gromov [2], Parker–Wolfson [33], Sikorav [36], Pansu [32] and Ye [41].

Remark. Note that if the homology class of u_n is constant, the limit cusp-curve must realise the same class. Also the ω_n-area of u_n converges to the ω-area of the cusp-curve, so that there is no energy loss. Further, if the cusp-curve consists of the components $C_1 \cup \cdots \cup C_k$ their cohomology classes $A_i = [C_i]$ provide a decomposition of A with $\omega(A_i) > 0$. A typical consequence is that, given a class $B \in H_2(M; \mathbf{Z})$, the set of J for which B may be represented by a J-holomorphic curve or cusp-curve is closed.

Another useful consequence is the following: a class $A \in H_2(M; \mathbf{Z})$ is said to be **indecomposable** (or **simple**) if it does not split as a sum $A_1 + A_2$ of integral classes on which ω is positive. The compactness theorem implies immediately that if A is indecomposable, the moduli space $\mathcal{M}(A, J)$ of unparametrized rational curves is compact.

4 J-holomorphic 2-spheres

In this section we investigate the diffeomorphism type of symplectic 4-manifolds M which contain a symplectically embedded 2-sphere C of nonnegative self-intersection. Since, as we remarked in §2, $M - C$ can be made minimal by blowing down a disjoint collection of exceptional spheres, it suffices to consider the case when $M - C$ is minimal.

Theorem 4.1 *Suppose that (M, ω) is a compact symplectic 4-manifold which contains a symplectically embedded 2-sphere C with nonnegative self-intersection number, and suppose further that $M - C$ is minimal. Then M contains a symplectically embedded 2-sphere C', with self-intersection number either 0 or 1. Moreover, if $C' \cdot C' = 1$, M is diffeomorphic to $\mathbf{CP}^2$, while if $C' \cdot C' = 0$, M is diffeomorphic to an S^2-bundle over a Riemann surface B with symplectic fibers, one of which is equal to C'.*

Here is a sketch of the proof. We will see below that we can assume that C has a J-holomorphic parametrization for some generic J. As mentioned above, the moduli space $\mathcal{M}(A, J)$ of unparametrized rational A-curves (where $A = [C]$) may be compactified by adding to it the set of all A-cusp-curves, together possibly with some multiply-covered spheres. The sphere C' which we are looking for will be a component of an A-cusp-curve S. We will show that, if $\mathcal{M}(A, J)$ is not itself compact, there is a J-indecomposable component C' with $p' = C' \cdot C' \geq 0$. The last step is to show that this is possible only if $p' = 0$ or 1 and M has the stated diffeomorphism type.

4.1 Structure of cusp-curves

We begin the detailed proof by analysing the structure of an A-cusp-curve $S = S_1 \cup \cdots \cup S_m$, using the adjunction formula, and Theorem 3.4 on positivity of intersections. Here the S_i are the components of S. We remind the reader that the word "curve" always denotes the image of a somewhere injective J-holomorphic map, while a component might be multiply-covered. Further, if a component S_i is a multiple covering of a curve T, we will say that J is regular for the class $[S_i]$ if and only if it is regular for $[T]$.

First some simple lemmas. We will assume that all the curves mentioned in this section have genus 0.

Lemma 4.2 *$(u, J) \in \mathcal{M}_p(A, \mathcal{J})$ is regular if and only if $c_1(A) \geq 1$.*

Proof: The "if" statement is proved in Proposition 3.7. To prove the converse, observe that, when $A \neq 0$, the dimension of any nonempty moduli space $\mathcal{M}_p(A, J)$ is at least 6, because the reparametrization group $\mathrm{G} = \mathrm{PSL}(2, \mathbf{C})$ acts freely and has dimension 6. But this dimension equals the index $4 + 2c_1(A)$ of P_A. Hence, we need $c_1(A) \geq 1$. □

Lemma 4.3 *Suppose that A and B are elements of $H_2(M, \mathbf{Z})$ such that*

$$A \cdot A = c_1(A) - 2, \quad B \cdot B \geq c_1(B) - 2,$$

and $A = kB$ for some integer $k \geq 2$. Then, if $c_1(A) \geq 1$, $k = 2$ and $1 = B \cdot B = c_1(B) - 2$.

Proof: This is an easy calculation: see [14, Lemma 2.9]. □

Corollary 4.4 *If S is a multiply-covered curve which is the limit of a sequence of J-holomorphic embedded spheres C_n with $c_1(C_n) \geq 1$ then S is a double covering of an embedded curve with self-intersection $+1$.*

The proof of the next key proposition uses homological calculations to derive a geometric result about the structure of a cusp-curve S which is homologous to C. It is an improved version of Proposition 2.11 in [14]. To explain the assumptions, observe that the minimality of $M - C$ implies that every exceptional sphere E in M meets C, so that the intersection number $E \cdot C = E \cdot S > 0$. Recall also that the adjunction formula says that

$$C \cdot C \geq c_1(C) - 2$$

with equality if and only if C is embedded.

Proposition 4.5 *Let $S_1, \ldots, S_m$ be the components of a J-holomorphic cusp-curve S such that $S \cdot S = c_1(S) - 2$ and $m \geq 2$. Suppose that $E \cdot S \geq 1$ for all exceptional spheres E in M, and that J is regular for each class $[S_i]$, and let $A_i = [S_i]$.*

- *(i) If $A_i \cdot A_i \geq 1$ for some i, then S_i is an embedded curve, and $A_i \neq A_j$ for any $i \neq j$ except possibly if $m = 2$ and $A_i \cdot A_i = 1$.*
- *(ii) If $A_i \cdot A_i = 0$ for some i, then S_i is a k_i-fold cover of an embedded curve for some $k_i \geq 1$.*
- *(iii) If $A_i \cdot A_i < 0$ for some i, then S_i is an exceptional sphere distinct from all other S_j.*
- *(iv) There is at least one component with $A_i \cdot A_i \geq 0$.*

Proof: As in Lemma 4.3 one can show that if S_i is a k-fold cover of a curve T such that $T \cdot T \geq 1$, then

$$S_i \cdot S_i \geq c_1(S_i) - 2 \tag{1}$$

Note that equality can occur here only if $k = 2$ and $T \cdot T = 1$.

We now rearrange the given decomposition of A in order to make as many components as possible satisfy this equation (1). Because J is $[S_i]$-regular for all i, Lemma 4.2 implies that $c(S_i) \geq 1$ for all i. Hence, by the adjunction formula, no component can be a (multiple cover of a) curve with self-intersection

< -1, and any J-holomorphic curve T with self-intersection -1 or 0 must be embedded. (When $T \cdot T = 0$, one must use the fact that $T \cdot T - c_1(T)$ is even. Note also that because curves are by definition somewhere injective, we are ruling out the possibility of singular curves in these classes, eg one with k double points and normal bundle of Chern number $-2k$ or $-2k-1$.) Thus any component with $S_i \cdot S_i < 0$ must either be an exceptional sphere (which satisfies equation (1)), or a multiple cover of an exceptional sphere. The latter do not satisfy equation (1), and we will deal with them later.

Components which are multiple covers of curves of self-intersection 0, do not satisfy equation (1) either. However, they may be dealt with by the following trick. Note that if T is a regular curve with $T \cdot T = 0$ it belongs to a 2-parameter family of J-holomorphic $[T]$-curves. Hence, we may replace any component of S which is a k-fold cover of such a T by k disjoint copies of T (which do not coincide with any other component). This does not disconnect S. For, because S has more than one component, either there is some component S_i such that $S_i \cdot T > 0$, in which case S_i will intersect all the copies of T, or all the components of S are multiple coverings of T. But, by Lemma 4.3, the latter case does not occur. Thus we may assume that all the components with zero self-intersection are distinct embedded curves and therefore satisfy (1).

We next rearrange the components S_i with positive self-intersection so that no two belong to homology classes which are multiples of each other. Thus, if $A_i = cA_j$, where $i \neq j$ and $c \geq 1$, we replace the components S_i and S_j by a single multiply-covered sphere S_i' in the class $A_i + A_j$. This does not disconnect S. Since this operation reduces the number m of components, it is possible that the result consists of a single multiply-covered curve kT. But then Corollary 4.4 implies that we started with a cusp-curve which had exactly two homologous embedded components of self-intersection $+1$. Since this possibility is covered in (i), the Proposition holds in this case.

