

Symplectic structures on fiber bundles

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Abstract

Let $\pi : P \rightarrow B$ be a locally trivial fiber bundle over a connected CW complex B with fiber equal to the closed symplectic manifold (M, ω) . Then π is said to be a *symplectic fiber bundle* if its structural group is the group of symplectomorphisms $\text{Symp}(M, \omega)$, and is called *Hamiltonian* if this group may be reduced to the group $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms. In this paper, building on prior work by Seidel and Lalonde, McDuff and Polterovich, we show that these bundles have interesting cohomological properties. In particular, for many bases B (for example when B is a sphere, a coadjoint orbit or a product of complex projective spaces) the rational cohomology of P is the tensor product of the cohomology of B with that of M . As a consequence the natural action of the rational homology $H_k(\text{Ham}(M))$ on $H_*(M)$ is trivial for all M and all $k > 0$.

Added: The erratum makes a small change to Theorem 1.1 concerning the characterization of Hamiltonian bundles.

keywords: symplectic fiber bundle, Hamiltonian fiber bundle, symplectomorphism group, group of Hamiltonian symplectomorphisms, rational cohomology of fiber bundles

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1 Introduction and main results

In this section we first discuss how to characterize Hamiltonian bundles and their automorphisms, and then describe their main properties, in particular deriving conditions under which the cohomology of the total space splits as a product. Finally we state some applications to the action of $\text{Ham}(M)$ on M and to nonHamiltonian symplectic bundles. This paper should be considered as a sequel to Lalonde–McDuff–Polterovich [13] and McDuff [21] which establish analogous results for Hamiltonian bundles over S^2 . Several of our results are well known for Hamiltonian bundles whose structural group is a compact Lie group. They therefore fit in with the idea mentioned by Reznikov [24] that the group of symplectomorphisms behaves cohomologically much like a Lie group.

1.1 Characterizing Hamiltonian bundles

A fiber bundle $M \rightarrow P \rightarrow B$ is said to be symplectic if its structural group reduces to the group of symplectomorphisms $\text{Symp}(M, \omega)$ of the closed symplectic manifold (M, ω) . In this case, each fiber $M_b = \pi^{-1}(b)$ is equipped with a well defined symplectic form ω_b such that (M_b, ω_b) is symplectomorphic to (M, ω) . Our first group of results establish geometric criteria for a symplectic bundle to be Hamiltonian, i.e. for the structural group to reduce to $\text{Ham}(M, \omega)$. Quite often we simplify the notation by writing $\text{Ham}(M)$ and $\text{Symp}_0(M)$ (or even Ham and Symp_0) instead of $\text{Ham}(M, \omega)$ and $\text{Symp}_0(M, \omega)$.

Recall that the group $\text{Ham}(M, \omega)$ is a connected normal subgroup of the identity component $\text{Symp}_0(M, \omega)$ of the group of symplectomorphisms, and fits into the exact sequence

$$\{id\} \rightarrow \text{Ham}(M, \omega) \rightarrow \text{Symp}_0(M, \omega) \xrightarrow{\text{Flux}} H^1(M, \mathbb{R})/\Gamma_\omega \rightarrow \{0\},$$

where Γ_ω is the flux group.¹ Because $\text{Ham}(M)$ is connected, every Hamiltonian bundle is symplectically trivial over the 1-skeleton of the base. The following proposition was proved in [18] Thm. 6.36 by a somewhat analytic argument. We give a more topological proof in §2.1 below.

Theorem 1.1 *A symplectic bundle $\pi : P \rightarrow B$ is Hamiltonian if and only if the following conditions hold:*

- (i) *the restriction of π to the 1-skeleton B_1 of B is symplectically trivial, and*
- (ii) *there is a cohomology class $a \in H^2(P, \mathbb{R})$ that restricts to $[\omega_b]$ on M_b .*

There is no loss of generality in assuming that the bundle $\pi : P \rightarrow B$ is smooth. Then recall from Guillemin–Lerman–Sternberg [8] (or [18] Chapter 6) that any 2-form τ on P that restricts to ω_b on each fiber M_b defines a connection ∇_τ on P whose horizontal distribution Hor_τ is just the τ -orthogonal complement of the tangent spaces to the fibers:

$$\text{Hor}_\tau(x) = \{v \in T_x P : \tau(v, w) = 0 \text{ for all } w \in T_x(M_{\pi(x)})\}.$$

Such forms τ are called *connection forms*. The closedness of τ is a sufficient (but not necessary) condition for the holonomy of ∇_τ to be symplectic, see Lemma 2.2. A simple argument due to Thurston ([18] Thm. 6.3 for instance) shows that the cohomological condition (ii) above is equivalent to the existence of a closed extension τ of the forms ω_b . Condition (i) is then equivalent to requiring that the holonomy of ∇_τ around any loop in B belongs to the identity component $\text{Symp}_0(M)$ of $\text{Symp}(M)$. Hence the above result can be rephrased in terms of such closed extensions τ as follows.

Proposition 1.2 *A symplectic bundle $\pi : P \rightarrow B$ is Hamiltonian if and only if the forms ω_b on the fibers have a closed extension τ such that the holonomy of ∇_τ around any loop in B lies in the identity component $\text{Symp}_0(M)$ of $\text{Symp}(M)$.*

¹It is not known whether this group is discrete in all cases, although its rank is always less than or equal to the first Betti number of M by the results of [13]. It is discrete if $[\omega]$ is a rational (or integral) class and in various other cases discussed in [12] and Kedra [9].

This is a slight extension of a result of Guillemin–Lerman–Sternberg, who called a specific choice of τ the *coupling form*: see also [18] Thm. 6.21. As we show in §3.2 the existence of τ is the key to the good behavior of Hamiltonian bundles under composition.

Remark 1.3 When M is simply connected, $\text{Ham}(M)$ is the identity component $\text{Symp}_0(M)$ of $\text{Symp}(M)$, and so a symplectic bundle is Hamiltonian if and only if condition (i) above is satisfied, i.e. if and only if it is trivial over the 1-skeleton B_1 . In this case, as observed by Gotay *et al.* in [5] Theorem 2, it is known that (i) implies (ii) for general topological reasons to do with the behavior of evaluation maps. (One can reconstruct their arguments from our Lemmas 2.3 and 2.4.) More generally, (i) implies (ii) for all symplectic bundles with fiber (M, ω) if and only if the flux group Γ_ω vanishes.

1.2 Hamiltonian structures and their automorphisms

The question then arises as to what a Hamiltonian structure on a fiber bundle actually is. How many Hamiltonian structures can one put on a given symplectic bundle $\pi : P \rightarrow B$? What does one mean by an automorphism of such a structure? These questions are discussed in detail in §2.2. We now summarize the results of that discussion.

In homotopy theoretic terms, a Hamiltonian structure on a symplectic bundle $\pi : P \rightarrow B$ is simply a lift \tilde{g} to $B\text{Ham}(M)$ of the classifying map $g : B \rightarrow B\text{Symp}(M, \omega)$ of the underlying symplectic bundle, i.e. it is a homotopy commutative diagram

$$\begin{array}{ccc} & B\text{Ham}(M) & \\ \tilde{g} \nearrow & & \downarrow \\ B & \xrightarrow{g} & B\text{Symp}(M). \end{array}$$

Hamiltonian structures are in bijective correspondence with homotopy classes of such lifts. There are two stages to choosing the lift \tilde{g} : one first lifts g to a map \hat{g} into $B\text{Symp}_0(M, \omega)$, where Symp_0 is the identity component of Symp , and then to a map \tilde{g} into $B\text{Ham}(M, \omega)$. As we show in §2.2, choosing \hat{g} is equivalent to fixing the isotopy class of an identification of (M, ω) with the fiber (M_{b_0}, ω_{b_0}) over the base point b_0 . If B is simply connected, in particular if B is a single point, there is then a unique Hamiltonian structure on P , i.e. a unique choice of lift \tilde{g} . Before describing what happens in the general case, we discuss properties of the extensions τ .

Let $\tau \in \Omega^2(P)$ be a closed extension of the symplectic forms on the fibers. Given a loop $\gamma : S^1 \rightarrow B$ based at b_0 , and a symplectic trivialisation $T_\gamma : \gamma^*(P) \rightarrow S^1 \times (M, \omega)$ that extends the given identification of M_{b_0} with M , push forward τ to a form $(T_\gamma)_*\tau$ on $S^1 \times (M, \omega)$. Its characteristic flow round S^1 is transverse to the fibers and defines a symplectic isotopy ϕ_t of $(M, \omega) = (M_{b_0}, \omega_{b_0})$ whose flux, as map from $H_1(M) \rightarrow \mathbb{R}$, is equal to $(T_\gamma)_*[\tau]([S^1] \otimes \cdot)$: see Lemma 2.8. This flux depends only on the cohomology class a of τ . Moreover, as we mentioned above, any extension a of the fiber class $[\omega]$ can be represented by a form τ that extends the ω_b . Thus, given T_γ and an extension $a = [\tau] \in H^2(P)$ of the fiber symplectic class $[\omega]$, it makes sense to define the *flux class* $f(T_\gamma, a) \in H^1(M, \mathbb{R})$ by

$$f(T_\gamma, a)(\delta) = (T_\gamma)_*(a)(\gamma \otimes \delta) \quad \text{for all } \delta \in H_1(M).$$

The equivalence class $[f(T_\gamma, a)] \in H^1(M, \mathbb{R})/\Gamma_\omega$ does not depend on the choice of T_γ : indeed two such choices differ by a loop ϕ in $\text{Symp}_0(M, \omega)$ and so the difference

$$f(T_\gamma, a) - f(T'_\gamma, a) = f(T_\gamma, a) \circ \text{tr}_\phi = \omega \circ \text{tr}_\phi$$

belongs to Γ_ω . The following lemma is elementary: see §2.2 :

Lemma 1.4 *If $\pi : P \rightarrow B$ is a symplectic bundle satisfying the conditions of Theorem B.2, there is an extension a of the symplectic fiber class that has trivial flux*

$$[f(T_\gamma, a)] = 0 \in H^1(M, \mathbb{R})/\Gamma_\omega$$

round each loop γ in B .

Definition 1.5 An extension a of the symplectic fiber class $[\omega_{b_0}]$ is *normalized* if it satisfies the conclusions of the above lemma. Two such extensions a and a' are equivalent (in symbols, $a \sim a'$) if and only if they have equal restrictions to $\pi^{-1}(B_1)$, or, equivalently, if and only if $a - a' \in \pi^*(H^2(B))$.

We show in §2.2 that Hamiltonian structures are in one-to-one correspondence with symplectic trivialisations of the 1-skeleton B_1 of B , with two such trivialisations being equivalent if and only if they differ by Hamiltonian loops. If two trivialisations T_γ, T'_γ differ by a Hamiltonian loop ϕ then $f(T_\gamma, a) - f(T'_\gamma, a) = 0$. In terms of fluxes of closed extensions, we therefore get:

Theorem 1.6 *Assume that a symplectic bundle $\pi : P \rightarrow B$ can be symplectically trivialized over B_1 . Then a Hamiltonian structure exists on P if and only if there is a normalized extension a of ω . Such a structure consists of an isotopy class of symplectomorphisms $(M, \omega) \rightarrow (M_{b_0}, \omega_{b_0})$ together with an equivalence class $\{a\}$ of normalized extensions of the fiber symplectic class.*

In other words, with respect to a fixed trivialization over B_1 , Hamiltonian structures are in one-to-one correspondence with homomorphisms $\pi_1(B) \rightarrow \Gamma_\omega$, given by the fluxes $f_\gamma(T, a)$ of monodromies round the loops of the base. We will call $\{a\}$ the *Hamiltonian extension class*, and will denote the Hamiltonian structure on P by the triple $(P, \pi, \{a\})$. A different description of a Hamiltonian structure is sketched in Appendix A.

We now turn to the question of describing automorphisms of Hamiltonian structures. It is convenient to distinguish between symplectic and Hamiltonian automorphisms, just as we distinguish between $\text{Symp}(M, \omega)$ and $\text{Ham}(M, \omega)$ in the case when $B = pt$. Notice that if $P \rightarrow B$ is a symplectic bundle, there is a natural notion of symplectic automorphism. This is a fiberwise diffeomorphism $\Phi : P \rightarrow P$ that covers the identity map on the base and restricts on each fiber to an element Φ_b of the group $\text{Symp}(M_b, \omega_b)$.² Because $\text{Ham}(M, \omega)$ is a normal subgroup of $\text{Symp}(M, \omega)$, it also makes sense to require that $\Phi_b \in \text{Ham}(M_b, \omega_b)$ for each b . Such automorphisms are called *Hamiltonian automorphisms* of the symplectic bundle $P \rightarrow B$. Let us write $\text{Symp}(P, \pi)$ and $\text{Ham}(P, \pi)$ for the groups of such automorphisms. Observe that the group $\text{Ham}(P, \pi)$ may not be connected. Because the fibers of Hamiltonian bundles are identified with (M, ω) up to isotopy, we shall also need to consider the (not necessarily connected) group $\text{Symp}_0(P, \pi)$ of symplectomorphisms of (P, π) where $\Phi_b \in \text{Symp}_0(M_b, \omega_b)$ for one and hence all b .

Now let us consider automorphisms of Hamiltonian bundles. As a guide note that in the trivial case when $B = pt$, a Hamiltonian structure on P is an identification of P with

²One could allow more general automorphisms of the base, but we will restrict to this simple case here.