Now suppose that $m > 1$, that $A_i \neq cA_j$ when $i \neq j$ and $A_i \cdot A_i > 0$, and that all the components of $S = S_1 \cup \cdots \cup S_m$ satisfy (1). Since $S \cdot S = c(S) - 2$, we have

$$\sum_i S_i \cdot S_i + 2 \sum_{i<j} S_i \cdot S_j = \left(\sum_i c(S_i) \right) - 2.$$

Also, because S is connected and the S_i are distinct, there must be at least $m-1$ pairs of components S_i, S_j which intersect. Thus $\sum_{i<j} S_i \cdot S_j \geq m-1$, and so

$$\sum_i S_i \cdot S_i \leq \sum_i (c(S_i) - 2).$$

Therefore, by (1), we must have $S_i \cdot S_i = c(S_i) - 2$ for each i. Further, $\sum_{i<j} S_i \cdot S_j = m-1 > 0$ which means that none of the components S_i with $S_i \cdot S_i > 0$ can be multiple covers, and so they are all embedded curves in distinct homology classes as required by part (i) of the Proposition. Finally, observe that if $A_i \cdot A_i < 0$ for all i, all the components are exceptional spheres. Because there are only $m-1$ points where the m components meet, there must be one component, say S_1, which meets only one other component. Then $S_1 \cdot S = 0$, which contradicts our assumption of minimality. Thus (iv) holds, and the Proposition is proved in this case.

It remains to consider the case when some of the S_i are k_i-fold covers of curves E_i of negative self-intersection so that (1) does not hold for these i. We remarked above that in this case the E_i must be exceptional spheres. Because S is connected, it is possible to reorder the S_i so that

(a) $S_q = k_q E_q, k_q > 1$ if $1 \leq q \leq r$,

(b) $E_q, 1 \leq q \leq r$ and $S_i, i > r$, satisfy (1);

(c) S_i intersects some E_q if and only if $r < i \leq s$;

(d) for $s < i \leq m$, S_i intersects some S_j where $j < i$.

Then, because $c_1(E_q) = 1$ and $E_q \cdot S \geq 1$ for each q by minimality,

$$\begin{aligned}
c_1(S) & - 2 \\
& = \sum_q k_q + \sum_{i>r} c_1(S_i) - 2 \\
& = S \cdot S \\
& = \left(\sum_q k_q E_q + \sum_{i=r+1}^{s} S_i + \sum_{i=s+1}^{m} S_i\right) \cdot \left(\sum_q k_q E_q + \sum_{i=r+1}^{s} S_i + \sum_{i=s+1}^{m} S_i\right) \\
& \geq \sum_q k_q E_q \cdot S + \sum_{i+r+1}^{s} S_i \cdot (\sum_q k_q E_q) + \sum_{i>r} S_i \cdot S_i + 2 \sum_{i>s} \left(S_i \cdot \sum_{j<i} S_j\right) \\
& \geq \sum_q k_q + K(s-r) + \sum_{i>r} S_i \cdot S_i + 2(m-s),
\end{aligned}$$

where $K = \min k_q \geq 2$. Thus

$$\sum_{i>r} c_1(S_i) - 2 \ \geq \ \sum_{i>r} S_i \cdot S_i + 2(m-r),$$

which contradicts (1). Thus this case does not occur. □

4.2 Indecomposability

Our next aim is to show that under the assumptions of Theorem 4.1, there always is a cusp-curve in class A which satisfies the conditions of the above proposition. We first amplify the definition of "indecomposable" given in at the end of §3 to take into account that in dimension 4 the connectedness of a cusp-curve can be described homologically. For simplicity, we continue to assume that all J-holomorphic curves considered have genus 0.

Definition 4.6 The class A is said to be ***J*-indecomposable** if it cannot be written as a sum $A_1 + \cdots + A_m$ for some $m \geq 2$ where

(a) each A_i has a spherical J-holomorphic representative S_i ; and

(b) if $[A_i] \neq [A_j]$ there is a sequence $A_i = A_{i_1}, A_{i_2}, \ldots, A_{i_n} = A_j$ such that $A_{i_p} \cdot A_{i_{p+1}} \geq 1$ for $p = 1, \ldots, n-1$.

Lemma 4.7 **(i)** *If A is J-indecomposable, the unparametrized moduli space $\mathcal{M}(A, J)$ is compact.*

(ii) *If A is J-indecomposable, the subset $\mathcal{D}$ of $\mathcal{J}(M, \omega)$ consisting of all J' for which A is J'-indecomposable is open.*

Proof: Observe that every cusp-curve $S = S_1 \cup \cdots \cup S_m$ and multiply-covered curve gives rise to a decomposition which satisfies conditions (a) and (b) above. (Note that condition (b) is phrased in such a way that the class $mB, m \geq 2$, is not J-indecomposable when B has a J-holomorphic representative, since this class has the allowable decomposition $B + \cdots + B$.) Therefore, if there are no such decompositions, there are no A-cusp-curves and so $\mathcal{M}(A, J)$ is compact by the compactness theorem 3.8.
(ii) is also an easy consequence of the compactness theorem. □

Lemma 4.8 *Let C be a symplectically embedded 2-sphere in (M, ω) in class A where $c_1(A) \geq 1$ and let $A_i, 1 \leq i \leq p$ be any finite subset of $H_2(M, \mathbf{Z})$. Then, there is an element $J \in \mathcal{J}$ which is regular for all A_i and is such that C has a J-holomorphic parametrization.*

Proof: Since ω-tameness is a pointwise condition, it is easy to construct an ω-tame J on M so that the tangent bundle TC is J-invariant. Since all almost complex structures on the 2-manifold C are integrable, it follows that C has a J-holomorphic parametrization. Moreover, by Proposition 3.7 any J-holomorphic parametrization u of C is J-regular. This implies that for all J' sufficiently close to J there is a J'-holomorphic A-curve u' which is close to u. Because the set $\mathcal{J}'$ of J which are A_i-regular for all i is dense, we may choose J' to be in $\mathcal{J}'$ and so close to J that the corresponding curve u' is isotopic to u. In other words, there is an isotopy $\psi_t : M \to M$ such that $u' = \psi_1 \circ u$. (In fact, because (u, J) is a regular point of P_A, the map P_A admits a section over some neighbourhood $\mathcal{N}$ of J_0 in $\mathcal{J}$. In particular, every path $J_t \in \mathcal{N}, 0 \leq t \leq 1$ starting at J lifts to a path u_t of J_t-holomorphic maps starting at $u_0 = u$. Thus we may take J' to be the endpoint of such a path.) It follows that u is J_1-holomorphic, where $J_1 = (\psi_1)^*(J')$. But, by Moser's theorem for pairs (see Chapter 3 of [27]), we may adjust the ψ_t to make them symplectomorphisms. Since $\mathcal{J}'$ is invariant under the action of the group of symplectomorphisms, $J_1 \in \mathcal{J}'$. Hence J_1 satisfies the desired conditions. □

Lemma 4.9 *Let C be a symplectically embedded 2-sphere with $C \cdot C = p \geq 0$ and suppose that $M - C$ is minimal. Then, there is an ω-tame J such that the class $[C]$ may be represented by a J-holomorphic cusp-curve $S = S_1 \cup \cdots \cup S_m$, where, for each i, the class $A_i = [S_i]$ is J-indecomposable and J-regular.*

Proof: Choose J so that C has a J-holomorphic parametrization. It is an easy consequence of the compactness theorem (see §3 and [26, Corollary 4.4.4]) that there is a neighbourhood $\mathcal{N}(J)$ of J in $\mathcal{J}$ such that only finitely many classes A' in $H_2(M;\mathbf{Z})$ with $\omega(A') \leq \omega(A)$ have J'-holomorphic representatives for some $J' \in \mathcal{N}(J)$. Thus, by the previous lemma, we may assume that J is a regular value for all such classes.

If A is J-indecomposable, there is nothing further to prove. Otherwise, consider the set of decompositions $A = A_1 + \cdots + A_m$ which satisfy conditions (a) and (b) in Definition 4.6. By the above, there are only finitely many of these. Each such decomposition gives rise either to a cusp-curve $S = S_1 \cup \cdots \cup S_m$ or to a representation of $[C]$ as a multiply-covered curve kB. But, by Lemma 4.3, in the latter case C also has a representation as the cusp-curve $B + B$. Thus every decomposition corresponds to a cusp-curve. Now observe that, among this finite set, there must be one which is maximal in the sense that it does not admit any further decompositions. The components of this must be J-indecomposable. □

Corollary 4.10 *If $M - C$ is minimal and M contains a symplectically embedded sphere with nonnegative self-intersection, there is a J-indecomposable class which is represented by an embedded J-holomorphic sphere C' with $C' \cdot C' \geq 0$.*

Proof: This follows by combining the above lemma with Proposition 4.5 (iv). Observe that since $M - C$ is minimal, every exceptional sphere E in M must intersect C. Moreover, because we can choose J such that C is J-holomorphic, and then represent the class $[E]$ by a J-holomorphic curve or cusp-curve (see [14]), it follows from Positivity of Intersections that the intersection number $E \cdot C \geq 1$, as is required by Proposition 4.5. □

We now further analyse this indecomposable class.

Proposition 4.11 *Let A be a J-indecomposable class which may be represented by an embedded J-holomorphic sphere, and suppose also that J is regular for the class A. Then $p = A \cdot A \leq 1$.*

Proof: Since A can be represented by an embedded curve C, $c_1(A) = 2 + A \cdot A$ and the dimension of the compact manifold $\mathcal{M}(A, J)$ is $2c_1(A) - 2 = 2p + 2$.

Suppose that $p > 1$, and let $\mathcal{M}_p(A, \mathcal{J})$ be the Fréchet manifold formed by the set of pairs (u, J') where $J' \in \mathcal{J}(M, \omega)$ and u is J-holomorphic. Then consider the evaluation map

$$e = e_{A,p+1} : \mathcal{M}_p(A, \mathcal{J}) \times_G (S^2 \times \cdots \times S^2) \to M \times \cdots \times M$$

given by $(u, J', z_0, \ldots, z_p) \mapsto (u(z_0), \ldots, u(z_p))$, where there are $p+1$ factors in each product and G acts diagonally. We first claim that e is transverse to the inclusion $\iota : M \to M \times \cdots \times M$ given by $x \mapsto (x, x_1, \ldots, x_p)$, where $x_1, \ldots, x_p$ are any distinct points. To prove this, it suffices to show that given tangent vectors v_i to M at x_i for $1 \leq i \leq p$, there is a tangent vector w_0 at x_0 and a family of diffeomorphisms ψ_t such that the path $e_{A,p}(\psi_t \circ u, (\psi_t)_* J, z_0, z_1, \ldots, z_p)$ is tangent to $(w_0, v_1, \dots, v_p)$ at $t = 0$. But this is obvious if $x_0 \neq$ any x_i, and it holds if $x_0 = x_i$ for some i since we may take $w_0 = v_i$. Therefore, because $P_A : \mathcal{M}_p(A, \mathcal{J}) \to \mathcal{J}$ is a Fredholm map, the intersection

$$R = R(J') = P_A^{-1}(J') \cap e^{-1}(\mathrm{Im}\ \iota)$$

is a manifold for generic J'. By Lemma 4.7, we may therefore assume that the given J is generic in this sense. Then $R = R(J)$ is a compact 4-manifold. Further, taking $(x_1, \dots, x_k)$ on the given A-curve, we see that R is nonempty. Moreover, it is easy to check that the composite

$$e_R = \mathrm{pr} \circ e : (u, J, z_0, \dots, z_p) \mapsto u(z_0),$$

induces a degree 1 map from R onto M. (Use the fact that two elements of R which both go through a point $y \neq$ any x_i correspond to two A-spheres which intersect in $p+1$ distinct points and hence must coincide in R because we have factored out by the reparametrization group G.)

Let $\rho : \mathcal{M}_p(A, J) \times_{\mathrm{G}} (S^2 \times \cdots \times S^2) \to \mathcal{M}_p(A, J)/\mathrm{G} = \mathcal{M}(A, J)$ be the obvious projection. We claim that ρ induces a submersion of R onto a submanifold $\rho(R)$ of $\mathcal{M}(A, J)$. In fact, if $p \geq 3$, then, for each $C \in \rho(R)$, there is a unique parametrization u_C of C which takes three fixed points w_1, w_2, w_3 to the three points x_1, x_2, x_3. Hence, $R \cap \rho^{-1}(C) = \{(u_C, z, w_1, \dots, w_p) : z \in S^2\}$, where $u_C^{-1}(x_i) = w_i$ for $3 < i \leq p$. Thus, in this case, R is diffeomorphic to the product of S^2 with the manifold $\rho(R)$. When $p = 2$, one can find a local product structure near a curve $C_0 \in \rho(R)$ as follows. Choose a third point x_3 on C_0 and let $T \in M$ be a little 2-disc which is transverse to C_0 at x_3. Then choose the parametrization u_C of the curves $C \in \rho(R)$ near C_0 by requiring that $u(w_i) = x_i$ for $i = 1, 2$ and $u(w_3) \in T$. Since u_C is unique, the previous argument shows that $\rho : R \to \rho(R)$ is a locally trivial fibration. (Note that a similar argument also works when $p = 0$ or 1.)

We now claim that $\rho(R) \cong S^2$. To see this, identify the tangent space to M at x_1 with $\mathbf{C}^2$ and consider the map $R \to \mathbf{C}P^1 = S^2$ given by:

$$(u, z_0, \dots, z_p) \mapsto \text{tangent space to } \mathrm{Im}\, u \text{ at } u(z_1) = x_1.$$

This map is well-defined since the adjunction formula implies that all the elements of R are embeddings. Further, it clearly factors through ρ, so that we get a map $\theta : \rho(R) \to S^2$. Since points of tangency of two curves contribute at least 2 to the intersection number, θ is injective. Hence it must be a homeomorphism.

Thus R is an S^2-bundle over S^2, and $e_R : R \to M$ is a degree 1 map which takes the fiber F of R to a curve in class A. Let us write $H^*(M)$ for the integral cohomology ring of M modulo torsion. Thus $H^*(M)$ is isomorphic to the image of the integral cohomology $H^*(M, \mathbf{Z})$ in the rational cohomology $H^*(M, \mathbf{Q})$. We first claim that $e_R^* : H^*(M) \to H^*(R)$ is an injection. To see this, observe that if $a \in H^i(M)$, there is by Poincaré duality an element $b \in H^{4-i}(M)$ such that $a \cup b$ generates $H^4(M)$. Therefore, because e_R has degree 1, $e_R^*(a) \cup e_R^*(b)$ generates $H^4(R)$, which implies that neither of $e_R^*(a), e_R^*(b)$ can vanish. Since $F \cdot F = 0$

and $A \cdot A = p > 1$, e_R does not induce an isomorphism on the ring $H^*(M)$, although it is bijective on H^4. It follows easily that the rank of $H^2(M, \mathbf{Q})$ is 1, so that $H^*(M)$ has one generator a of degree 2 such that $a \cup a$ is the generator b of $H^4(M)$. Since there is no injection of this ring into $H^*(S^2 \times S^2)$ which is an isomorphism on H^4, the bundle R must be nontrivial. Thus $H^2(R)$ is generated by two elements $u^{\pm}$ which satisfy the relations $(u^{\pm})^2 = \pm b'$, where b' generates $H^4(R)$. An easy calculation now shows that, if $p > 1$, there is no suitable injection in this case either. □

4.3 Diffeomorphism type of M

It remains to figure out what happens when p is 0 or 1. If $p = 0$ the above argument adapts easily to show that there is a degree 1 map

$$e_R : R \to M$$

from an S^2-bundle R onto M which takes each fiber of R to an A-curve. Since $A \cdot A = 0$, the images of these fibers are all disjoint. The next proposition shows that it is a diffeomorphism.

Proposition 4.12 *If A is a J-indecomposable class with $A \cdot A = 0$ such that $\mathcal{M}_p(A, J)$ is nonempty, then the evaluation map*

$$e = e_A(J) : \mathcal{M}_p(A, J) \times_G S^2 \to M$$

is a diffeomorphism. Thus M is an S^2-bundle over a sphere.

Proof: We saw above that e is smooth and bijective. Clearly, it suffices to show that its derivative has maximal rank everywhere. We will suppose that this fails to hold at (u_0, z_0), and derive a contradiction.

Note first that, by the adjunction formula, the fact that there is one embedded J-holomorphic A-sphere implies that all such spheres are embedded. Therefore, if the rank of $de(u_0, z_0)$ is not maximal, there is a nonzero tangent vector Y to $\mathcal{M}_p(A, J)$ at u_0 such that $Y(z_0)$ is tangent to the curve $C_0 = \operatorname{Im} u_0$. Let u_t be a path in $\mathcal{M}_p(A, J)$ which is tangent to Y at $t = 0$, choose a sequence $t_i \to 0$, and let $z_i \in S^2$ be a point at which the distance $\mu_i = d(u_{t_i}(z_i), C_0)$ is a maximum (where the distance is measured with respect to some metric on M). Then, by construction, the ratio

$$\frac{d(u_{t_i}(z_0), C_0)}{d(u_{t_i}(z_i), C_0)} = \frac{d(u_{t_i}(z_0), C_0)}{\mu_i} \to 0.$$

To see that this is impossible, let us identify a neighbourhood of C_0 with the product $S^2 \times D^2$ in such a way that J equals the standard split almost complex structure J_0 along $C_0 = S^2 \times \{0\}$. By the results of [30] (see also [22]), we may suppose that the discs $\{w\} \times D^2$ are J-holomorphic. Define the expansion maps $\psi_i : S^2 \times D^2(\mu_i) \to S^2 \times D^2$ by

$$\psi_i(w, v) = (w, \frac{v}{\mu_i}),$$

where $D^2(\mu_i)$ denotes the disc of radius μ_i and we assume that $D^2 = D^2(1)$. Then, setting $J_i = (\psi_i)_*(J)$, we claim that J_i converges in the C^∞-topology to an almost complex structure $\tilde{J} = J_0 + B_0$, where J_0 is the product structure and $B_0(w, v)$ is linear in the fiber coordinate $v \in D^2$.