M up to symplectic isotopy. Hence the group of automorphisms of this structure can be identified with $\text{Symp}_0(M, \omega)$. In general, if $\{a\}$ is a Hamiltonian structure on (P, π) and $\Phi \in \text{Symp}_0(P, \pi)$ then $\Phi^*(\{a\}) = \{a\}$ if and only if $\Phi^*(a) = a$ for some a in the class $\{a\}$, because Φ induces the identity map on the base and $a - a' \in \pi^*(H^2(B))$ when $a \sim a'$. We therefore make the following definition.

Definition 1.7 Let $(P, \pi, \{a\})$ be a Hamiltonian structure on the symplectic bundle $P \rightarrow B$ and let $\Phi \in \text{Symp}(P, \pi)$. Then Φ is an *automorphism of the Hamiltonian structure* $(P, \pi, \{a\})$ if $\Phi \in \text{Symp}_0(P, \pi)$ and $\Phi^*(\{a\}) = \{a\}$. The group formed by these elements is denoted by $\text{Aut}(P, \pi, \{a\})$.

The following result is not hard to prove, but is easiest to see in the context of a discussion of the action of $\text{Ham}(M)$ on $H^*(M)$. Therefore the proof is deferred to §5.

Proposition 1.8 Let $P \rightarrow B$ be a Hamiltonian bundle and $\Phi \in \text{Symp}_0(P, \pi)$. Then the following statements are equivalent:

- (i) Φ is isotopic to an element of $\text{Ham}(P, \pi)$;
- (ii) $\Phi^*(\{a\}) = \{a\}$ for some Hamiltonian structure $\{a\}$ on P ;
- (iii) $\Phi^*(\{a\}) = \{a\}$ for all Hamiltonian structures $\{a\}$ on P .

Corollary 1.9 For any Hamiltonian bundle $P \rightarrow B$, the group $\text{Aut}(P, \pi, \{a\})$ does not depend on the choice of the Hamiltonian structure $\{a\}$ put on P . Moreover, it contains $\text{Ham}(P, \pi)$ and each element of $\text{Aut}(P, \pi, \{a\})$ is isotopic to an element in $\text{Ham}(P, \pi)$.

The following characterization is now obvious:

Lemma 1.10 Let P be the product $B \times M$ and $\{a\}$ any Hamiltonian structure. Then:

- (i) $\text{Ham}(P, \pi)$ consists of all maps from B to $\text{Ham}(M, \omega)$.
- (ii) $\text{Aut}(P, \pi, \{a\})$ consists of all maps $\Phi : B \rightarrow \text{Symp}_0(M, \omega)$ for which the composite

$$\pi_1(B) \xrightarrow{\Phi_*} \pi_1(\text{Symp}_0(M)) \xrightarrow{\text{Flux}_\omega} H^1(M, \mathbb{R})$$

is trivial.

The basic reason why Proposition 1.8 holds is that Hamiltonian automorphisms of (P, π) act trivially on the set of extensions of the fiber symplectic class. This need not be true for symplectic automorphisms. For example, if $\pi : P = S^1 \times M \rightarrow S^1$ is a trivial bundle and Φ is given by a nonHamiltonian loop ϕ in $\text{Symp}_0(M)$, then Φ is in $\text{Symp}_0(P, \pi)$ but it preserves no Hamiltonian structure on P since $\Phi^*(a) = a + [dt] \otimes \text{Flux}(\phi)$.

In general, if we choose a trivialization of P over B_1 , there are exact sequences

$$\begin{aligned} \{id\} &\rightarrow \text{Aut}(P, \pi, \{a\}) \rightarrow \text{Symp}_0(P, \pi) \rightarrow \text{Hom}(\pi_1(B), \Gamma_\omega) \rightarrow \{id\}, \\ \{id\} &\rightarrow \text{Ham}(P, \pi, \{a\}) \rightarrow \text{Aut}(P, \pi, \{a\}) \rightarrow H^1(M, \mathbb{R})/\Gamma_\omega \rightarrow \{0\}. \end{aligned}$$

In particular, the subgroup of $\text{Aut}(P, \pi, \{a\})$ consisting of automorphisms that belong to $\text{Ham}(M_{b_0}, \omega_{b_0})$ at the base point b_0 retracts to $\text{Ham}(P, \pi, \{a\})$.

1.3 Stability

Another important property of Hamiltonian bundles is stability.

Definition 1.11 A symplectic (resp. Hamiltonian) bundle $\pi : P \rightarrow B$ with fiber (M, ω) is said to be *stable* if π may be given a symplectic (resp. Hamiltonian) structure with respect to any symplectic form ω' on M that is sufficiently close to (but not necessarily cohomologous to) ω , in such a way that the structure depends continuously on ω' .

Using Moser's homotopy argument, it is easy to prove that any symplectic bundle is stable (see Corollary 3.2). The following characterization of Hamiltonian stability is an almost immediate consequence of Theorem B.2. It is proved in §3.1 below.

Lemma 1.12 *A Hamiltonian bundle $\pi : P \rightarrow B$ is stable if and only if the restriction map $H^2(P, \mathbb{R}) \rightarrow H^2(M, \mathbb{R})$ is surjective.*

The following result is less immediate.

Theorem 1.13 *Every Hamiltonian bundle is stable.*

The proof uses the (difficult) stability property for Hamiltonian bundles over S^2 that was established in [13, 21] as well as the (easy) fact that the image of the evaluation map $\pi_2(\text{Ham}(M)) \rightarrow \pi_2(M)$ lies in the kernel of $[\omega]$: see Lemma 2.4.

1.4 Cohomological splitting

We next extend the splitting results of Lalonde–McDuff–Polterovich [13] and McDuff [21]. These papers prove that the rational cohomology of every Hamiltonian bundle $\pi : P \rightarrow S^2$ splits additively, i.e. there is an additive isomorphism

$$H^*(P) \cong H^*(S^2) \otimes H^*(M).$$

For short we will say in this situation that π is *c-split*.³ This is a deep result, that requires the use of Gromov–Witten invariants for its proof. The results of the present paper provide some answers to the following question:

Does any Hamiltonian fiber bundle over a compact CW-complex c-split?

A special case is when the structural group of $P \rightarrow B$ can be reduced to a compact Lie subgroup G of $\text{Ham}(M)$. Here c-splitting over any base follows from the work of Atiyah–Bott [2] or ours. In this context, one usually discusses the universal Hamiltonian G -bundle with fiber M

$$M \longrightarrow M_G = EG \times_G M \longrightarrow BG.$$

The cohomology of $P = M_G$ is known as the equivariant cohomology $H_G^*(M)$ of M . Atiyah–Bott show that if G is a torus T that acts in a Hamiltonian way on M then the bundle $M_T \rightarrow BT$ is c-split. We prove a generalization of this in Corollary 4.8. The result for a general compact Lie group G follows by standard arguments: see Corollary 4.10.

The following theorem describes conditions on the base B that imply c-splitting.

³In some literature (see for example Thomas [26]) this condition is called T.N.C.Z. (totally noncohomologous to zero), because it is equivalent to requiring that the inclusion of the fiber M into P induce an injection on rational homology. The paper [21] also discusses situations in which the ring structure of $H^*(P)$ splits.

Theorem 1.14 *Let (M, ω) be a closed symplectic manifold, and $M \hookrightarrow P \rightarrow B$ a bundle with structure group $\text{Ham}(M)$ and with base a compact CW-complex B . Then the rational cohomology of P splits in each of the following cases:*

- (i) *the base has the homotopy type of a coadjoint orbit or of a product of spheres with at most three of dimension 1;*
- (ii) *the base has the homotopy type of a complex blow up of a product of complex projective spaces;*
- (iii) $\dim(B) \leq 3$.

Case (ii) is a generalization of the foundational example $B = S^2$ and is proved by similar analytic methods. The idea is to show that the map $\iota : H_*(M) \rightarrow H_*(P)$ is injective by showing that the image $\iota(a)$ in P of any class $a \in H_*(M)$ can be detected by a nonzero Gromov–Witten invariant of the form $n_P(\iota(a), c_1, \dots, c_n; \sigma)$, where $c_i \in H_*(P)$ and $\sigma \in H_2(P)$ is a spherical class with nonzero image in $H_2(B)$. The proof should generalize to the case when all one assumes about the base is that there is a nonzero invariant of the form $n_B(pt, pt, c_1, \dots, c_k; A)$: see [10] and the discussion in §4.2 below.

The proofs of parts (i) and (iii) start from the fact of c -splitting over S^2 and proceed using purely topological methods. The following fact about compositions of Hamiltonian bundles is especially useful. Let $M \hookrightarrow P \rightarrow B$ be a Hamiltonian bundle over a simply connected base B and assume that all Hamiltonian bundles over M as well as over B c -split. Then any Hamiltonian bundle over P c -splits too. (This fact is based on the characterization of Hamiltonian bundles in terms of closed extensions of the symplectic form). This provides a powerful recursive argument which allows one to establish c -splitting over $\mathbb{C}P^n$ by induction on n , and is an essential tool in all our arguments.⁴

The question whether all Hamiltonian bundles over symplectic 4-manifolds c -split is still unresolved, despite our previous claims (cf. McDuff [19] for example). However, even when the base has no symplectic structure and is only a 4-dimensional CW-complex, our methods still yield some results about c -splitting when additional restrictions are placed on the fiber: see [11].

It is not clear whether one should expect that c -splitting always occurs. This question is closely related to Halperin’s conjecture, a slightly simplified version of which proposes that a fibration in the rational homotopy category whose fiber and base are simply connected c -splits if the fiber F is elliptic (that is $\pi_*(F) \otimes \mathbb{Q}$ has finite dimension) and its rational cohomology $H^i(F)$ vanishes for odd i . These hypotheses imply in particular that the fiber is formal. Clearly, the validity of Halperin’s conjecture with respect to a given fiber $F = (M, \omega)$ implies that all Hamiltonian fibrations with that fiber are c -split. However, note that his hypotheses are somewhat different from ours since many symplectic manifolds are neither elliptic nor formal. Meier shows in [23] Lemma 2.5 that if F is a simply connected and formal space such that all homotopy fibrations over spheres with fiber F are c -split then *all* fibrations with fiber F and simply connected base are c -split. Since there may be homotopy fibrations that are not Hamiltonian, the fact that Hamiltonian fibrations c -split over spheres is not enough to imply that all Hamiltonian fibrations with simply connected and formal fiber are c -split. Nevertheless, Meier’s result is an interesting complement to ours.

⁴A similar property has been exploited in the context of the Halperin conjecture discussed below: see for example Markl [22].

Although to our knowledge Halperin’s conjecture is still not resolved, there has been quite a bit of work that establishes its validity when the fiber satisfies additional properties. In particular, it holds when the cohomology ring $H^*(F, \mathbb{Q})$ has at most 3 even dimensional generators (see Lupton [15]) or when its generators all have the same even dimension (see Belegradek–Kapovitch [3].) In this paper we have concentrated on establishing results on c-splitting that hold for all fibers (M, ω) . However, there are some simple arguments that apply for special M . For example, in section 4.3 we present an argument due to Blanchard that establishes c-splitting when the cohomology of the fiber satisfies the hard Lefschetz condition. A modification due to Kedra shows that c-splitting holds whenever M has dimension 4. Moreover, the Belegradek–Kapovitch theorem has a Hamiltonian analog: we show in Lemma 4.14 that c-splitting occurs whenever $H^*(M)$ is generated by $H^2(M)$.

In view of this, it is natural to wonder whether c-splitting is a purely homotopy-theoretic property. A c-symplectic manifold (M, a_M) is defined to be a $2n$ -manifold together with a class $a_M \in H^2(M)$ such that $a_M^n > 0$.⁵ In view of Theorem B.2 one could define a c-Hamiltonian bundle over a simply connected base manifold B to be a bundle $P \rightarrow B$ with c-symplectic fiber (M, a_M) in which the symplectic class a_M extends to a class $a \in H^2(P)$. In [1], Allday discusses a variety of results about symplectic torus actions, some of which do extend to the c-symplectic case and some of which do not. The next lemma shows that c-splitting in general is a geometric rather than a homotopy-theoretic property. Its proof may be found in §4.3.

Lemma 1.15 *There is a c-Hamiltonian bundle over S^2 that is not c-split.*

It is also worth noting that it is essential to restrict to finite dimensional spaces: c-splitting does not always hold for “Hamiltonian” bundles with infinite dimensional fiber. (See the footnote to Lemma 2.4.)

1.5 The homological action of $\text{Ham}(M)$ on M

The action $\text{Ham}(M) \times M \rightarrow M$ gives rise to maps

$$H_k(\text{Ham}(M)) \otimes H_*(M) \longrightarrow H_{k+*}(M) : (\phi, Z) \mapsto \text{tr}_\phi(Z),$$

and dually

$$\text{tr}_\phi^* : H_k(\text{Ham}(M)) \rightarrow \text{Hom}(H^*(M), H^{*-k}(M)), \quad k \geq 0.$$

In this language, the flux of a loop $\phi \in \pi_1(\text{Ham}(M))$ is precisely the element $\text{tr}_\phi^*([\omega]) \in H^1(M)$. (Here we should use real rather than rational coefficients so that $[\omega] \in H^*(M)$.) The following result is a consequence of Theorem 1.14.