To prove this, we can work locally in a coordinate chart $U \times D^2$ such that the maps $w \mapsto (w, 0)$ and $v \mapsto (w, v)$ are holomorphic. Then the matrix representing the automorphism J of the tangent bundle has the form:

$$J = J_0 + A + B = J_0 + \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ B_{31} & B_{32} & 0 & 0 \\ B_{41} & B_{42} & 0 & 0 \end{pmatrix}$$

where $A = B = 0$ when $v = 0$. (Sikorav derives a more precise normal form in [36, Corollary 3.1.2].) Hence

$$J_i(w, v) = J_0 + A(w, \mu_i v) + \frac{1}{\mu_i} B(w, \mu_i v),$$

and the claim follows.

Now observe that $\tilde{J}$ is tamed by the symplectic form $\tau = \kappa\sigma_0 + dv_1 \wedge dv_2$, where σ_0 is the standard area form on S^2 and $\kappa > 0$ is suitably large. Hence τ tames the J_i for large i, and we may apply Gromov's compactness theorem to the sequence of J_i-holomorphic curves $\psi_i \circ u_{t_i}$. Thus this sequence has a subsequence which converges to some $\tilde{J}$-holomorphic curve or cusp-curve in $S^2 \times D^2$ which represents the homology class $[C_0]$. For topological reasons, the limit has to be a curve, $\tilde{C}$ say. By construction, $\tilde{C}$ goes through the point $(z_0, 0)$ where the original vector field Y vanishes, and, by choice of μ_i, it also goes through some point on $S^2 \times \partial D^2$. Therefore, $\tilde{C}$ is distinct from C_0 and intersects it at least once. Thus $\tilde{C} \cdot C_0 > 0$ by Positivity of Intersections. But this is impossible since $\tilde{C} \cdot C_0 = C_0 \cdot C_0 = 0$. □

Remark The above proof was taken from McDuff [17], Lemma 3.5. A more analytic proof would show that the elements Y of the tangent space $T_u\mathcal{M}_p(A, J)$ represent J'-holomorphic curves for some suitable J'. It then follows that Y cannot be tangent to the zero section by positivity of intersections. For more details of this approach see Hofer–Lizan–Sikorav [3] and Lorek [11].

Proposition 4.13 *If A is a J-indecomposable class with $A \cdot A = 1$ such that $\mathcal{M}_p(A, J)$ is nonempty, then M is diffeomorphic to* $\mathbf{CP}^2$.

Proof: We saw in Proposition 4.11 that there is a degree 1 map e_R from an S^2-bundle R to M which maps the fibers of R into a family of spheres through the fixed point x_1. Moreover R must be the nontrivial bundle over S^2, which is diffeomorphic to the blow-up of $\mathbf{CP}^2$ at a single point, and the task is to show that e_R is diffeomorphic to the blow-down map. Observe that e_R is injective over $M - \{x_1\}$, and that the spheres intersect transversally at x_1. Moreover one can prove that e_R is a diffeomorphism over $M - \{x_1\}$ by adapting the proof of Proposition 4.12. Let B be a closed 4-ball centered at $x_1 \in M$ and chosen so small that its boundary ∂B is fibered by its intersection with the spheres through x_1, and let Σ, U be the inverse images under e_R of x_1 and B. Then Σ is a smoothly embedded section of R and has a neighbourhood with boundary diffeomorphic to $\partial B = S^3$. Therefore Σ has self-intersection 1. Further, the disc bundle $R - \text{Int}\, U$ may be smoothly identified with the complement of a ball B' centered at x_0 in CP^2 in such a way that each disc fiber gets mapped into a line through x_0. Therefore M may be obtained from $M - \text{Int}\, B \cong \mathbf{CP}^2 - \text{Int} B'$ by attaching the ball B by a diffeomorphism $\partial B \to \partial B'$ which preserves the circle fibers. Thus this attaching map lifts a map of the 2-sphere, and so is isotopic to the obvious map obtained from the given identifications of B and B' with the 4-ball. This shows that M is diffeomorphic to $\mathbf{CP}^2$ and that e_R is the blow-down map $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2 \to \mathbf{CP}^2$, as required. □

Proof of Theorem 4.1 We showed in Corollary 4.10 that under the given hypotheses $M - C$ contains a J-holomorphic sphere C with $p = C \cdot C \geq 0$ which is J-indecomposable. Proposition 4.11 shows that p must be 0 or 1. The result now follows from Propositions 4.12 and 4.13. □

Remark. In [18] we generalize Theorem 4.1 to the case of immersed J-spheres. However, the argument breaks down in the case of curves of higher genus. One difference between the two cases is that in the case of (immersed) spheres there always is an evaluation map e of positive degree, while this may not be so for curves of higher genus. The central fact is that the dimension of a family of embedded J-holomorphic spheres of self-intersection p is precisely what is needed for there to be a finite number of curves through a generic set of $p+1$ points. By positivity of intersections, this curve must be unique. Therefore, the degree of the corresponding evaluation map e is exactly 1. We gave in [18] an indirect argument which used this fact to show that each immersed sphere counts with +1. (Lorek gives a direct analytical proof of this in [12].) But for curves of genus g and self-intersection p, all one can say is that there is a finite number of curves through a generic set of $p - g + 1$ points. In [11] Lorek gives examples in which the evaluation map for curves of higher genus is not orientation preserving. Thus cancellations may occur, and so one cannot say anything about the degree of the evaluation map.

Even if the degree is positive one cannot necessarily conclude that M is rational or ruled. In fact, using Taubes's identification of the Gromov invariants with Seiberg–Witten invariants, one can find examples of minimal manifolds M which are not rational or ruled and which support nonempty positive dimensional moduli spaces of curves that map to M by an evaluation map of positive degree. In Taubes's language, the existence of such a moduli space just means that there is an element $A \in H_2(M; \mathbf{Z})$ with nonzero Gromov invariant $\text{Gr}(A)$ for which the formal dimension $d(A) = c_1(A) + A^2$ is > 0. (Thus these manifolds do not have simple type and so must have $B_2^+ = 1$.) Hence we may take M to be the Barlow surface. However, in these examples we must have $c_1(A) \leq 0$, since Liu has shown in [10] that any minimal symplectic manifold that has a symplectic submanifold C with $c_1(C) > 0$ must be rational or ruled.

5 Symplectic forms on rational 4-manifolds and on $\mathbf{R}^4$

In this section we work out some of the results from Gromov's seminal paper [2], and extend them to all ruled surfaces over the sphere. We will begin by studying symplectic structures on the product $M = S^2 \times S^2$. Fix a point $z_0 \in S^2$, and let W be the wedge of two copies of S^2:

$$W = \{z_0\} \times S^2 \ \cup \ S^2 \times \{z_0\}.$$

We will say that a form ω on the product $U \times V$ **splits** if it is the sum of pull-backs of forms from each factor. The usual area form on S^2 (normalized so that the total area is π) will be called σ_0.

5.1 Gromov's results

Theorem 5.1 (Gromov) *Let ω be a symplectic form on $S^2 \times S^2$ which equals the standard split form $\omega_0 = \sigma_0 \oplus \sigma_0$ near W. Then there is a diffeomorphism ϕ which is the identity near W such that $\phi^*(\omega) = \omega_0$.*

Remark. One has to state this theorem rather carefully because it is not known whether or not the group of compactly supported diffeomorphisms of $\mathbf{R}^4$ is connected. Thus the diffeomorphism ϕ above may not be isotopic to the identity. This means that ω splits with respect to *some* product structure on $S^2 \times S^2$ which equals the standard product structure near W but may not be isotopic to it.

Proof of Theorem 5.1 Define the homology classes A, B by setting $A = [S^2 \times \{z_0\}]$ and $B = [\{z_0\} \times S^2]$. Then, because $\omega(A) = \omega(B)$ is the smallest positive value of ω on $H_2(M; \mathbf{Z})$, the classes A and B are both indecomposable. Therefore, by Proposition 4.12 above, both evaluation maps $e_A(J)$ and $e_B(J)$ are diffeomorphisms for all ω-tame J.

Choose $J \in \mathcal{J}(M, \omega)$ so that it equals the standard split structure near W and, for each $z \in S^2$ let $C_A(z)$ (resp. $C_B(z)$) denote the unique A-curve (resp. B-curve) through the point (z_0, z) (resp. (z, z_0)) in W. Consider the map:

$$\Phi : S^2 \times S^2 \to M : (z_1, z_2) \mapsto C_A(z_2) \cap C_B(z_1).$$

It is easy to check that this is a diffeomorphism. Moreover, $\Phi(x) = x$ for all $x \in W$ and for all x sufficiently close to the point $x_0 = (z_0, z_0)$ of intersection of the two spheres in W, since the spheres $C_A(z), C_B(z)$ are standard when z is sufficiently close to z_0.