Theorem 1.16 *The maps tr_ϕ and tr_ϕ^* are zero for all $\phi \in H_k(\text{Ham}(M)), k > 0$.*

The argument goes as follows. Recall that the cohomology ring of $\text{Ham}(M)$ is generated by elements dual to its homotopy. It therefore suffices to consider the restriction of tr_k to the spherical elements ϕ . But in this case it is not hard to see that the tr_k are precisely the connecting homomorphisms in the Wang sequence of the bundle $P_\phi \rightarrow S^{k+1}$ with clutching

⁵Caution: the letter “c” here also stands for “cohomologically” but the meaning here is somewhat different from its use in the word “c-split”.

function ϕ . These vanish because all Hamiltonian bundles over spheres are c-split by part (i) of Theorem 1.14. Details may be found in §5.

In particular, looking at the action on $H_0(M)$, we see that the point evaluation map

$$\text{ev} : \text{Ham}(M) \rightarrow M : \psi \mapsto \psi(x)$$

induces the trivial map on rational (co)homology. It also induces the trivial map on π_1 .⁶ However, the map on $\pi_k, k > 1$, need not be trivial. To see this, consider the action of $\text{Ham}(M)$ on the symplectic frame bundle $\text{SFr}(M)$ of M and the corresponding point evaluation maps. The obvious action of $\text{SO}(3) \simeq \text{Ham}(S^2)$ on $\text{SFr}(S^2) \simeq \mathbb{R}P^3$ induces an isomorphism

$$H_3(\text{SO}(3)) \cong H_3(\text{SFr}(S^2)),$$

showing that these evaluation maps are not homologically trivial. Moreover, its composite with the projection $\text{SFr}(S^2) \rightarrow S^2$ gives rise to a nonzero map

$$\pi_3(\text{SO}(3)) = \pi_3(\text{Ham}(S^2)) \rightarrow \pi_3(S^2).$$

Thus the corresponding Hamiltonian fibration over S^4 with fiber S^2 , though c-split, does not have a section.

Note, however, that the evaluation map

$$\pi_{2\ell}(X^X) \xrightarrow{\text{ev}} \pi_{2\ell}(X) \rightarrow H_{2\ell}(X, \mathbb{Q}), \quad \ell > 0,$$

is always zero, if X is a finite CW complex and X^X is its space of self-maps. Indeed, because the cohomology ring $H^*(X^X, \mathbb{Q})$ is freely generated by elements dual to $\pi_*(X^X) \otimes \mathbb{Q}$, there would otherwise be an element $a \in H^{2\ell}(X)$ that would pull back to an element of infinite order in the cohomology ring of the H -space X^X . Hence a itself would have to have infinite order, which is impossible. A more delicate argument shows that the integral evaluation $\pi_{2\ell}(X^X) \rightarrow H_{2\ell}(X, \mathbb{Z})$ is zero: see [6].

By Lemma 1.10, a Hamiltonian automorphism of the product Hamiltonian bundle $B \times M \rightarrow B$ is simply a map $B \rightarrow B \times \text{Ham}(M)$ of the form $b \mapsto (b, \phi_b)$. If B is a closed manifold we will see that Theorem 1.16 implies that any Hamiltonian automorphism of the product bundle acts as the identity map on the rational cohomology of $B \times M$: see Proposition 5.2. The natural generalization of this result would claim that a Hamiltonian automorphism of a bundle P acts as the identity map on the rational cohomology of P . We do not know yet whether this is true in general. However, we can show that it is closely related to the c-splitting of Hamiltonian bundles. Thus we can establish it only under conditions similar to the conditions under which c-splitting holds. See Proposition 5.4 below.

1.6 Implications for general symplectic bundles

Consider the Wang sequence for a symplectic bundle $\pi : P \rightarrow S^2$ with clutching map $\phi \in \pi_1(\text{Symp}(M))$:

$$\dots \longrightarrow H^k(M) \xrightarrow{\partial} H^{k-1}(M) \xrightarrow{u} H^{k+1}(P) \xrightarrow{\text{rest}} H^{k+1}(M) \longrightarrow \dots$$

⁶This is a consequence of the proof of the Arnold conjecture: see [12] §1.3. It is equivalent to the existence of a section of every Hamiltonian bundle over S^2 and so also follows from the results in [13, 21].

Here the map u may be realized in de Rham cohomology by choosing any extension of a given closed form α on M and then wedging it with the pullback of a normalized area form on the base. Further, as pointed out above, the boundary map $\partial = \partial_\phi$ is just tr_ϕ^* . Thus the bundle is Hamiltonian if and only if $\text{tr}_\phi^*([\omega]) = \partial([\omega]) = 0$. In the Hamiltonian case Theorem 1.14 implies that ∂ is identically 0. In the general case, we know that the map $\partial : H^*(M) \rightarrow H^{*-1}(M)$ is a derivation: i.e.

$$\partial(ab) = \partial(a)b + (-1)^{\text{deg}(a)}a\partial(b).$$

The following result is an easy consequence of the fact that the action of $\pi_2(\text{Ham}(M)) = \pi_2(\text{Symp}(M))$ on $H^*(M)$ is trivial.

Proposition 1.17 *The boundary map ∂ in the Wang rational cohomology sequence of a symplectic bundle over S^2 has the property that $\partial \circ \partial = 0$.*

The proof is given in §6. The above result holds trivially when ϕ corresponds to a smooth (not necessarily symplectic) S^1 -action since then ∂ is given in deRham cohomology by contraction ι_X by the generating vector field X . Moreover, the authors know of no smooth bundle over S^2 for which the above proposition does not hold, though it is likely that they exist. By the remarks in §6 such a bundle would have no extension over $\mathbb{C}P^2$.

One consequence is the following result about the boundary map $\partial = \partial_\phi$ in the case when the loop ϕ is far from being Hamiltonian. Recall (e.g. from [13]) that $\pi_1(\text{Ham}(M))$ is included in (but not necessarily equal to) the kernel of the evaluation map $\pi_1(\text{Symp}(M)) \rightarrow \pi_1(M)$. Any loop whose evaluation is homologically essential can therefore be thought of as “very nonHamiltonian”.

Corollary 1.18 *$\ker \partial = \text{im } \partial$ if and only if the image of ϕ under the evaluation map $\pi_1(\text{Symp}(M)) \rightarrow H_1(M, \mathbb{Q})$ is nonzero.*

A similar result was obtained by Allday concerning S^1 actions on c-symplectic manifolds: see statement (d) in [1]. He was considering manifolds M that satisfy the weak Lefschetz condition, i.e. manifolds such that

$$\wedge[\omega]^{n-1} : H^1(M, \mathbb{R}) \rightarrow H^{2n-1}(M, \mathbb{R})$$

is an isomorphism, in which case every nonHamiltonian loop is “very nonHamiltonian.”

2 The characterization of Hamiltonian bundles

This section contains proofs of the basic results on the existence and classification of Hamiltonian bundles, namely Theorems B.2 and 1.6 and Proposition B.3.

2.1 Existence of Hamiltonian structures

We begin by giving a proof of Theorem B.2 using as little analysis as possible. We will repeat some of the arguments in [18] Ch. 6 for the sake of clarity. The main new point is the replacement of the Guillemin–Lerman–Sternberg construction of the coupling form by the more topological Lemma 2.4.

Geometric proofs (such as those in [18]) apply when P and B are smooth manifolds and π is a smooth surjection. However, as the following lemma makes clear, this is no restriction.

Lemma 2.1 *Suppose that $\pi : Q \rightarrow W$ is a locally trivial bundle over a finite CW complex W with compact fiber (M, ω) and suppose that the structural group G is equal either to $\text{Symp}(M, \omega)$ or to $\text{Ham}(M, \omega)$. Then there is a smooth bundle $\pi : P \rightarrow B$ as above with structural group G and a homeomorphism f of W onto a closed subset of B such that $\pi : Q \rightarrow W$ is homeomorphic to the pullback bundle $f^*(P) \rightarrow W$.*

Proof: Embed W into some Euclidean space and let B be a suitable small neighborhood of W . Then W is a retract of B so that the classifying map $W \rightarrow BG$ extends to B . It remains to approximate this map $B \rightarrow BG$ by a smooth map. \square

First let us sketch the proof of Theorem B.2 when the base is simply connected. We use the minimum amount of geometry: nevertheless to get a relation between the existence of the class a and the structural group it seems necessary to use the idea of a symplectic connection. We begin with an easy lemma.

Lemma 2.2 *Let $P \rightarrow B$ be a symplectic bundle with closed connection form τ . Then the holonomy of the corresponding connection ∇_τ round any contractible loop in B is Hamiltonian.*

Proof: It suffices to consider the case when $B = D^2$. Then the bundle $\pi : P \rightarrow D^2$ is symplectically trivial and so may be identified with the product $D^2 \times M$ in such a way that the symplectic form on each fiber is simply ω . Use this trivialization to identify the holonomy round the loop $s \mapsto e^{2\pi is} \in \partial D^2$ with a family of diffeomorphisms $\Phi_s : M \rightarrow M, s \in [0, 1]$. Since this holonomy is simply the flow along the null directions (or characteristics) of the closed form τ on the hypersurface ∂P , a standard calculation shows that the Φ_s are symplectomorphisms. Given a 1-cycle $\delta : S^1 \rightarrow M$ in the fiber M over $1 \in \partial D^2$, consider the closed 2-cycle that is the union of the following two cylinders:

$$\begin{aligned} c_1 : [0, 1] \times S^1 &\rightarrow \partial D^2 \times M : & (s, t) &\mapsto (e^{2\pi is}, \Phi_s(\delta(t))), \\ c_2 : [0, 1] \times S^1 &\rightarrow 1 \times M : & (s, t) &\mapsto (1, \Phi_{1-s}(\delta(t))). \end{aligned}$$

This cycle is obviously contractible. Hence,

$$\tau(c_1) = -\tau(c_2) = \text{Flux}(\{\Phi_s\})(\delta).$$

But $\tau(c_1) = 0$ since the characteristics of $\tau|_{\partial P}$ are tangent to c_1 . Applying this to all δ , we see that the holonomy round ∂D^2 has zero flux and so is Hamiltonian. \square

Lemma 2.3 *If $\pi_1(B) = 0$ then a symplectic bundle $\pi : P \rightarrow B$ is Hamiltonian if and only if the class $[\omega_b] \in H^2(M)$ extends to $a \in H^*(P)$.*

Proof: Suppose first that the class a exists. By Lemma 2.1, we can work in the smooth category. Then Thurston's convexity argument allows us to construct a closed connection form τ on P and hence a horizontal distribution Hor_τ . The previous lemma shows that the holonomy around every contractible loop in B is Hamiltonian. Since B is simply connected, the holonomy round *all* loops is Hamiltonian. Using this, it is easy to reduce the structural group of $P \rightarrow B$ to $\text{Ham}(M)$. For more details, see [18].

Next, suppose that the bundle is Hamiltonian. We need to show that the fiber symplectic class extends to P . The proof in [18] does this by the method of Guillemin–Lerman–Sternberg and constructs a closed connection form τ (called the coupling form) starting from a connection with Hamiltonian holonomy. This construction uses the curvature of the connection and is quite analytic. In contrast, we shall now use topological arguments to reduce to the cases $B = S^2$ and $B = S^3$. These cases are then dealt with by elementary arguments.

Consider the Leray–Serre cohomology spectral sequence for $M \rightarrow P \rightarrow B$. Its E_2 term is a product: $E_2^{p,q} = H^p(B) \otimes H^q(M)$. (Here H^* denotes cohomology over \mathbb{R} .) We need to show that the class $[\omega] \in E_2^{0,2}$ survives into the E_∞ term, which happens if and only if it is in the kernel of the two differentials $d_2^{0,2}, d_3^{0,2}$. Now

$$d_2^{0,2} : H^2(M) \rightarrow H^2(B) \otimes H^1(M)$$

is essentially the same as the flux homomorphism. More precisely, if $c : S^2 \rightarrow B$ represents some element (also called c) in $H_2(B)$, then the pullback of the bundle $\pi : P \rightarrow B$ by c is a bundle over S^2 that is determined by a loop $\phi_c \in \pi_1(\text{Ham}(M))$ that is well defined up to conjugacy. Moreover, for each $\lambda \in H_1(M)$,

$$d_2^{0,2}([\omega])(c, \lambda) = \text{tr}_{\phi_c}^*(\lambda),$$

where tr^* is as in §1.5. Hence $d_2^{0,2}([\omega]) = 0$ because ϕ_c is Hamiltonian.

To deal with d_3 observe first that because the inclusion of the 3-skeleton B_3 into B induces an injection $H^q(B) \rightarrow H^q(B_3)$ for $q \leq 3$, $d_3^{0,2}([\omega])$ vanishes in the spectral sequence for $P \rightarrow B$ if it vanishes for the pullback bundle over B_3 . Therefore we may suppose that B is a 3-dimensional CW-complex whose 2-skeleton B_2 is a wedge of 2 spheres. (Recall that $\pi_1(B) = 0$.) Further, we can choose the cell decomposition so that the first k 3-cells span the kernel of the boundary map $C_3 \rightarrow C_2$ in the cellular chain complex of B_3 . Because $H_2(B_2) = \pi_2(B_2)$, the attaching maps of these first k -cells are null homotopic. Hence there is a wedge B' of 2-spheres and 3-spheres and a map $B' \rightarrow B_3$ that induces a surjection on H_3 . It therefore suffices to show that $d_3^{0,2}([\omega])$ vanishes in the pullback bundle over B' . This will clearly be the case if it vanishes in every Hamiltonian bundle over S^3 .

Now, a Hamiltonian fiber bundle over S^3 is determined by a map

$$I^2/\partial I^2 = S^2 \rightarrow \text{Ham}(M) : (s, t) \mapsto \phi_{s,t},$$

and it is easy to see that $d_3^{0,2}([\omega]) = 0$ exactly when the the evaluation map

$$\text{ev}_x : \text{Ham}(M) \rightarrow M : \phi \mapsto \phi(x)$$

takes $\pi_2(\text{Ham}(M))$ into the kernel of ω .

The result now follows from Lemma 2.4 below. □

Lemma 2.4 *Given a smooth map $\Psi : (I^2, \partial I^2) \rightarrow (\text{Ham}(M), id)$ and $x \in M$, let $\Psi^x : (I^2, \partial I^2) \rightarrow M$ be the composite of Ψ with evaluation at x . Then*

$$\int_{I^2} (\Psi^x)^* \omega = 0,$$

for all $x \in M$.

Proof: For each s, t let $X_{s,t}$ (resp. $Y_{s,t}$) be the Hamiltonian vector field on M that is tangent to the flow of the isotopy $s \mapsto \Psi^x(s, t)$, (resp. $t \mapsto \Psi^x(s, t)$.) Then

$$\int_{I^2} (\Psi^x)^* \omega = \iint \omega(X_{s,t}(\Psi^x(s, t)), Y_{s,t}(\Psi^x(s, t))) ds dt.$$

The first observation is that this integral is a constant c that is independent of x , since the maps $\Psi^x : S^2 \rightarrow M$ are all homotopic. Secondly, recall that for any Hamiltonian vector fields X, Y on M

$$\int_M \omega(X, Y) \omega^n = n \int_M \omega(X, \cdot) \omega(Y, \cdot) \omega^{n-1} = 0,$$

since $\omega(X, \cdot), \omega(Y, \cdot)$ are exact 1-forms. Taking $X_{s,t} = X_{s,t}(\Psi^x(s, t))$ and similarly for Y , we have

$$\int_{I^2} c \omega^n = \int_{I^2} \left(\int_M \omega(X_{s,t}, Y_{s,t}) \omega^n \right) ds dt = 0.$$

Hence $c = 0$.

This lemma can also be proved by purely topological methods. In fact, as remarked in the discussion after Theorem 1.16, the evaluation map $\pi_2(X^X) \rightarrow H_2(X, \mathbb{R})$ vanishes for any finite CW complex X .⁷

□

This completes the proof of Theorem B.2 for simply connected bases.

Lemma 2.5 *Theorem B.2 holds for all B .*

Proof: Suppose that $\pi_B : P \rightarrow B$ is Hamiltonian. It is classified by a map $B \rightarrow B \text{Ham}(M)$. Because $B \text{Ham}(M)$ is simply connected this factors through a map $C \rightarrow B \text{Ham}(M)$, where C is obtained by collapsing the 1-skeleton of B to a point. In particular condition (i) is satisfied. To verify (ii), let $\pi_C : Q \rightarrow C$ be the corresponding Hamiltonian bundle, so that there is a commutative diagram

$$\begin{array}{ccc} P & \rightarrow & Q \\ \pi_B \downarrow & & \pi_C \downarrow \\ B & \rightarrow & C = B/B_1. \end{array}$$

Lemma 2.3 applied to π_C tells us that there is a class $a_C \in H^2(Q)$ that restricts to $[\omega]$ on the fibers. Its pullback to P is the desired class a .

Conversely, suppose that conditions (i) and (ii) are satisfied. By (i), the classifying map $B \rightarrow B \text{Symp}(M)$ factors through a map $f : C \rightarrow B \text{Symp}(M)$, where C is as above. This map f depends on the choice of a symplectic trivialization of π over the 1-skeleton B_1 of B . We now show that f can be chosen so that (ii) holds for the associated symplectic bundle $Q_f \rightarrow C$.

⁷The following example due to Gotay *et al* [5] demonstrates the importance of this finiteness hypothesis. Let \mathcal{H} be a complex Hilbert space with unitary group $\mathbb{U}(\mathcal{H})$ and consider the exact sequence $S^1 \rightarrow \mathbb{U}(\mathcal{H}) \rightarrow \mathbb{P}\mathbb{U}(\mathcal{H})$, where $\mathbb{P}\mathbb{U}(\mathcal{H})$ is the projective unitary group. Since $\mathbb{U}(\mathcal{H})$ is contractible, $\mathbb{P}\mathbb{U}(\mathcal{H}) \simeq \mathbb{C}P^\infty$. Since $\mathbb{P}\mathbb{U}(\mathcal{H})$ can be considered as a subgroup of the symplectomorphism group of $\mathbb{C}P(\mathcal{H})$, the generator ϕ of $\pi_2(\mathbb{P}\mathbb{U}(\mathcal{H}))$ gives rise to a ‘‘symplectic’’ fibration $\mathbb{C}P(\mathcal{H}) \rightarrow P_\phi \rightarrow S^3$, which is ‘‘Hamiltonian’’ because $\mathbb{C}P(\mathcal{H}) \simeq \mathbb{C}P^\infty$ is simply connected. It is easy to check that the evaluation map $\mathbb{P}\mathbb{U}(\mathcal{H}) \rightarrow \mathbb{C}P(\mathcal{H}) : A \mapsto A(x)$ is a homotopy equivalence. Hence Lemma 2.3 fails in this case.

As in the proof of Lemma 2.3. we need to show that the differentials $(d_C)_2^{0,2}, (d_C)_3^{0,2}$ in the spectral sequence for $Q_f \rightarrow C$ both vanish on $[\omega]$. Let

$$\cdots \rightarrow C_k(B) \xrightarrow{\partial} C_{k-1}(B) \rightarrow \cdots$$

be the cellular chain complex for B , and choose 2-cells e_1, \dots, e_k in B whose attaching maps $\alpha_1, \dots, \alpha_k$ form a basis over \mathbb{Q} for the image of ∂ in $C_1(B)$. Then the obvious maps $C_k(B) \rightarrow C_k(C)$ (which are the identity for $k > 1$) give rise to an isomorphism

$$H_2(B, \mathbb{Q}) \bigoplus \oplus_i \mathbb{Q}[e_i] \cong H_2(C, \mathbb{Q}).$$

By the naturality of spectral sequences, the vanishing of $(d_B)_2^{0,2}([\omega])$ implies that $(d_C)_2^{0,2}([\omega])$ vanishes on all cycles in $H_2(C, \mathbb{Q})$ coming from $H_2(B, \mathbb{Q})$. Therefore we just need to check that it vanishes on the cycles e_i . For this, we have to choose the trivialization over B_1 so that its pullback by each α_i gives rise to a Hamiltonian bundle over e_i . For this it would suffice that its pullback by each α_i is the “natural trivialization”, i.e the one that extends over the 2-cell e_i . To arrange this, choose any symplectic trivialization over $B_1 = \vee_j \gamma_j$. Then comparing this with the natural trivialization gives rise to a homomorphism

$$\Phi : \oplus_i \mathbb{Z}e_i \rightarrow \pi_1 \text{Symp}(M, \mathbb{Z}) \xrightarrow{\text{Flux}} H^1(M, \mathbb{R}).$$

Since the boundary map $\oplus_i \mathbb{Z}e_i \rightarrow C_1(B) \otimes \mathbb{Q}$ is injective, we can now change the chosen trivializations over the 1-cells γ_j in B_1 to make $\Phi = 0$.

This ensures that $d_2^{0,2} = 0$ in the bundle over C . Since the map $H^q(C) \rightarrow H^q(B)$ is an isomorphism when $q \geq 3$, the vanishing of $d_3^{0,2}$ for B implies that it vanishes for C . Therefore (ii) holds for $Q \rightarrow C$. By the previous result, this implies that the structural group of $Q \rightarrow C$ reduces to $\text{Ham}(M)$. Therefore, the same holds for $P \rightarrow B$. \square

In the course of the above proof we established the following useful result.

Corollary 2.6 *Let C be the CW complex obtained by collapsing the 1-skeleton of B to a point and $f : B \rightarrow C$ be the obvious map. Then any Hamiltonian bundle $P \rightarrow B$ is the pullback by f of some Hamiltonian bundle over C .*

Theorem B.2 shows that there are two obstructions to the existence of a Hamiltonian structure on a symplectic bundle. Firstly, the bundle must be symplectically trivial over the 1-skeleton B_1 , and secondly the symplectic class on the fiber must extend. The first obstruction obviously depends on the 1-skeleton B_1 while the second, in principle, depends on its 3-skeleton (since we need d_2 and d_3 to vanish on $[\omega]$). However, in fact, it only depends on the 2-skeleton, as is shown in the next lemma.

Lemma 2.7 *Every symplectic bundle over a 2-connected base B is Hamiltonian.*

Proof: We give two proofs. First, observe that as in Lemma 2.3 we just have to show that $d_3^{0,2}([\omega]) = 0$. The arguments of that lemma apply to show that this is the case.

Alternatively, let $\widetilde{\text{Symp}}_0$ (resp. $\widetilde{\text{Ham}}$) denote the universal cover of the group $\text{Symp}_0 = \text{Symp}_0(M, \omega)$ (resp $\text{Ham}(M)$), and set $\pi_S = \pi_1(\text{Symp}_0)$ so that there are fibrations

$$\widetilde{\text{Ham}} \rightarrow \widetilde{\text{Symp}}_0 \xrightarrow{\text{Flux}} H^1(M, \mathbb{R}), \quad B(\pi_S) \rightarrow B\widetilde{\text{Symp}}_0 \rightarrow B\text{Symp}_0.$$

The existence of the first fibration shows that $\widetilde{\text{Ham}}$ is homotopy equivalent to $\widetilde{\text{Symp}}_0$ so that $B\widetilde{\text{Ham}} \simeq B\widetilde{\text{Symp}}_0$, while the second implies that there is a fibration

$$B\widetilde{\text{Symp}}_0 \rightarrow B\text{Symp}_0 \rightarrow K(\pi_S, 2),$$

where $K(\pi_S, 2)$ is an Eilenberg–MacLane space. A symplectic bundle over B is equivalent to a homotopy class of maps $B \rightarrow B\text{Symp}_0$. If B is 2-connected, the composite $B \rightarrow B\text{Symp}_0 \rightarrow K(\pi_S, 2)$ is null homotopic, so that the map $B \rightarrow B\text{Symp}_0$ lifts to $B\widetilde{\text{Symp}}_0$ and hence to the homotopic space $B\widetilde{\text{Ham}}$. Composing this map $B \rightarrow B\widetilde{\text{Ham}}$ with the projection $B\widetilde{\text{Ham}} \rightarrow B\text{Ham}$ we get a Hamiltonian structure on the given bundle over B .

Equivalently, use the existence of the fibration $\widetilde{\text{Ham}} \rightarrow \widetilde{\text{Symp}}_0 \rightarrow H^1(M, \mathbb{R})$ to deduce that the subgroup $\pi_1(\text{Ham})$ of $\widetilde{\text{Ham}}$ injects into $\pi_1(\text{Symp}_0)$. This implies that the relative homotopy groups $\pi_i(\text{Symp}_0, \text{Ham})$ vanish for $i > 1$, so that

$$\pi_i(B\text{Symp}_0, B\text{Ham}) = \pi_{i-1}(\text{Symp}_0, \text{Ham}) = 0, \quad i > 2.$$

The desired conclusion now follows by obstruction theory. \square

The second proof does not directly use the sequence $0 \rightarrow \text{Ham} \rightarrow \text{Symp}_0 \rightarrow H^1/\Gamma_\omega \rightarrow 0$ since the flux group Γ_ω may not be a discrete subgroup of H^1 .

2.2 The classification of Hamiltonian structures

The previous subsection discussed the question of the existence of Hamiltonian structures on a given bundle. We now look at the problem of describing and classifying them. We begin by proving Lemma 1.4 that states that any closed extension of the fiber class can be normalized.

Proof of Lemma 1.4 Let $\pi : P \rightarrow B$ be a symplectic bundle satisfying the conditions of Theorem B.2 and fix an identification of (M, ω) with (M_{b_0}, ω_{b_0}) . Let a be any closed extension of $[\omega]$, $\gamma_1, \dots, \gamma_k$ be a set of generators of the first rational homology group of B , $\{c_i\}$ the dual basis of $H^1(B)$ and T_1, \dots, T_k symplectic trivializations round the γ_i . Assume for the moment that each class $f(T_i, a) \in H^1(M_{b_0}) = H^1(M)$ has an extension $\tilde{f}(T_i, a)$ to P . Subtracting from a the class $\sum_{i=1}^k \pi^*(c_i) \cup \tilde{f}(T_i, a)$, we get a closed extension a' whose corresponding classes $f(T_i, a')$ belong to Γ_ω .

There remains to prove that the extensions of the $f(T_i, a)$'s exist in Hamiltonian bundles. It is enough to prove that the fiber inclusion $M \rightarrow P$ induces an injection on the first homology group. One only needs to prove this over the 2-skeleton B_2 of B , and, by Corollary 2.6 we can assume as well that B_2 is a wedge of 2-spheres. Hence this is a consequence of the easy fact that the evaluation of a Hamiltonian loop on a point of M gives a 1-cycle of M that is trivial in rational homology, i.e. that the differential $d_2^{0,1}$ vanishes in the cohomology spectral sequence for $P \rightarrow B$: see for instance [13] where this is proved by elementary methods. \square

The next result extends Lemma 2.2.