The map Φ is not quite a symplectomorphism. However, a simple calculation shows that it pushes the split form $\omega_0 = \sigma_0 \oplus \sigma_0$ forward to a form ω'

which is tamed by J. Hence the family $\tau_t = t\omega + (1-t)\omega', 0 \le t \le 1$ consists of forms which are tamed by J and hence nondegenerate. Thus they are symplectic, and so Moser's method yields an isotopy ϕ_t such that $\Phi' = \phi_1 \circ \Phi$ is a symplectomorphism. Because $\Phi(x) = x$ for $x \in W$ and x near x_0, the isotopy τ_t is constant when restricted to W and near x_0. Therefore, we may assume that $\phi_t(x) = x$ for these x too. Thus $\Phi'(x) = x$ for all these x, and it remains to make a final adjustment in the symplectomorphism Φ' to make it equal the identity at all points near W. This may be accomplished using the techniques which prove that the group of symplectomorphisms is locally contractible: see [27] Proposition 3.33. □

The following corollaries are all due to Gromov. The first shows that there are no compact perturbations of the standard symplectic form

$$\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

on $\mathbf{R}^4$.

Corollary 5.2 *If ω is a symplectic form on $D^2 \times D^2$ which equals the standard form ω_0 near the boundary, then there is a diffeomorphism ϕ which equals the identity near $\partial(D^2 \times D^2)$ such that $\phi^*(\omega) = \omega_0$. A similar result holds for forms ω on the unit 4-ball B which are standard near the boundary.*

Proof: To prove the first statement, compactify $D^2 \times D^2$ to $S^2 \times S^2$, and apply the previous result. To prove the second, we isotop the given form ω to a form ω_1 which is standard outside a polydisc $D' \times D'$ which fits inside the ball B, and then apply the previous result. A suitable isotopy is given by

$$\begin{aligned} \omega_t &= \frac{1}{t^2}\phi_t^*(\omega) \text{ on } \phi^{-1}(B) \\ &= \omega_0 \text{ on } B - \phi^{-1}(B), \end{aligned}$$

where ϕ_t is the homothety $\phi_t(x) = tx, 1 \le t \le 2$. □

Corollary 5.3 (Uniqueness on $\mathbf{CP}^2$) *If ω is a symplectic form on $\mathbf{CP}^2$ which is nondegenerate on an embedded 2-sphere S with self-intersection 1 then ω is diffeomorphic to the standard Kähler form.*

Proof: By the symplectic neighbourhood theorem (see [27, Chapter 3]), a neighbourhood of S is symplectomorphic to a neighbourhood of a line $\mathbf{CP}^1$ in $\mathbf{CP}^2$ with its Kähler form τ_0. It is easy to check that the complement $(\mathbf{CP}^2 - \mathbf{CP}^1, \tau_0)$ is symplectomorphic to the open ball in $\mathbf{R}^4$ of radius r where $\pi r^2 = \tau_0(\mathbf{CP}^1) = \omega(S)$. (See Example 6.43 in [27].) Therefore, the result follows from the previous corollary. □

Another interesting corollary is that the group of compactly supported symplectomorphisms of $(\mathbf{R}^4, \omega_0)$ is contractible. This follows immediately from:

Corollary 5.4 *Let $\mathcal{G}$ be the group of symplectomorphisms of $(D^2 \times D^2, \sigma_0 \oplus \sigma_0)$ which equal the identity near the boundary. Then $\mathcal{G}$ is contractible.*

Proof: Let $\mathcal{J}_0$ be the space of all ω_0-compatible J which equal J_0 near $\partial(D^2 \times D^2)$, and consider the map

$$\iota : \mathcal{G} \to \mathcal{J}_0 : \quad \phi \mapsto \phi_*(J_0).$$

Because the projections from $D^2 \times D^2$ to its factors are J_0-holomorphic, any diffeomorphism which preserves J_0 must be a product of diffeomorphisms of the two factors. But the only such diffeomorphism in $\mathcal{G}$ is the identity. Hence ι is injective.

What we want to do now is construct an inverse $r : \mathcal{J}_0 \to \mathcal{G}$ by setting $r(J)$ equal to the diffeomorphism $\Phi(J)$ constructed as in the proof of Theorem 5.1 above. The result will then follow because the space $\mathcal{J}_0$ is contractible. Of course this does not quite work, first because $\Phi(J)$ is not a symplectomorphism, and second because $\Phi(J)$ equals the identity only on the boundary, not near it.

To deal with these points, consider the group $\mathcal{G}'$ of symplectomorphisms of $(D^2 \times D^2, \omega_0)$ which equal the identity on Y, where Y is the union of the boundary $\partial(D^2 \times D^2)$ with a neighbourhood of the corner $\partial D^2 \times \partial D^2$. Then, a variant of the Moser argument (in Chapter 3 of [27] for example) shows that $\mathcal{G}'$ deformation retracts onto its subgroup $\mathcal{G}$. Further, the modification $\phi_1(J) \circ \Phi(J)$ of $\Phi(J)$ which is produced by Moser's method yields an element of $\mathcal{G}'$. It thus remains to check that this isotopy $\phi_t(J)$ may be chosen to vary continuously with J. But the path τ_t of symplectic forms $\tau_t(J) = t\omega + (1-t)\omega'(J)$ depends continuously on Φ and hence on J, and so the map $J \mapsto \phi_1(J)$ will be continuous provided that we solve the equation

$$\frac{d}{dt}\tau_t = d\alpha_t, \quad \alpha_t = 0 \ \text{ near } \ Y$$

by a procedure under which α_t varies smoothly with the family τ_t. This is possible, for example using Hodge theory.

The result now follows because $\mathcal{J}_0$ is contractible. □

Exercise 5.5 (Gromov) Show that the group of symplectomorphisms of the manifold $(S^2 \times S^2, \sigma_0 \oplus \sigma_0)$ deformation retracts onto the subgroup consisting of symplectomorphisms which also preserve J_0. (This subgroup is an extension of order 2 of the product $\mathrm{SO}(3) \times \mathrm{SO}(3)$.) Gromov observed that this result no longer holds when the symplectic form on $S^2 \times S^2$ is changed to $\lambda\sigma_0 \oplus \sigma_0$ for $\lambda > 1$. This is the basis of the construction of two symplectomorphic but nonisotopic forms on the manifold $S^2 \times S^2 \times T^2$ in [13].

Remark 5.6 There are two generalizations of Theorem 5.1 which can be considered: what happens if we assume only that the restriction $\omega|_X$ is nondegenerate, and what happens if we change the cohomology class of ω. The effect of changing the cohomology class is dealt with below. For now, let us consider the hypothesis that $\omega|_X$ is nondegenerate. Then, by Moser's theorem, there is a split isotopy $\phi_t \times \psi_t$ of $M = S^2 \times S^2$ such that $\omega' = (\phi_1 \times \psi_1)^*\omega$ is a multiple of σ_0 on each sphere in W, say

$$\omega' = \begin{cases} \lambda\sigma_0 & \text{on } S^2 \times \{z_0\} \\ \mu\sigma_0 & \text{on } \{z_0\} \times S^2. \end{cases}$$

If the two spheres in W are ω'-orthogonal at their intersection point $x_0 = (z_0, z_0)$, so that $\omega' = \lambda\sigma_0 + \mu\sigma_0$ on T_xM, then it is not hard to prove that there is an isotopy ψ_t which is the identity on W such that $\psi_1^*\omega'$ splits: see McRae [28]. However, McRae showed that in general this does not hold because there are local invariants of the symplectic structure near the point x_0. Nevertheless, McDuff and Polterovich show in [25, Proposition 4.1.C] that one can isotop ω' in any arbitrarily small neighbourhood of x_0 to a form ω'' which is still nondegenerate on each sphere in W and is such that these spheres meet ω''-orthogonally at x.

5.1.1 Symplectic forms on $S^2 \times S^2$

Our aim here is to prove the following result from [14]:

Theorem 5.7 *Let ω be a symplectic form on $S^2 \times S^2$ which splits near $W = \{z_0\} \times S^2 \cup S^2 \times \{z_0\}$. Then ω is isotopic to a split form.*

Proof: We may assume without loss of generality that $\omega = \lambda\sigma_0 \oplus \sigma_0$ near W, where $\lambda \geq 1$. If $\lambda > 1$ then the class $A = [S^2 \times \{z_0\}]$ is no longer indecomposable since it splits as $(A-B)+B$. However, we claim that the classes A and B are J-indecomposable for any J which equals J_0 near W. Indeed, in this case both A and B have J-holomorphic representatives, and so, by positivity of intersections, the only other classes with such representatives have the form $kA + \ell B$ where $k, \ell \geq 0$. Of course, the almost complex structures J which equal J_0 near W are not generic in the set of all J. However, by Proposition 3.7, any J is regular for the classes A and B. Therefore, the proof of Theorem 5.1 goes through without essential change. □

5.2 Classification of ruled surfaces over spheres

We now ready to prove Theorem 2.3. First consider the case $M = S^2 \times S^2$ with form ω. It suffices to show that the fibration $M \to S^2$ has a symplectic section in class $S^2 \times \mathrm{pt}$, since the result then follows from Theorem 5.7. To find the desired section, one first uses a simple geometric procedure to construct a section of (M, ω'), where ω' is deformation equivalent to ω and is obtained by increasing the size of the base as described in the next section. (Briefly: one cuts M open along a fiber so that it is diffeomorphic to a ruled surface over a square, and then constructs the global section from a section over a boundary edge by flowing it through the square along a Hamiltonian flow. The section so obtained may not close up at the boundary, but one can make it do so if one increases the size of the square base.) The second step is to isotop this section of (M, ω') back to (M, ω) using the theory of J-spheres. For more details, see Proposition 4.5 of [14]. □

Remarks. (1) The first step of the above argument works for base manifolds B of arbitrary genus since it is geometric. However the second step only works when B has genus ≤ 1 since these are the only cases in which the homology class $[B \times \mathrm{pt}]$ can be represented by a regular curve. When $B = T^2$ one cannot then establish uniqueness by arguing as in Theorem 5.7 since one cannot include this section into a whole second family of curves using J-holomorphic methods: it is not a generic phenomenon to have a family of tori of self-intersection 0. Nevertheless, as we shall see below, the existence of this section is the essential step in the proof of uniqueness for $B = T^2$.
(2) It is also possible to establish the existence of a symplectic section in class $[B \times \mathrm{pt}]$ by using Taubes's work since the corresponding Seiberg–Witten invariant is nonzero. Again, this works only when B has genus ≤ 1.

Next, consider the nontrivial bundle $\pi : M_S \to S^2$. Recall that M_S is diffeomorphic to the blow-up X of $\mathbf{CP}^2$ at a point, and so in each cohomology class has a standard symplectic form τ_λ which we can describe as follows. As in Corollary 5.3, identify $\mathbf{CP}^2 - \mathbf{CP}^1$ with an open ball in $\mathbf{R}^4$ (which, by rescaling, we can assume to be the unit ball) and then blow up the origin by removing a ball of radius λ centered at 0. Thus, if Σ is the exceptional divisor in X, $X - (\mathbf{CP}^1 \cup \Sigma)$ is symplectomorphic to an open annulus in $\mathbf{R}^4$. It will be convenient for our purposes here to use another picture of this manifold (X, τ_λ). Namely, take the blow-up point x_0 to be on the removed line $\mathbf{CP}^1$ rather than at the origin, and blow this point up by removing a metric ball B centered at x_0 (where we use the Fubini-Study metric). Then it is not hard to see that the image U of $\mathbf{CP}^2 - (\mathbf{CP}^1 \cup B)$ in the open ball $B^4(1)$ is star-shaped. Hence it follows as in Corollary 5.2 that any symplectic form on U which equals the standard form ω_0 outside a compact set is isotopic to ω_0 by an isotopy which is fixed outside a compact set. Translating this result back to the blow-up X of $\mathbf{CP}^2$ we see that any form on X which is standard near the union of a fiber F with the exceptional divisor Σ is isotopic to the standard form. (Note that the proper transform of a line in $\mathbf{CP}^2$ is a fiber.) Therefore, in order to prove that the given form ω on M_S is standard (ie diffeomorphic to τ_λ) we just have to show that it is standard near the union of a fiber with an exceptional sphere.

All we know so far about the form ω on M_S is that it is nondegenerate on the fibers. Therefore, the first task is to show that the class of the exceptional sphere is represented by a J-curve. This follows in the usual way using Taubes's theory: see Liu [10]. However, it can also be shown in the context of pseudoholomorphic curves. To do this, one first constructs a symplectic section of $M_S \to S$ in the class of the line L by the same geometric method that was mentioned above in the case of the trivial bundle. It then follows from Proposition 4.13 that the exceptional sphere must have a J-holomorphic representative C_E , since otherwise the class $[L]$ would be indecomposable. One now puts $F \cup C_E$ into standard position using the arguments described in Remark 5.6. □

Remark The above proof is taken from [20]. This result was first proved in [17, §4] by a slightly different argument which was based on the fact that $X - (\mathbf{CP}^1 \cup F \cup \Sigma)$ may be identified with an open annulus in $\mathbf{R}^4 = \mathbf{C}^2$ minus a complex line which retracts conformally towards the inner boundary (ie towards Σ.)

6 Symplectic forms on irrational ruled manifolds

In this section, we prove Theorem 2.4 which classifies the irrational ruled symplectic 4-manifolds. We recall that a symplectic 4-manifold (M, ω) is said to be **ruled** if ω is nondegenerate on the fibers of some S^2-fibration $M \to B$, and it is irrational if the base B has genus > 0. Such a fibration is also called a **ruling** of M, and forms which are nondegenerate on its fibers are said to be compatible with the ruling.

One first shows that:

Proposition 6.1 *On a ruled manifold, all symplectic forms compatible with the given ruling are deformation equivalent.*

Proof: For this, one uses cutting and pasting arguments to reduce to the case when $B = S^2$. More precisely, let $\pi : (M, \omega) \to \Sigma_g$ be a symplectic S^2-bundle over Σ_g with $g > 0$. Let Λ_g be a $2g$-wedge of loops such that Σ_g cut along Λ_g is a $4g$-polygon, and let $x_0 \in \Lambda_g$ be the attaching point of the loops. Then $\pi^{-1}(\Lambda_g)$ is a coisotropic subspace, homeomorphic to $\Lambda_g \times S^2$ and foliated by symplectic spheres, and it is enough to prove that ω is deformation equivalent to a form $\bar{\omega}$ which splits in a neighbourhood of $\pi^{-1}(\Lambda_g)$. Indeed, the compactification of $(M - \pi^{-1}(\Lambda_g), \bar{\omega})$ may then be compactified to a symplectic S^2-bundle (M', ω') over S^2 by the addition of a fiber F_0, and ω' would then be isotopic to a standard form by Theorem 2.3. Further, since ω' is already split in a neighbourhood $D \times F_0$ of F_0, the proof of Theorem 2.3 shows that the isotopy to a standard form can be chosen constant on $D \times F_0$. Thus this isotopy may be pulled-back to the original manifold M. This gives an isotopy between $\bar{\omega}$ and a standard Kahler form.

Since $\pi^{-1}(\Lambda_g)$ is foliated by the fibers of π, the characteristic flow of $\pi^{-1}(\Lambda_g)$ is everywhere transversal to the fibers and defines, for each loop $\gamma_i \subset \Lambda_g$, a symplectic diffeomorphism ϕ_i from $S^2_{x_0} = \pi^{-1}(x_0)$ to itself. Here ϕ_i is the monodromy of the foliation round the hypersurface $\pi^{-1}(\gamma_i)$. By the coisotropic neighbourhood theorem, the restriction of ω to $\pi^{-1}(\gamma_i)$ determines its structure near $\pi^{-1}(\gamma_i)$. Therefore, in order to show that ω splits near $\pi^{-1}(\Lambda_g)$ it suffices to show that its characteristic foliation on $\pi^{-1}(\Lambda_g)$ is topologically trivial, i.e. that the monodromy $\phi_i = \mathrm{id}$. Thus the proof boils down to constructing a deformation between ω and a form having trivial monodromy round all loops γ_i.

For this, we take a symplectic product neighbourhood $U \times S^2_{x_0}$ of the fiber and deform ω by adding $t\pi^*\rho$, where ρ is a nonnegative 2-form on the base with support in U. When t is large enough the new form $\omega_t = \omega + t\pi^*\rho$ will be a product on a set whose volume is large compared to the rest of M. This will give lots of room for modifying each hypersurface $\pi^{-1}(\gamma_i) \cap (U \times S^2_{x_0})$ so that the one-turn monodromy is trivial. See [14] or [5], Proposition I.4, for the details. (Actually, the increase in the area of the base needed to trivialize each monodromy ϕ_i is equal to the Hofer norm of ϕ_i).

It remains to show:

Proposition 6.2 *Let ω, ω' be two cohomologous symplectic forms compatible with a given ruling. If they are deformation equivalent, then they are isotopic.*

Proof: First, we explain how to use a suitable family of symplectic submanifolds Z_t to change a deformation into a genuine isotopy. The submanifolds Z_t are constructed in the next paragraph.

Recall that for each base B, there are exactly two S^2-bundles up to fiberwise diffeomorphism: the trivial bundle $\pi : B \times S^2 \to B$ and the non-trivial one $\pi : M_B \to B$. (If we think of M as the projectivization of a rank 2 complex vector bundle E over B, then these bundles may be distinguished by the Stiefel-Whitney class $w_2(E)$, which is zero in the trivial case and nonzero otherwise.) We will suppose that both the fiber F and base B are oriented and, without loss of generality, will only consider symplectic forms compatible with these orientations.

For simplicity we will first explain the proof for the trivial bundle, and will write $\{a_F, a_B\}$ for the basis of $H^2(B \times S^2; \mathbf{R})$ which is dual to the homology basis $\{[F] = [\mathrm{pt} \times S^2], [B] = [B \times \mathrm{pt}]\}$. Thus

$$a_F(F) = 1, \quad a_F(B) = 0, \quad a_B(F) = 0, \quad a_B(B) = 1.$$

Clearly, there is a (compatibly oriented) symplectic form in the class $xa_F + ya_B$ if and only if $x, y > 0$.