Lemma 2.8 *Let $P \rightarrow B$ be a symplectic bundle with a given symplectic trivialization of P over B_1 , and let $a \in H^2(P)$ be a normalized extension of the fiber symplectic class. Then*

the restriction of a to $\pi^{-1}(B_1)$ defines and is defined by a homomorphism Φ from $\pi_1(B)$ to Γ_ω .

Proof: As in Lemma 2.2, we can use the given trivialization to identify the holonomy round some loop $s \mapsto \gamma(s) \in B_1$ with a family of symplectomorphisms $\Phi_s^\gamma : M \rightarrow M, s \in [0, 1]$. Given a 1-cycle $\delta : S^1 \rightarrow M$ in the fiber M over $1 \in \partial D^2$, consider the closed 2-cycle $C(\gamma, \delta) = c_1 \cup c_2$ as before. Since $\tau(c_1) = 0$,

$$\tau(C(\gamma, \delta)) = \tau(c_2) = -\text{Flux}(\{\Phi_s^\gamma\})(\delta).$$

If we now set

$$\Phi(\gamma) = -\text{Flux}(\{\Phi_s^\gamma\}),$$

it is easy to check that Φ is a homomorphism. Its values are in Γ_ω by the definition of normalized extension classes. The result follows. \square

The next task is to prove Theorem 1.6 that characterizes Hamiltonian structures. Thus we need to understand the homotopy classes of lifts \tilde{g} of the classifying map $g : B \rightarrow B \text{Symp}(M, \omega)$ of the underlying symplectic bundle to $B \text{Ham}(M)$. We first consider the intermediate lift \hat{g} of g into $B \text{Symp}_0(M, \omega)$. In view of the fibration sequence

$$\pi_0(\text{Symp}) \rightarrow B \text{Symp}_0 \rightarrow B \text{Symp} \rightarrow B(\pi_0(\text{Symp}))$$

in which each space is mapped to the homotopy fiber of the subsequent map, a map $g : B \rightarrow B \text{Symp}$ lifts to $\hat{g} : B \rightarrow B \text{Symp}_0$ if and only if the symplectic bundle given by g can be trivialized over the 1-skeleton B_1 of B . Moreover such lifts are in bijective correspondence with the elements of $\pi_0(\text{Symp})$ and so correspond to an identification (up to symplectic isotopy) of (M, ω) with the fiber (M_{b_0}, ω_{b_0}) at the base point b_0 . (Recall that B is always assumed to be connected.)

To understand the full lift \tilde{g} , recall the exact sequence

$$\{id\} \longrightarrow \text{Ham}(M, \omega) \longrightarrow \text{Symp}_0(M, \omega) \xrightarrow{\text{Flux}} H^1(M, \mathbb{R})/\Gamma_\omega \longrightarrow \{0\}. \quad (*)$$

If Γ_ω is discrete, then the space $H^1(M, \mathbb{R})/\Gamma_\omega$ is homotopy equivalent to a torus and we can investigate the liftings \tilde{g} by homotopy theoretic arguments about the fibration

$$H^1(M, \mathbb{R})/\Gamma_\omega \rightarrow B \text{Ham}(M, \omega) \rightarrow B \text{Symp}_0(M, \omega).$$

However, in general, we need to argue more geometrically.

Suppose that a symplectic bundle $\pi : P \rightarrow B$ is given that satisfies the conditions of Theorem B.2. Fix an identification of (M, ω) with (M_{b_0}, ω_{b_0}) . We have to show that lifts from $B \text{Symp}_0$ to $B \text{Ham}$ are in bijective correspondence with equivalence classes of normalized extensions a of the fiber symplectic class. By Theorem B.2 and Lemma 1.4, there is a lift if and only if there is a normalized extension class a . Therefore, it remains to show that the equivalence relations correspond. The essential reason why this is true is that the induced map

$$\pi_i(\text{Ham}(M, \omega)) \rightarrow \pi_i(\text{Symp}_0(M, \omega))$$

is an injection for $i = 1$ and an isomorphism for $i > 1$. This, in turn, follows from the exactness of the sequence (*).

Let us spell out a few more details, first when B is simply connected. Then the classifying map from the 2-skeleton B_2 to $B\text{Symp}_0$ has a lift to $B\text{Ham}$ if and only if the image of the induced map

$$\pi_2(B_2) \rightarrow \pi_2(B\text{Symp}_0(M)) = \pi_1\text{Symp}_0(M)$$

lies in the kernel of the flux homomorphism

$$\text{Flux} : \pi_1(\text{Symp}_0(M)) \longrightarrow \Gamma_\omega.$$

Since $\pi_1(\text{Ham}(M, \omega))$ injects into $\pi_1(\text{Symp}_0(M, \omega))$ there is only one such lift up to homotopy. Standard arguments now show that this lift can be extended uniquely to the rest of B . Hence in this case there is a unique lift. Correspondingly there is a unique equivalence class of extensions a .

Now let us consider the general case. We are given a map $g : B \rightarrow B\text{Symp}_0$ and want to identify the different homotopy classes of liftings of g to $B\text{Ham}$. Let $C = B/B_1$ as above. By Corollary 2.6, there is a symplectic trivivialization T over B_1 that is compatible with the given identification of the base fiber and induces a map $C \rightarrow B\text{Symp}_0$ which lifts to $B\text{Ham}$. Since this lifting $g_{T,C}$ of $C \rightarrow B\text{Symp}_0$ is unique, each isotopy class T of such trivivializations over B_1 gives rise to a unique homotopy class g_T of maps $B \rightarrow B\text{Ham}$, namely

$$g_T : B \longrightarrow C \xrightarrow{g_{T,C}} B\text{Ham}.$$

Note that g_T is a lifting of f and that every lifting occurs this way.

Standard arguments show that two such isotopy classes differ by a homomorphism

$$\pi_1(B) \longrightarrow \pi_1(\text{Symp}_0).$$

Moreover, the corresponding maps g_T and $g_{T'}$ are homotopic if and only if T and T' differ by a homomorphism with values in $\pi_1(\text{Ham})$. Thus homotopy classes of liftings of g to $B\text{Ham}$ are classified by homomorphisms $\pi_1(B) \rightarrow \Gamma_\omega$. By Lemma 2.8 these homomorphisms are precisely what defines the equivalence classes of extensions a . \square

3 Properties of Hamiltonian bundles

The key to extending results about Hamiltonian bundles over S^2 to higher dimensional bases is their functorial properties, in particular their behavior under composition. Before discussing this, it is useful to establish the fact that this class of bundles is stable under small perturbations of the symplectic form on M .

3.1 Stability

Moser's argument implies that for every symplectic structure ω on M there is a Serre fibration

$$\text{Symp}(M, \omega) \longrightarrow \text{Diff}(M) \longrightarrow \mathcal{S}_\omega,$$

where \mathcal{S}_ω is the space of all symplectic structures on M that are diffeomorphic to ω . At the level of classifying spaces, this gives a homotopy fibration

$$\mathcal{S}_\omega \hookrightarrow B\text{Symp}(M, \omega) \longrightarrow B\text{Diff}(M).$$

Any smooth fiber bundle $P \rightarrow B$ with fiber M is classified by a map $B \rightarrow B\text{Diff}(M)$, and isomorphism classes of symplectic structures on it with fiber (M, ω) correspond to homotopy classes of sections of the associated fibration $W(\omega) \rightarrow B$ with fiber \mathcal{S}_ω . We will suppose that π is described by a finite set of local trivializations $T_i : \pi^{-1}(V_i) \rightarrow V_i \times M$ with the transition functions $\phi_{ij} : V_i \cap V_j \rightarrow \text{Diff}(M)$.

Lemma 3.1 *Suppose that $M \rightarrow P \rightarrow B$ is a smooth fibration constructed from a cocycle (T, ϕ_{ij}) with the following property: there is a symplectic form ω on M such that for each $x \in M$ the convex hull of the finite set of forms*

$$\{\phi_{ij}^*(\omega) : x \in V_i \cap V_j\}$$

lies in the set \mathcal{S}_ω of symplectic forms diffeomorphic to ω . Then $(M, \omega) \rightarrow P \rightarrow B$ may be given the structure of a symplectic bundle.

Proof: It suffices to construct a section σ of $W(\omega) \rightarrow B$. The hypothesis implies that for each x the convex hull of the set of forms $T_i(x)^*(\omega), x \in V_i$, lies in the fiber of $W(\omega)$ at x . Hence we may take

$$\sigma(x) = \sum_i \rho_i T_i^*(\omega),$$

where ρ_i is a partition of unity subordinate to the cover V_i . □

Corollary 3.2 *Let $P \rightarrow B$ be a symplectic bundle with closed fiber (M, ω) and compact base B . There is an open neighborhood \mathcal{U} of ω in the space $\mathcal{S}(M)$ of all symplectic forms on M such that, for all $\omega' \in \mathcal{U}$, the structural group of $\pi : P \rightarrow B$ may be reduced to $\text{Symp}(M, \omega')$.*

Proof: Trivialize $P \rightarrow B$ so that $\phi_{ij}^*(\omega) = \omega$ for all i, j . Then the hypothesis of the lemma is satisfied for all ω' sufficiently close to ω by the openness of the symplectic condition. □

Thus the set $\mathcal{S}_\pi(M)$ of symplectic forms on M , with respect to which π is symplectic, is open. The aim of this section is to show that a corresponding statement is true for Hamiltonian bundles. The following example shows that the Hamiltonian property need not survive under *large* perturbations of ω because condition (i) in Theorem B.2 can fail. However, it follows from the proof of stability that condition (ii) never fails under any perturbation.

Example 3.3 Here is an example of a smooth family of symplectic bundles that is Hamiltonian at all times $0 \leq t < 1$ but is nonHamiltonian at time 1. Let $h_t, 0 \leq t \leq 1$, be a family of diffeomorphisms of M with $h_0 = id$ and define

$$Q = M \times [0, 1] \times [0, 1] / (x, 1, t) \equiv (h_t(x), 0, t).$$

Thus we can think of Q as a family of bundles $\pi : P_t \rightarrow S^1$ with monodromy h_t at time t . Seidel [25] has shown that there are smooth families of symplectic forms ω_t and diffeomorphisms $h_t \in \text{Symp}(M, \omega_t)$ for $t \in [0, 1]$ such that h_t is not in the identity component of $\text{Symp}(M, \omega_t)$ for $t = 1$ but is in this component for $t < 1$. For such h_t each bundle $P_t \rightarrow S^1$ is symplectic. Moreover, it is symplectically trivial and hence Hamiltonian for $t < 1$ but is nonHamiltonian at $t = 1$. This example shows why the verification of condition (i) in the next proof is somewhat delicate.

Lemma 3.4 *A Hamiltonian bundle $\pi : P \rightarrow B$ is stable if and only if the restriction map $H^2(P) \rightarrow H^2(M)$ is surjective.*

Proof: If $\pi : P \rightarrow B$ is Hamiltonian with respect to ω' then by Theorem B.2 $[\omega']$ is in the image of $H^2(P) \rightarrow H^2(M)$. If π is stable, then $[\omega']$ fills out a neighborhood of $[\omega]$ which implies surjectivity. Conversely, suppose that we have surjectivity. Then the second condition of Theorem B.2 is satisfied. To check (i) let $\gamma : S^1 \rightarrow B$ be a loop in B and suppose that $\gamma^*(P)$ is identified symplectically with the product bundle $S^1 \times (M, \omega)$. Let $\omega_t, 0 \leq t \leq \varepsilon$, be a (short) smooth path with $\omega_0 = \omega$. Then, because $P \rightarrow B$ has the structure of an ω_t -symplectic bundle for each t , each fiber M_b has a corresponding smooth family of symplectic forms $\omega_{b,t}$ of the form $g_{b,t}^* \psi_b^*(\omega_t)$, where ψ_b is a symplectomorphism $(M_b, \omega_b) \rightarrow (M, \omega)$. Hence, for each t , $\gamma^*(P)$ can be symplectically identified with

$$\bigcup_{s \in [0,1]} (\{s\} \times (M, g_{s,t}^*(\omega_t))),$$

where $g_{1,t}^*(\omega_t) = \omega_t$ and the $g_{s,t}$ lie in an arbitrarily small neighborhood U of the identity in $\text{Diff}(M)$. By Moser's homotopy argument, we can choose U so small that each $g_{1,t}$ is isotopic to the identity in the group $\text{Symp}(M, \omega_t)$. This proves (i). \square

Corollary 3.5 *The pullback of a stable Hamiltonian bundle is stable.*

Proof: Suppose that $P \rightarrow B$ is the pullback of $P' \rightarrow B'$ via $B \rightarrow B'$ so that there is a diagram

$$\begin{array}{ccc} P & \rightarrow & P' \\ \downarrow & & \downarrow \\ B & \rightarrow & B'. \end{array}$$

By hypothesis, the restriction $H^2(P') \rightarrow H^2(M)$ is surjective. But this map factors as $H^2(P') \rightarrow H^2(P) \rightarrow H^2(M)$. Hence $H^2(P) \rightarrow H^2(M)$ is also surjective. \square

Lemma 3.6 (i) *Every Hamiltonian bundle over S^2 is stable.*

(ii) *Every symplectic bundle over a 2-connected base B is Hamiltonian stable.*

Proof: (i) holds because every Hamiltonian bundle over S^2 is c-split, in particular the restriction map $H^2(P) \rightarrow H^2(M)$ is surjective. (ii) follows immediately from Lemma 2.7. \square

Proof of Theorem 1.13.