Proposition 6.3 *Let $\pi : M \to B$ be a topologically trivial S^2-bundle. Let τ_t, $t \in [0,1]$, be any deformation path joining two π-compatible symplectic forms with $[\tau_0] = [\tau_1] = xa_F + ya_B$, where $x, y > 0$. Assume that there is a smooth 1-parameter family of embedded τ_t-symplectic submanifolds Z_t in a class $m[F] + n[B]$ which satisfies $n > mx/y$. Then the path τ_t can be transformed into an isotopy through a family of deformation paths which join the same endpoints τ_0, τ_1.*

Proof: Let S_t be a 1-parameter family of τ_t-symplectic 2-spheres in class $[F]$. (These can be constructed as J_t-holomorphic curves, where J_t is a generic family of τ_t-tame almost complex structures.) Following an argument of [5], we use the two families S_t and Z_t to transform the deformation path τ_t into a genuine isotopy with same endpoints τ_0, τ_1.

The first step is to construct smooth families of forms σ_t, ρ_t which represent the Poincaré duals of S_t, Z_t respectively, and are such that the forms $\tau_t + s\sigma_t$ and $\tau_t + r\rho_t$ are symplectic for all $s, r \geq 0$ and all t. Here is a construction for the forms σ_t. Because S_t has trivial normal bundle, the symplectic neighbourhood theorem (see [27] for example) implies that there is a smooth family of diffeomorphisms ψ_t of a neighbourhood $\mathcal{N}(S_t)$ of S_t into a neighbourhood of $S^2 \times \{0\}$ in the product $S^2 \times D^2$ which pushes τ_t forward to the product form $\alpha_0 + \alpha_1$. Choose an open neighbourhood V of $\{0\}$ in the disc D^2 such that $S^2 \times V$ is in the image of ψ_t for all t, let f be a bump function with support in V and then set

$$\sigma_t = \psi_t^*(\alpha_0 + f\alpha_1).$$

The forms ρ_t may be constructed using a similar (but slightly more complicated) normal form for the symplectic neighbourhood of Z_t: see [21, Lemma 3.7].

Next, take a_F, a_B as ordered basis for $H^2(M, \mathbf{R})$ and identify $H^2(M, \mathbf{R})$ with $\mathbf{R}^2$. For all t, $[\tau_t]$ belongs to the open sector of $H^2(M, \mathbf{R})$ comprised between the positive horizontal axis (generated by a_F) and the positive vertical axis (generated by a_B)), because the nondegeneracy condition of these forms imply that their cohomology classes cannot cross the axes. The half-line $L = \{C[\tau_0] : C > 0\}$, which contains both endpoints of the path $[\tau_t]$, divides this quadrant into two open sectors, say Q_1, Q_2, respectively the sector comprised between the horizontal axis and L, and the one comprised between L and the vertical axis. When $[\tau_t]$ belongs to Q_1, there is a unique positive real number s_t such that $[\omega_t = \tau_t + s_t\sigma_t]$ belongs to L, and when $[\tau_t]$ belongs to Q_2, there is a unique positive real number s_t such that $[\omega_t = \tau_t + s_t\rho_t]$ belongs to L because the above inequality $n > m\frac{x}{y}$ means that the slope m/n of $[\rho_t]$ is smaller than the slope y/x of L. Clearly s_t is a piecewise smooth function of t. Finally, let $\kappa_t > 0$ be such that $\kappa_t[\omega_t] = [\tau_0]$. Then the 1-parameter family $\kappa_t\omega_t$ (reparametrized in t to make it smooth) is a genuine isotopy between $\omega_0 = \tau_0$ and $\kappa_1\omega_1 = \tau_1$. □

A similar argument works in the case of the nontrivial bundle. Before describing it we must introduce some notation. Let us denote by $[F], [B_k] \in H_2(M, \mathbf{Z})$ the classes of the fiber and of the section of self-intersection k. (If the bundle is topologically nontrivial, the self-intersections of sections are always odd, and any odd integers can appear in this way.) Thus $[F], [S_1]$ is a basis for $H_2(M, \mathbf{Z})$, and $[F], [B] = \frac{1}{2}([B_{-1}] + [B_1])$ is a basis for $H_2(M, \mathbf{R})$. If the genus g of the base is > 0, McDuff showed in [14] that a class $\alpha \in H^2(M, \mathbf{R})$ contains a π-compatible symplectic form exactly when $\alpha([F])$ and $\alpha([B])$ are strictly positive.[2] Therefore, if we define $\{a_F, a_B\}$ as before to be the dual basis to $\{[F], [B]\}$ the previous argument applies word for word. This proves:

Proposition 6.4 *With the above definition of the classes $a_F, a_B \in H^2(M)$ and $[F], [B] \in H_2(M)$, Proposition 6.3 also holds for the nontrivial bundle.*

6.1 Gromov invariants

The Gromov invariants of the symplectic 4-manifold (M, ω) defined by Taubes in [37] roughly speaking count the number of (possibly *disconnected*) J-holomorphic curves in a given homology class A. More precisely, given $A \in H_2(M; \mathbf{Z})$ put $k(A) = \frac{1}{2}(c_1(A) + A \cdot A)$ and choose a set Ω of $k(A)$ distinct points in M. Then, for each ω-tame almost complex stucture J, let $\mathcal{H}_J(A)$ be the set of all pairs (ϕ, C) which satisfy the following conditions:

• ϕ is a J-holomorphic map from the possibly disconnected but closed Riemann surface Σ to M such that $\phi_*([\Sigma]) = A$;

[2] Interestingly enough, this is not true when the base is S^2 since in this case the class $[B_{-1}]$ is always represented by a symplectically embedded submanifold (in fact, by an exceptional sphere) and so we must have $\alpha(B_{-1}) > 0$. Thus the class α contains a compatible form if and only if $\alpha(F)$ and $\alpha(B_{-1})$ are both strictly positive.

• $\Omega \subset \phi(\Sigma)$.

Taubes shows that if M is minimal and J is generic there is a finite number of elements of $\mathcal{H}_J(A)$. Moreover, if $\Sigma_i, i = 1, \ldots, k$, are the components of Σ, the restriction ϕ_i of ϕ to the component Σ_i is an embedding, except possibly if Σ_i is a torus. (In this case, ϕ_i could be the multiple covering of an embedding, and extra information is needed to count them: see [39]. However, this case is not relevant to our work here.) Each map ϕ_i inherits a natural sign $\varepsilon_i = \pm 1$ from the moduli space to which it belongs, and we define the sign of ϕ itself to be

$$\varepsilon(\phi) = \prod_i \phi_i.$$

The Gromov invariant is then defined as:

$$\mathrm{Gr}(A) = \sum_{(\phi,\Sigma)\in H_J(A)} \varepsilon(\phi), \quad \text{for generic } J.$$

When M is not minimal one has to check that multiply covered exceptional spheres do not cause problems. This question is discussed in [24].

The next proposition shows that all we have to do to find the submanifolds Z_t needed in Proposition 6.3 is calculate some Gromov invariant.

Proposition 6.5 *Let (M, τ_0) be a symplectically ruled surface over a base manifold B of genus $g > 0$, and let $A = (g-1)[F] + n[B] \in H_2(M, \mathbf{Z})$ where $n \geq g$. In the case $g = 1$, assume that $n = 1$ if M is the topologically trivial S^2-bundle and that $n = 2$ otherwise. Suppose that $\mathrm{Gr}(A) \neq 0$. Then any deformation path τ_t can be reparametrized so that there is a family of τ_t-symplectic submanifolds Z_t in class A as required by Propositions 6.3 and 6.4.*

Proof: Let $\mathcal{J}$ be the space of all C^∞ almost complex structures on M which are τ_t-tame for some $t \in [0,1]$, and let J_t be a path in $\mathcal{J}$ which joins a generic τ_0-tame almost complex structure J_0 to a generic τ_1-tame element J_1. We show below that for all t the set $\mathcal{H}_{J_t}(A)$ contains only connected curves. This means that $\mathcal{H}_{J_t}(A)$ is a subset of the moduli space of J_t-holomorphic A-curves (it consists of all curves through the $k(A)$ points of Ω), and the theory of J-holomorphic curves shows that for generic path J_t (with fixed endpoints), the set

$$\mathcal{H} = \bigcup_t \mathcal{H}_{J_t}(A)$$

is a compact oriented 1-dimensional manifold which provides a cobordism between $\mathcal{H}_{J_0}(A)$ and $\mathcal{H}_{J_1}(A)$. (See, for example, [26, Proposition 7.2.1] and [34].) Since $\mathrm{Gr}(A) \neq 0$ there must be at least one arc $(\phi_s, C_s), s \in [0,1]$, in $\mathcal{H}$ which starts on $\mathcal{H}_{J_0}(A)$ and ends on $\mathcal{H}_{J_1}(A)$. Clearly, there is a continuous map $\beta : [0,1] \to [0,1]$ with $\beta(0) = 0, \beta(1) = 1$ such that ϕ_s is J-holomorphic for some J which is $\tau_{\beta(s)}$-tame. By slightly perturbing everything if necessary, we may assume that β is smooth. Then reparametrize τ_t to be the family

$$t \mapsto \tau_{\beta(t)}.$$

and set Z_t equal to the image of ϕ_t. Note that, by Taubes's definition of the Gromov invariant, each ϕ_t must be an embedding when $g > 1$. When $g = 1$, the choice of n in the statement of the proposition means that the class A is primitive, hence the elliptic curve ϕ_t cannot be a multiple covering and must therefore be embedded.