This states that every Hamiltonian bundle is stable. To prove this, first observe that we can restrict to the case when B is simply connected. For the map $B \rightarrow B\text{Ham}(M)$ classifying P factors through a map $C \rightarrow B\text{Ham}(M)$, where $C = B/B_1$ as before, and the stability of the induced bundle over C implies that for the original bundle by Corollary 3.5.

Next observe that by Lemma 3.4 a Hamiltonian bundle $P \rightarrow B$ is stable if and only if the differentials $d_k^{0,2} : E_k^{0,2} \rightarrow E_k^{k,3-k}$ in its Leray cohomology spectral sequence vanish on the whole of $H^2(M)$ for $k = 2, 3$. Exactly as in the proof of Lemma 2.3 we can reduce the statement for $d_2^{0,2}$ to the case $B = S^2$. Thus $d_2^{0,2} = 0$ by Lemma 3.6(i). Similarly, we can reduce the statement for $d_3^{0,2}$ to the case $B = S^3$ and then use Lemma 3.6(ii). \square

3.2 Functorial properties

We begin with some trivial observations and then discuss composites of Hamiltonian bundles. The first lemma is true for any class of bundles with specified structural group.

Lemma 3.7 *Suppose that $\pi : P \rightarrow B$ is Hamiltonian and that $g : B' \rightarrow B$ is a continuous map. Then the induced bundle $\pi' : g^*(P) \rightarrow B'$ is Hamiltonian.*

Recall from §1.1 that any extension τ of the forms on the fibers is called a connection form.

Lemma 3.8 *If $P \rightarrow B$ is a smooth Hamiltonian fiber bundle over a symplectic base (B, σ) and if P is compact then there is a connection form Ω^κ on P that is symplectic.*

Proof: By Proposition B.3, the bundle P carries a closed connection form τ . Since P is compact, the form $\Omega^\kappa = \tau + \kappa\pi^*(\sigma)$ is symplectic for large κ . \square

Observe that the deformation type of the form Ω^κ is unique for sufficiently large κ since given any two closed connection forms τ, τ' the linear isotopy

$$t\tau + (1-t)\tau' + \kappa\pi^*(\sigma), \quad 0 \leq t \leq 1,$$

consists of symplectic forms for sufficiently large κ . However, it can happen that there is a symplectic connection form τ such that $\tau + \kappa\pi^*(\sigma)$ is not symplectic for small $\kappa > 0$, even though it is symplectic for large κ . (For example, suppose $P = M \times B$ and that τ is the sum $\omega + \pi^*(\omega_B)$ where $\omega_B + \sigma$ is not symplectic.)

Let us now consider the behavior of Hamiltonian bundles under composition. If

$$(M, \omega) \rightarrow P \xrightarrow{\pi_P} X, \quad \text{and} \quad (F, \sigma) \rightarrow X \xrightarrow{\pi_X} B$$

are Hamiltonian fiber bundles, then the restriction

$$\pi_P : W = \pi_P^{-1}(F) \longrightarrow F$$

is a Hamiltonian fiber bundle. Since F is a manifold, we can assume without loss of generality that $W \rightarrow F$ is smooth: see Lemma 2.1. Moreover, since (F, σ) is symplectic Lemma 3.8 implies that the manifold W carries a symplectic connection form Ω_W^κ , and it is natural to ask when the composite map $\pi : P \rightarrow B$ with fiber (W, Ω_W^κ) is itself Hamiltonian.

Lemma 3.9 *Suppose in the above situation that B is simply connected and that P is compact. Then $\pi = \pi_X \circ \pi_P : P \rightarrow B$ is a Hamiltonian fiber bundle with fiber (W, Ω_W^κ) , where $\Omega_W^\kappa = \tau_W + \kappa\pi_P^*(\sigma)$, τ_W is any symplectic connection form on W , and κ is sufficiently large.*

Proof: By Lemma 2.1, we may assume that the base B as well as the fibrations are smooth. We first show that there is some symplectic form on W for which π is Hamiltonian and then show that it is Hamiltonian with respect to the given form Ω_W^κ .

Let τ_P (resp. τ_X) be a closed connection form for the bundle π_P , (resp. π_X), and let τ_W be its restriction to W . Then Ω_W^κ is the restriction to W of the closed form

$$\Omega_P^\kappa = \tau_P + \kappa\pi_P^*(\tau_X).$$

By increasing κ if necessary we can ensure that Ω_P^κ restricts to a symplectic form on every fiber of π not just on the chosen fiber W . This shows firstly that $\pi : P \rightarrow B$ is symplectic, because there is a well defined symplectic form on each of its fibers, and secondly that it is Hamiltonian with respect to this form Ω_W^κ on the fiber W . Hence Lemma 3.4 implies that $H^2(P)$ surjects onto $H^2(W)$.

Now suppose that τ_W is *any* closed connection form on $\pi_P : W \rightarrow F$. Because the restriction map $H^2(P) \rightarrow H^2(W)$ is surjective, the cohomology class $[\tau_W]$ is the restriction of a class on P and so, by Thurston's construction, the form τ_W can be extended to a closed connection form τ_P for the bundle π_P . Therefore the previous argument applies in this case too. \square

Now let us consider the general situation, when $\pi_1(B) \neq 0$. The proof of the lemma above applies to show that the composite bundle $\pi : P \rightarrow B$ is symplectic with respect to suitable Ω_W^κ and that it has a symplectic connection form. However, even though $\pi_X : X \rightarrow B$ is symplectically trivial over the 1-skeleton of B the same may not be true of the composite map $\pi : P \rightarrow B$. Moreover, in general it is not clear whether triviality with respect to one form Ω_W^κ implies it for another. Therefore, we may conclude the following:

Proposition 3.10 *If $(M, \omega) \rightarrow P \xrightarrow{\pi_P} X$, and $(F, \sigma) \rightarrow X \xrightarrow{\pi_X} B$ are Hamiltonian fiber bundles and P is compact, then the composite $\pi = \pi_X \circ \pi_P : P \rightarrow B$ is a symplectic fiber bundle with respect to any form Ω_W^κ on its fiber $W = \pi^{-1}(pt)$, provided that κ is sufficiently large. Moreover if π is symplectically trivial over the 1-skeleton of B with respect to Ω_W^κ then π is Hamiltonian.*

In practice, we will apply these results in cases where $\pi_1(B) = 0$. We will not specify the precise form on W , assuming that it is Ω_W^κ for a suitable κ .

4 Splitting of rational cohomology

We write $H_*(X), H^*(X)$ for the rational (co)homology of X . Recall that a bundle $\pi : P \rightarrow B$ with fiber M is said to be *c-split* if

$$H^*(P) \cong H^*(B) \otimes H^*(M).$$

This happens if and only if $H_*(M)$ injects into $H_*(P)$. Dually, it happens if and only if the restriction map $H^*(P) \rightarrow H^*(M)$ is onto. Note also that a bundle $P \rightarrow B$ c-splits if and only if the E_2 term of its cohomology spectral sequence is a product and all the differentials $d_k, k \geq 2$, vanish.

In this section we prove all parts of Theorem 1.14. We begin by using topological arguments that are based on the fact that bundles over S^2 are c-split. This was proved in [13, 21] by geometric arguments using Gromov–Witten invariants. In §4.2 we discuss the extent to which these geometric arguments generalize. Finally in §4.3 we discuss c-splitting in a homotopy-theoretic context.

4.1 A topological discussion of c-splitting

The first lemma is obvious but useful. We will often refer to its second part as the Surjection Lemma.

Lemma 4.1 Consider a commutative diagram

$$\begin{array}{ccc} P' & \rightarrow & P \\ \downarrow & & \downarrow \\ B' & \rightarrow & B \end{array}$$

where P' is the induced bundle. Then:

(i) If $P \rightarrow B$ is c -split so is $P' \rightarrow B'$.

(ii) **(Surjection Lemma)** If $P' \rightarrow B'$ is c -split and $H_*(B') \rightarrow H_*(B)$ is surjective, then $P \rightarrow B$ is c -split.

Proof: (i): Use the fact that $P \rightarrow B$ is c -split if and only if the map $H_*(M) \rightarrow H_*(P)$ is injective.

(ii): The induced map on the E_2 -term of the cohomology spectral sequences is injective. Therefore the existence of a nonzero differential in the spectral sequence $P \rightarrow B$ implies one for the pullback bundle $P' \rightarrow B'$. \square

Corollary 4.2 Suppose that $P \rightarrow W$ is a Hamiltonian fiber bundle over a symplectic manifold W and that its pullback to some blowup \widehat{W} of W is c -split. Then $P \rightarrow W$ is c -split.

Proof: This follows immediately from (ii) above since the map $H_*(\widehat{W}) \rightarrow H_*(W)$ is surjective. \square

Lemma 4.3 If $(M, \omega) \xrightarrow{\pi} P \rightarrow B$ is a compact Hamiltonian bundle over a simply connected CW-complex B and if every Hamiltonian fiber bundle over M and B is c -split, then every Hamiltonian bundle over P is c -split.

Proof: Let $\pi_E : E \rightarrow P$ be a Hamiltonian bundle with fiber F and let

$$F \rightarrow W \rightarrow M$$

be its restriction over M . Then by assumption the latter bundle c -splits so that $H_*(F)$ injects into $H_*(W)$. Lemma 3.9 implies that the composite bundle $E \rightarrow B$ is Hamiltonian with fiber W and therefore also c -splits. Hence $H_*(W)$ injects into $H_*(E)$. Thus $H_*(F)$ injects into $H_*(E)$, as required. \square

Lemma 4.4 If Σ is a closed orientable surface then any Hamiltonian bundle over $S^2 \times \dots \times S^2 \times \Sigma$ is c -split.

Proof: Consider any degree one map f from $\Sigma \rightarrow S^2$. Because $\text{Ham}(M, \omega)$ is connected, $B\text{Ham}(M, \omega)$ is simply connected, and therefore any homotopy class of maps from $\Sigma \rightarrow B\text{Ham}(M, \omega)$ factors through f . Thus any Hamiltonian bundle over Σ is the pullback by f of a Hamiltonian bundle over S^2 . Because such bundles c -split over S^2 , the same is true over Σ by Lemma 4.1(i).

The statement for $S^2 \times \dots \times S^2 \times \Sigma$ is now a direct consequence of iterative applications of Lemma 4.3 applied to the trivial bundles $S^2 \times \dots \times S^2 \times \Sigma \rightarrow S^2$. \square

Corollary 4.5 *Any Hamiltonian bundle over $S^2 \times \dots \times S^2 \times S^1$ is c-split.*

Proposition 4.6 *For each $k \geq 1$, every Hamiltonian bundle over S^k c-splits.*

Proof: By Lemma 4.4 and Corollary 4.5 there is for each k a k -dimensional closed manifold X such that every Hamiltonian bundle over X c-splits. Given any Hamiltonian bundle $P \rightarrow S^k$ consider its pullback to X by a map $f : X \rightarrow S^k$ of degree 1. Since the pullback c-splits, the original bundle does too by the surjection lemma. \square

As we shall see this result implies that the action of the homology groups of $\text{Ham}(M)$ on $H_*(M)$ is always trivial. Here are some other examples of situations in which Hamiltonian bundles are c-split.

Lemma 4.7 *Every Hamiltonian bundle over $\mathbb{C}P^{n_1} \times \dots \times \mathbb{C}P^{n_k}$ c-splits.*

Proof: Let us prove first that it splits over $\mathbb{C}P^n$. Use induction over n . Again it holds when $n = 1$. Assuming the result for n let us prove it for $n + 1$. Let B be the blowup of $\mathbb{C}P^{n+1}$ at one point. Then B fibers over $\mathbb{C}P^n$ with fiber $\mathbb{C}P^1$. Thus every Hamiltonian bundle over B c-splits by Lemma 4.3. The result for $\mathbb{C}P^{n+1}$ now follows from Corollary 4.2. Finally Hamiltonian bundles c-split over products of projective spaces by repeated applications of Lemma 4.3. \square

Corollary 4.8 *Every Hamiltonian bundle whose structural group reduces to a subtorus $T \subset \text{Ham}(M)$ c-splits.*

Proof: It suffices to consider the universal model

$$M \longrightarrow ET \times_T M \longrightarrow BT,$$

and hence to show that all Hamiltonian bundles over BT are c-split. But this is equal to $\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty$ and the proof that the i^{th} group of homology of the fiber injects in $P \rightarrow \mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty$ may be reduced to the proof that it injects in the restriction of the bundle P over $\mathbb{C}P^j \times \dots \times \mathbb{C}P^j$ for a sufficiently large j . But this is Lemma 4.7. \square

Remark 4.9 Observe that the proof of the above corollary shows that every Hamiltonian bundle over $\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty$ c-splits. Since the structural group of such a bundle can be larger than the torus T , our result extends the Atiyah-Bott splitting theorem for Hamiltonian bundles with structural group T .

For completeness, we show how the above corollary leads to a proof of the splitting of G -equivariant cohomology where G is a Lie subgroup of $\text{Ham}(M, \omega)$.

Corollary 4.10 *If G is a compact connected Lie group that acts in a Hamiltonian way on M then any bundle $P \rightarrow B$ with fiber M and structural group G is c-split. In particular,*

$$H_G^*(M) \cong H^*(M) \otimes H^*(BG).$$

Proof: By Lemma 4.1(i) we only need to prove the second statement, since

$$M_G = EG \times_G M \longrightarrow BG$$

is the universal bundle. Every compact connected Lie group G is the image of a homomorphism $T \times H \rightarrow G$, where the torus T maps onto the identity component of the center of G and H is the semi-simple Lie group corresponding to the commutator subalgebra $[\text{Lie}(G), \text{Lie}(G)]$ in the Lie algebra $\text{Lie}(G)$. It is easy to see that this homomorphism induces a surjection on rational homology $BT \times BH \rightarrow BG$. Therefore, we may suppose that $G = T \times H$. Let $T_{max} = (S^1)^k$ be the maximal torus of the semi-simple group H . Then the induced map on cohomology $H^*(BH) \rightarrow H^*(BT_{max}) = \mathbb{Q}[a_1, \dots, a_k]$ takes $H^*(BH)$ bijectively onto the set of polynomials in $H^*(BT_{max})$ that are invariant under the action of the Weyl group, by the Borel-Hirzebruch theorem. Hence the maps $BT_{max} \rightarrow BH$ and $BT \times BT_{max} \rightarrow BG$ induce a surjection on homology. Therefore the desired result follows from the surjection lemma and the last corollary (or the Atiyah–Bott theorem.) \square

Lemma 4.11 *Every Hamiltonian bundle over a coadjoint orbit c-splits.*

Proof: This is an immediate consequence of the results by Grossberg–Karshon [7]§3 on Bott towers. A Bott tower is an iterated fibration $M_k \rightarrow M_{k-1} \rightarrow \dots \rightarrow M_1 = S^2$ of Kähler manifolds where each map $M_{i+1} \rightarrow M_i$ is a fibration with fiber S^2 . They show that any coadjoint orbit X can be blown up to a manifold that is diffeomorphic to a Bott tower M_k . Moreover the blow-down map $M_k \rightarrow X$ induces a surjection on rational homology. Every Hamiltonian bundle over M_k c-splits by repeated applications of Lemma 4.3. Hence the result follows from the surjection lemma. \square

Lemma 4.12 *Every Hamiltonian bundle over a 3-complex X c-splits.*

Proof: As in the proof of stability given in Theorem 1.13 we can reduce this to the cases $X = S^2$ and $X = S^3$ and then use Proposition 4.6. The only difference from the stability result is that we now require the differentials $d_2^{0,q}, d_3^{0,q}$ to vanish for all q rather than just at $q = 2$. \square

Lemma 4.13 *Every Hamiltonian bundle over a product of spheres c-splits, provided that there are no more than 3 copies of S^1 .*

Proof: By hypothesis $B = \prod_{i \in I} S^{2m_i} \times \prod_{j \in J} S^{2n_j+1} \times T^k$, where $n_i > 0$ and $0 \leq k \leq 3$. Set

$$B' = \prod_{i \in I} \mathbb{C}P^{m_i} \times \prod_{j \in J} \mathbb{C}P^{n_j} \times T^{|J|} \times T^\ell,$$

where $\ell = k$ if $k + |J|$ is even and $= k + 1$ otherwise. Since $\mathbb{C}P^{n_i} \times S^1$ maps onto S^{2n_i+1} by a map of degree 1, there is a homology surjection $B' \rightarrow B$ that maps the factor T^ℓ to T^k . By the surjection lemma, it suffices to show that the pullback bundle $P' \rightarrow B'$ is c-split.

Consider the fibration

$$T^{|J|} \times T^\ell \rightarrow B' \rightarrow \prod_{i \in I} \mathbb{C}P^{m_i} \times \prod_{j \in J} \mathbb{C}P^{n_j}.$$

Since $|J|+\ell$ is even, we can think of this as a Hamiltonian bundle. Moreover, by construction, the restriction of the bundle $P' \rightarrow B'$ to $T^{|J|} \times T^\ell$ is the pullback of a bundle over T^k , since the map $T^{|J|} \rightarrow B$ is nullhomotopic. (Note that each S^1 factor in $T^{|J|}$ goes into a different sphere in B .) Because $k \leq 3$, the bundle over T^k c-splits. Hence we can apply the argument in Lemma 4.3 to conclude that $P' \rightarrow B'$ c-splits. \square

Lemma 4.14 *Every Hamiltonian bundle whose fiber has cohomology generated by H^2 is c-split.*

Proof: This is an immediate consequence of Theorem 1.13. \square

Proof of Theorem 1.14

Parts (i) and (iii) are proved in the Lemmas 4.11, 4.12 and 4.13 above, and part (ii) is proved in §4.2 below. \square

Of course, the results in this section can be extended further by applying the surjection lemma and variants of Lemma 4.3. For example, any Hamiltonian fibration c-splits if its base B is the image of a homology surjection from a product of spheres and projective spaces, provided that there are no more than three S^1 factors. One can also consider iterated fibrations of projective spaces, rather than simply products. However, we have not yet managed to deal with arbitrary products of spheres. In order to do this, it would suffice to show that every Hamiltonian bundle over a torus T^m c-splits. This question has not yet been resolved for $m \geq 4$.

4.2 Hamiltonian bundles and Gromov–Witten invariants

We begin by sketching an alternative proof that every Hamiltonian bundle over $B = \mathbb{C}P^n$ is c-split that generalizes the arguments in [21]. We will use the language of [20], which is based on the Liu–Tian [14] approach to general Gromov–Witten invariants. No doubt any treatment of general Gromov–Witten invariants could be used instead.

Proof that every Hamiltonian bundle over $\mathbb{C}P^n$ is c-split.

The basic idea is to show that the inclusion $\iota : H_*(M) \rightarrow H_*(P)$ is injective by showing that for every nonzero $a \in H_*(M)$ there is $b \in H_*(M)$ and $\sigma \in H_2(P; \mathbb{Z})$ for which the Gromov–Witten invariant $n_P(\iota(a), \iota(b); \sigma)$ is nonzero. Intuitively speaking this invariant “counts the number of isolated J -holomorphic curves in P that represent the class σ and meet the classes $\iota(a), \iota(b)$.” More correctly, it is defined to be the intersection number of the image of the evaluation map

$$ev : \overline{\mathcal{M}}'_{0,2}(P, J, \sigma) \longrightarrow P \times P$$

with the class $\iota(a) \times \iota(b)$, where $\overline{\mathcal{M}}'_{0,2}(P, J, \sigma)$ is a virtual moduli cycle made from perturbed J -holomorphic curves with 2 marked points, and ev is given by evaluating at these two points. As explained in [20, 21], $\overline{\mathcal{M}}^\nu = \overline{\mathcal{M}}'_{0,2}(P, J, \sigma)$ is a branched pseudomanifold, i.e. a kind of stratified space whose top dimensional strata are oriented and have rational labels. Roughly speaking, one can think of it as a finite simplicial complex, whose dimension d equals the “formal dimension” of the moduli space, i.e. the index of the relevant operator. The elements of $\overline{\mathcal{M}}^\nu$ are stable maps $[\Sigma, h, z_1, z_2]$ where z_1, z_2 are two marked points on the

nodal, genus 0, Riemann surface Σ , and the map $h : \Sigma \rightarrow P$ satisfies a perturbed Cauchy–Riemann equation $\bar{\partial}_J h = \nu_h$. The perturbation ν can be arbitrarily small, and is chosen so that each stable map in $\overline{\mathcal{M}}^\nu$ is a regular point for the appropriate Fredholm operator. Hence $\overline{\mathcal{M}}^\nu$ is often called a *regularization* of the unperturbed moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,2}(P, J, \sigma)$ of all J -holomorphic stable maps.

Given any Hamiltonian bundle $P_S \rightarrow S^2$ and any $a \in H_*(M)$, it was shown in [13, 21] that there is $b \in H_*(M)$ and a lift $\sigma_S \in H_2(P_S; \mathbb{Z})$ of the fundamental class of S^2 to P_S such that

$$n_{P_S}(\iota_S(a), \iota_S(b); \sigma_S) \neq 0,$$

where ι_S denotes the inclusion into P_S . Therefore, if P_S is identified with the restriction of P to a complex line L_0 in B and if a, b and σ_S are as above, it suffices to prove that

$$n_{P_S}(\iota_S(a), \iota_S(b); \sigma_S) = n_P(\iota(a), \iota(b); \sigma)$$

where σ is the image of σ_S in P . Note that a direct count shows that the dimensions of the appropriate virtual moduli spaces $\overline{\mathcal{M}}_{0,2}^\nu(P_S, J_S, \sigma_S)$ and $\overline{\mathcal{M}}_{0,2}^\nu(P, J, \sigma)$ differ by the codimension of $P_S \times P_S$ in $P \times P$ (which equals the codimension of $\mathbb{C}P^1 \times \mathbb{C}P^1$ in $\mathbb{C}P^n \times \mathbb{C}P^n$) so that the both sides are well-defined.

As was shown in [21] Corollary 4.11, one can construct the virtual moduli cycle $\overline{\mathcal{M}}^\nu(P_S) = \overline{\mathcal{M}}_{0,2}^\nu(P_S, J_S, \sigma_S)$ using an almost complex structure J_S and a perturbation ν that are compatible with the bundle. In particular, this implies that the projection $P_S \rightarrow S^2$ is (J_S, j) -holomorphic (where j is the usual complex structure on S^2) and that every element of $\overline{\mathcal{M}}^\nu(P_S)$ projects to a j -holomorphic stable map in S^2 .

We claim that this is also true for the bundle $P \rightarrow B$. In other words, we can choose J so that the projection $(P, J) \rightarrow (B, j)$ is holomorphic, where j is the usual complex structure on $B = \mathbb{C}P^n$, and we can choose ν so that every element in $\overline{\mathcal{M}}^\nu(P)$ projects to a j -holomorphic stable map in B . The proof is exactly as before: see [21] Lemma 4.9. The essential point is that every element of the unperturbed moduli space $\overline{\mathcal{M}}_{0,2}(\mathbb{C}P^n, j, [\mathbb{C}P^1])$ is regular. In fact, the top stratum in $\overline{\mathcal{M}}_{0,2}(\mathbb{C}P^n, j, [\mathbb{C}P^1])$ is the space $\mathcal{L} = \mathcal{M}_{0,2}(\mathbb{C}P^n, j, [\mathbb{C}P^1])$ of all lines in $\mathbb{C}P^n$ with 2 distinct marked points. The other stratum completes this space by adding in the lines with two coincident marked points, which are represented as stable maps by a line together with a ghost bubble containing the two points.

It follows that there is a projection map

$$proj : \overline{\mathcal{M}}_{0,2}^\nu(P, J, \sigma) \longrightarrow \overline{\mathcal{M}}_{0,2}(\mathbb{C}P^n, j, [\mathbb{C}P^1]).$$

Moreover the inverse image of a line $L \in \mathcal{L}$ can morally speaking be identified with $\overline{\mathcal{M}}_{0,2}^\nu(P_S, J_S, \sigma_S)$. The latter statement would be correct if we were considering ordinary moduli spaces of stable maps, but the virtual moduli space is not usually built in such a way that the fibers $(proj)^{-1}(L)$ have the needed structure of a branched pseudomanifold. However, we can choose to construct $\overline{\mathcal{M}}_{0,2}^\nu(P, J, \sigma)$ so that this is true for all lines near a fixed line L_0 . In [21] (see also [20]) a detailed recipe is given for constructing $\overline{\mathcal{M}}^\nu$ from the unperturbed moduli space $\overline{\mathcal{M}}$. The construction is based on the choice of suitable covers $\{U_i\}, \{V_I\}$ of $\overline{\mathcal{M}}$ and of perturbations ν_i over each U_i . Because regularity is an open condition, one can make these choices first for all stable maps that project to the fixed line L_0 and then extend to the set of stable maps that project to nearby lines in such a way that

$\overline{\mathcal{M}}^\nu$ is locally a product near the fiber over L_0 : see the proof of [21] Proposition 4.6 for a very similar construction.

Once this is done, the rest of the argument is easy. If we identify P_S with $\pi^{-1}(L_0)$ and choose a representative $\alpha \times \beta$ of $\iota_S(a) \times \iota_S(b)$ in $P_S \times P_S$ that is transverse to the evaluation map from $\overline{\mathcal{M}}'_{0,2}(P_S, J_S, \sigma_S)$, its image in $P \times P$ will be transverse to the evaluation map from $\overline{\mathcal{M}}'_{0,2}(P, J, \sigma)$ because *proj* is a submersion at L_0 . Moreover, by [21] Lemma 4.14, we may suppose that α and β lie in distinct fibers of the projection $P_S \rightarrow S^2$. Let x_a, x_b be the corresponding points of $\mathbb{C}P^n$ under the identification $S^2 = L_0$. Then every stable map that contributes to $n_P(\iota(a), \iota(b); \sigma)$ projects to an element of $\overline{\mathcal{M}}_{0,2}(\mathbb{C}P^n, j, [\mathbb{C}P^1])$ whose marked points map to the distinct points x_a, x_b . Since there is a unique line in $\mathbb{C}P^n$ through two given points, in this case L_0 , every stable map that contributes to $n_P(\iota(a), \iota(b); \sigma)$ must project to L_0 and hence be contained in $\overline{\mathcal{M}}'_{0,2}(P_S, J_S, \sigma_S)$. One can then check that

$$n_{P_S}(\iota_S(a), \iota_S(b); \sigma_S) = n_P(\iota(a), \iota(b); \sigma),$$

as claimed. The only delicate point here is to verify that the sign of each stable map on the left hand side is the same as the sign of the corresponding map on the right hand side. But this is also a consequence of the local triviality of the above projection map (see [10] for more details). \square

The above argument generalizes easily to the case when B is a complex blowup of $\mathbb{C}P^n$.