It remains to show that $\mathcal{H}_J(A)$ consists of connected curves. Suppose that $(\phi, C) \in \mathcal{H}_J(A)$ has ℓ connected components in the (nonzero) classes $A_1, \ldots, A_\ell$ where $A_i = m_i[F] + n_i[B]$. Then because the components are disjoint we need $A_i \cdot A_j = 0$ for $i \neq j$. But it is easy to see that $A = (g-1)[F] + n[B]$ has no such decomposition with $\ell > 1$ except possibly in the case $g = 1$. But when $g = 1$ we assumed that $n = 1$ for the trivial bindle, and $n = 2$ otherwise. Since $[B]$ is not an integral class in the latter case, the result follows. □

One way to calculate Gromov invariants is to find a generic almost complex structure (if one is lucky it will be integrable) and then count the relevant J-holomorphic curves. This approach is manageable when A is the class of a section, and was used in [21, 23] to classify symplectic ruled surfaces over the torus. When the base has genus > 1 the above proposition shows that we need to look at more general classes A. No doubt, a similar approach via complex analysis would work for these classes. However, since Taubes's approach leads to the result we need now, we will go by that route.

6.2 Taubes's theory

Taubes's main result is that the Gromov invariants coincide with certain Seiberg–Witten invariants. For the convenience of the reader, we quickly review the relevant parts of Taubes's theory before calculating the invariant we need.

We consider perturbed Seiberg–Witten equations parametrized by a pair (g, η) where g is a metric on X and η is an imaginary-valued g-self-dual 2-form. Given an integral homology class $\beta \in H_2(X)$, let Γ_β be the spin^c-structure with determinant bundle L_β where $c_1(L_\beta) = -K + 2\mathrm{PD}(\beta)$. (Here, $\mathrm{PD}(\beta)$ denotes the Poincaré dual of β and $K = -c_1(\omega)$ is the canonical divisor. Also, we will denote a general homology class by β since in this context the letter A is used to denote a connection.) Then the equations are:

$$D_A(\Phi) = 0, \quad F_A^+ = \sigma(\Phi) - \eta, \tag{2}$$

where D_A is the Dirac operator on plus spinors, F_A^+ is the g-self-dual part of the curvature of a connection A on L_β and σ is a quadratic function of the spinor Φ. When $b_2^+(M) > 1$ the number of solutions of these equations (when appropriately counted) is independent of (g, η) and is called the Seiberg–Witten invariant $SW(M, \Gamma_\beta)$.[3] Note that this can be non-zero only when the index (or formal dimension of the moduli space) of the problem is nonnegative, ie when

$$d(\beta) = -K(\beta) + \beta^2 \geq 0.$$

[3]When the dimension $d(\beta)$ of the solution space is > 0 the Seiberg–Witten invariant is given by the homology class represented by the solution space in a suitable ambient space, rather than by the algebraic number of its points.

On the other hand, when $b_2^+(M) = 1$, the space of pairs (g, η) is divided into two halves by a hyperplane (the wall) and (provided that $d(\beta) \geq 0$) the number of solutions *jumps* as one crosses the wall. Thus there are now two Seiberg–Witten invariants $SW^{\pm}(M, \Gamma_\beta)$ which differ by the wall-crossing number $w(\beta)$. The wall has the equation

$$\varepsilon_\beta(g, \eta) = 0,$$

where

$$\varepsilon_\beta(g, \eta) = \pi(K - 2\mathrm{PD}(\beta)) \cdot \omega_g - i\eta \cdot \omega_g.$$

(Here ω_g is the unique g-self-dual 2-form of norm 1 and the operation $\cdot$ on cohomology classes evaluates their cup product on the fundamental class $[M]$.) Hence one can detect where one is by looking at the sign of $\varepsilon_\beta(g, \eta)$.

Taubes considers pairs (g, η_r) of the form

$$\eta_r = 4ri\omega_g + F^+_{A_0}, \quad r \to \infty,$$

where A_0 is the connection on L_0 given by g and where g is a metric defined in terms of the symplectic form ω and an almost complex structure J by $g(x, y) = \omega(x, Jy)$. He shows that there is a 1-to-1 correspondence between the number of solutions to these equations (for large r) and the Gromov invariant $\mathrm{Gr}(\beta)$ which counts the number of J-holomorphic representatives of the class β. Since

$$\varepsilon_\beta(g, \eta_r) > 0 \quad \text{for large } r,$$

his theorem can be stated as:

Theorem 6.6 (Taubes)

$$\pm\mathrm{Gr}(\beta) = SW^+(M, \Gamma_\beta).$$

Strictly speaking, Taubes proved this only when $\beta \neq 0$, because one cannot define $\mathrm{Gr}(\beta)$ in terms of holomorphic curves when $\beta = 0$. However, he also showed that $SW^+(M, \Gamma_0) = 1$ and so we extend the above to all β by simply defining $\mathrm{Gr}(0)$ to be 1.

It is clear from the definition of Gromov invariants in terms of counting J-holomorphic curves that they only depend on ω up to deformation. The above theorem shows that in fact $\mathrm{Gr}(\beta)$ depends only on the canonical class K. No-one has as yet found a 4-manifold which supports symplectic structures with different canonical classes K, but it is not impossible that such exist.

In order to show that certain Gromov invariants do not vanish, we need a mechanism which produces nonzero Seiberg–Witten invariants. Because rational and ruled manifolds have $b_2^+ = 1$, we can do this by calculating the jump $w(\beta)$ in the number of solutions of the Seiberg–Witten equations as (g, η) crosses the wall in the positive direction. Kronheimer and Mrowka showed in [4] that this number is 1 when M is simply connected (or, more generally, when $b_1(M) = 1$). The general case has been worked out by several people: see Li-Liu [9], Ohta–Ono [31] and Salamon [35].

Here is a statement of the formula. Since $b_1(M)+b_2^+(M)$ must be odd when M is symplectic, we know that $b_1(M)$ is even, say $b_1(M) = 2m$. Let $\mathbf{T}$ denote the $2m$-torus $H^1(M, \mathbf{R}/\mathbf{Z})$ which parametrizes the space of flat connections on line bundles on M, and define $\Omega \in H^2(M \times \mathbf{T})$ by

$$\Omega = \sum_i x_i \cup y_i$$

where x_i is a basis for $H^1(M, \mathbf{Z})$ and y_i is the corresponding basis for $H^1(\mathbf{T})$. Then, provided that the formal dimension $d(\beta) \geq 0$,

$$w(\beta) = \frac{(-1)^m}{4^m m!} \left(\int_M \Omega^2 \cup (-K + 2\mathrm{PD}(\beta)) \right)^m [\mathbf{T}],$$

where $\int_M$ integrates the element $\Omega^2 \cup (-K + 2\mathrm{PD}(\beta)) \in H^6(M \times \mathbf{T})$ over the fiber M yielding an element in $H^2(\mathbf{T})$.

Proposition 6.7 (Li–Liu, Ohta–Ono) *Let (M, ω) be a symplectically ruled surface over B where $g = \mathrm{genus}\,(B) > 0$, and let $\beta = m[F] + n[B]$, where the fiber class $[F]$ and base $[B]$ are defined as above. Then, if $d(\beta) \geq 0$,*

$$\pm \mathrm{Gr}\,(\beta) = SW^+(M, \Gamma_\beta) = (n+1)^g.$$

Proof: (Sketch) Every such manifold M admits a metric g of positive scalar curvature which is compatible with some symplectic structure. (Just make the fibers very "small" compared to the base so that their curvature is large.) Witten showed in [40] that the Seiberg–Witten equations for the pair $(g, 0)$ have no solutions at all in this case. It is also easy to check that for the given β,

$$\varepsilon_\beta(g, 0) = \pi(K - 2\mathrm{PD}(\beta)) \cdot \omega < 0.$$

Therefore as r increases along the Taubes perturbation, one crosses the wall and picks up $w(\beta)$ solutions. Hence $SW^+(M, \Gamma_\beta) = w(\beta) = (n+1)^g$. □

This completes the proof of Theorem 2.4.

To finish we remark that Theorem 2.2 is proved by elaborating these ideas. One has to get some handle on the possible values for K. For example, when $M = \mathbf{CP}^2$ Taubes showed that $\omega \cdot K$ has to be negative. It then remains to show that there is a class β with nonzero Gromov invariant such that $-K(\beta) > \beta^2$. For a holomorphic representative of such a class must have at least one component that is a sphere, since all embedded curves of higher genus have $-K(\beta) \leq \beta^2$ by the adjunction formula. Morover, this sphere has nonnegative self-intersection because M is minimal.

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