Proposition 4.15 *Let B be a blowup of $\mathbb{C}P^n$ along a disjoint union $Q = \coprod Q_i$ of complex submanifolds, each of complex codimension ≥ 2 . Then every Hamiltonian bundle B is c-split.*

Proof: The above argument applies almost verbatim in the case when Q is a finite set of points. The top stratum of $\overline{\mathcal{M}}_{0,2}(B, j, [\mathbb{C}P^1])$ still consists of lines marked by two distinct points, and again all elements of this unperturbed moduli space are regular.

In the general case, there is a blow-down map $\psi : B \rightarrow \mathbb{C}P^n$ which is bijective over $\mathbb{C}P^n - Q$, and we can choose j on B so that the exceptional divisors $\psi^{-1}(Q)$ are j -holomorphic, and so that j is pulled back from the usual structure on $\mathbb{C}P^n$ outside a small neighborhood of $\psi^{-1}(Q)$. Let L_0 be a complex line in $\mathbb{C}P^n - Q$. Then its pullback to B is still j -holomorphic. Hence, although the unperturbed moduli space $\overline{\mathcal{M}}_{0,2}(B, j, [\mathbb{C}P^1])$ may contain nonregular and hence “bad” elements, its top stratum does contain an open set U_{L_0} consisting of marked lines near L_0 that are regular. Moreover, if we fix two points x_a, x_b on L_0 , every element of $\overline{\mathcal{M}}_{0,2}(B, j, [\mathbb{C}P^1])$ whose marked points map sufficiently close to x_a, x_b actually lies in this open set U_{L_0} . We can then regularize $\overline{\mathcal{M}}_{0,2}(B, j, [\mathbb{C}P^1])$ to a virtual moduli cycle that contains the open set U_{L_0} as part of its top stratum. Moreover, because the construction of the regularization is local with respect to $\overline{\mathcal{M}}_{0,2}(B, j, [\mathbb{C}P^1])$, this regularization $\overline{\mathcal{M}}'_{0,2}(B, j, [\mathbb{C}P^1])$ will still have the property that each of its elements whose marked points map sufficiently close to x_a, x_b actually lies in this open set U_{L_0} .

We can now carry out the previous argument, choosing J on P to be fibered, and constructing ν to be compatible with the fibration on that part of $\overline{\mathcal{M}}_{0,2}(P, J, \sigma)$ that projects to U_{L_0} . Further details will be left to the reader. \square

Corollary 4.16 *Let $X = \#k\mathbb{C}P^2 \# \ell \overline{\mathbb{C}P}^2$ be the connected sum of k copies of $\mathbb{C}P^2$ with ℓ copies of $\overline{\mathbb{C}P}^2$. If one of k, ℓ is ≤ 1 then every bundle over X is c-split.*

Proof: By reversing the orientation of X we can suppose that $k \leq 1$. The case $k = 1$ is covered in the previous proposition. When $k = 0$, pull the bundle back over the blowup of X at one point and then use the Surjection Lemma 4.1(ii). \square

The previous proof can easily be generalized to the case of a symplectic base B that has a spherical 2-class A with Gromov-Witten invariant of the form $n_B(pt, pt, c_1, \dots, c_k; A)$ absolutely equal to 1. (By this we mean that for some generic j on B the relevant moduli space contains exactly one element, which moreover parametrizes an embedded curve in B .)

Here c_1, \dots, c_k are arbitrary homology classes of B and we assume that $k \geq 0$. Again the idea is to construct the regularizations $\overline{\mathcal{M}}_{0,2+k}^\nu(P, J, \sigma)$ and $\overline{\mathcal{M}}_{0,2+k}^\nu(B, j, A)$ so that there is a projection from one to the other which is a fibration at least near the element of $\overline{\mathcal{M}}_{0,2+k}^\nu(B, j, A)$ that contributes to $n_B(pt, pt, c_1, \dots, c_k; A)$. Thus B could be the blowup of $\mathbb{C}P^n$ along a symplectic submanifold Q that is disjoint from a complex line. One could also take similar blowups of products of projective spaces, or, more generally, of iterated fibrations of projective spaces. For example, if $B = \mathbb{C}P^m \times \mathbb{C}P^n$ with the standard complex structure then there is one complex line in the diagonal class $[\mathbb{C}P^1] + [\mathbb{C}P^1]$ passing through any two points and a cycle $H_1 \times H_2$, where H_i is the hyperplane class, and one could blow up along any symplectic submanifold that did not meet one such line.

It is also very likely that this argument can be extended to apply when all we know about B is that some Gromov-Witten invariant $n_B(pt, pt, c_1, \dots, c_k; A)$ is nonzero, for example, if B is a blowup of $\mathbb{C}P^n$ along arbitrary symplectic submanifolds. There are two new problems here: (a) we must control the construction of $\overline{\mathcal{M}}_{0,2+k}^\nu(P, J, \sigma)$ in a neighborhood of all the curves that contribute to $n_B(pt, pt, c_1, \dots, c_k; A)$ and (b) we must make sure that the orientations are compatible so that curves in P projecting over different and noncancelling curves in B do not cancel each other in the global count of the Gromov-Witten invariant in P . Note that the bundles given by restricting P to the different curves counted in $n_B(pt, pt, c_1, \dots, c_k; A)$ are diffeomorphic, since, this being a homotopy theoretic question, we can always replace X by the simply connected space $X/(X_1)$ in which these curves are homotopic: see Corollary 2.6. Thus what is needed for this generalization to hold is to develop further the theory of fibered GW-invariants that was begun in [21]. See [10].

4.3 Homotopy-theoretic reasons for c-splitting

In this section we discuss c-splitting in a homotopy-theoretic context. Recall that a c-Hamiltonian bundle is a smooth bundle $P \rightarrow B$ together with a class $a \in H^2(P)$ whose restriction a_M to the fiber M is c-symplectic, i.e. $(a_M)^n \neq 0$ where $2n = \dim(M)$. Further a closed manifold M is said to satisfy the hard Lefschetz condition with respect to the class $a_M \in H^2(M, \mathbb{R})$ if the maps

$$\cup(a_M)^k : H^{n-k}(M, \mathbb{R}) \rightarrow H^{n+k}(M, \mathbb{R}), \quad 1 \leq k \leq n,$$

are isomorphisms. In this case, elements in $H^{n-k}(M)$ that vanish when cupped with $(a_M)^{k+1}$ are called primitive, and the cohomology of M has an additive basis consisting of elements of the form $b \cup (a_M)^\ell$ where b is primitive and $\ell \geq 0$. (These manifolds are sometimes called ‘‘cohomologically Kähler.’’)

Lemma 4.17 (Blanchard [4]) *Let $M \rightarrow P \rightarrow B$ be a c-Hamiltonian bundle such that $\pi_1(B)$ acts trivially on $H^*(M, \mathbb{R})$. If in addition M satisfies the hard Lefschetz condition with respect to the c-symplectic class a_M , then the bundle c-splits.*

Proof: The proof is by contradiction. Consider the Leray spectral sequence in cohomology and suppose that d_p is the first non zero differential. Then, $p \geq 2$ and the E_p term in the spectral sequence is isomorphic to the E_2 term and so can be identified with the tensor product $H^*(B) \otimes H^*(M)$. Because of the product structure on the spectral sequence, one of the differentials $d_p^{0,i}$ must be nonzero. So there is $b \in E_p^{0,i} \cong H^i(M)$ such that $d_p^{0,i}(b) \neq 0$. We may assume that b is primitive (since these elements together with a_M generate $H^*(M)$.) Then $b \cup a_M^{n-i} \neq 0$ but $b \cup a_M^{n-i+1} = 0$.

We can write $d_p(b) = \sum_j e_j \otimes f_j$ where $e_j \in H^*(B)$ and $f_j \in H^\ell(M)$ where $\ell < i$. Hence $f_j \cup a_M^{n-i+1} \neq 0$ for all j by the Lefschetz property. Moreover, because the E_p term is a tensor product

$$(d_p(b)) \cup a_M^{n-i+1} = \sum_j e_j \otimes (f_j \cup a_M^{n-i+1}) \neq 0.$$

But this is impossible since this element is the image via d_p of the trivial element $b \cup a_M^{n-i+1}$. \square

Here is a related argument due to Kedra.⁸

Lemma 4.18 *Every Hamiltonian bundle with 4-dimensional fiber c-splits.*

Proof: Consider the spectral sequence as above. We know as in the proof of Lemma 4.12 that $d_2 = 0$ and $d_3 = 0$. Consider d_4 . We just have to check that $d_4^{0,3} = 0$ since $d_4^{0,i} = 0$ for $i = 1, 2$ for dimensional reasons, and $= 0$ for $i = 4$ since the top class survives.

Suppose $d_4(b) \neq 0$ for some $b \in H^3(M)$. Let $c \in H^1(M)$ be such that $b \cup c \neq 0$. Then $d_4(c) = 0$ and $d_4(b \cup c) = d_4(b) \cup c \neq 0$ since $d_4(b) \in H^4(B) \otimes H^0(M)$. But we need $d_4(b \cup c) = 0$ since the top class survives. So $d_4 = 0$ and then $d_k = 0$, $k > 4$ for reasons of dimension. \square

Here is an example of a c-Hamiltonian bundle over S^2 that is not c-split. This shows that c-splitting is a geometric rather than a topological (or homotopy-theoretic) property.

Proof of Lemma 1.15

The idea is very simple. First observe that if S^1 acts on manifolds X, Y with fixed points p_X, p_Y then we can extend the S^1 action to the connected sum $X \# Y^{opp}$ at p_X, p_Y whenever the S^1 actions on the tangent spaces at p_X and p_Y are the same. (Here Y^{opp} denotes Y with the opposite orientation.) Now let S^1 act on $X = S^2 \times S^2 \times S^2$ by the diagonal action in the first two spheres (and trivially on the third) and let the S^1 action on Y be the example constructed in [16] of a nonHamiltonian S^1 action that has fixed points. The fixed points in Y form a disjoint union of 2-tori and the S^1 action in the normal directions has index ± 1 . In other words, there is a fixed point p_Y in Y at which we can identify $T_{p_Y} Y$ with $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, where $\theta \in S^1$ acts by multiplication by $e^{i\theta}$ in the first factor, by multiplication by $e^{-i\theta}$ in the second and trivially in the third. Since there is a fixed point on X with the same local structure, the connected sum $Z = X \# Y^{opp}$ does support an S^1 -action. Moreover Z is a

⁸Private communication

c-symplectic manifold. There are many possible choices of c-symplectic class: under the obvious identification of $H^2(Z)$ with $H^2(X) + H^2(Y)$ we will take the c-symplectic class on Z to be given by the class of the symplectic form on X .

Let $P_X \rightarrow S^2$, $P_Y \rightarrow S^2$ and $P_Z \rightarrow S^2$ be the corresponding bundles. Then P_Z can be thought of as the connect sum of P_X with P_Y along the sections corresponding to the fixed points. By analyzing the corresponding Mayer–Vietoris sequence, it is easy to check that the c-symplectic class on Z extends to P_Z . Further, the fact that the symplectic class in Y does not extend to P_Y implies that it does not extend to P_Z either. Hence $P_Z \rightarrow S^2$ is not c-split. \square

5 Action of the homology of $\text{Ham}(M)$ on $H_*(M)$

The action $\text{Ham}(M) \times M \rightarrow M$ gives rise to maps

$$H_k(\text{Ham}(M)) \times H_*(M) \rightarrow H_{*+k}(M) : (\phi, Z) \mapsto \text{tr}_\phi(Z).$$

Theorem 1.16 states:

Proposition 5.1 *These maps are trivial when $k \geq 1$.*

Proof: To see this, let us first consider the action of a spherical element

$$\phi : S^n \rightarrow \text{Ham}(M).$$

It is not hard to check that the homomorphisms

$$\text{tr}_\phi : H_k(M) \rightarrow H_{k+n}(M)$$

are precisely the connecting homomorphisms in the Wang sequence of the bundle $P_\phi \rightarrow S^{k+1}$ with clutching function ϕ : i.e. there is an exact sequence

$$\dots H_k(M) \xrightarrow{\text{tr}_\phi} H_{k+n}(M) \rightarrow H_{k+n}(P) \xrightarrow{\cap^{[M]}} H_{k-1}(M) \rightarrow \dots$$

Thus the fact that $P_\phi \rightarrow S^{k+1}$ is c-split immediately implies that the tr_ϕ are trivial.

Next recall that in a H -space the rational cohomology ring is generated by elements dual to the rational homotopy. It follows that there is a basis for $H_*(\text{Ham}(M))$ that is represented by cycles of the form

$$\phi_1 \times \dots \times \phi_k : S_1 \times \dots \times S_k \rightarrow \text{Ham}(M),$$

where the S_j s are spheres and one defines the product of maps by using the product structure in $\text{Ham}(M)$. Therefore it suffices to show that these product elements act trivially. However, if $a \in H_*(M)$ is represented by the cycle α , then $\text{tr}_{S_k}(\alpha)$ is null-homologous, and so equals the boundary $\partial\beta$ of some chain β . Therefore:

$$\partial(\text{tr}_{S_1 \times \dots \times S_{k-1}}(\beta)) = \pm \text{tr}_{S_1 \times \dots \times S_{k-1}}(\partial\beta) = \pm \text{tr}_{S_1 \times \dots \times S_{k-1}}(\text{tr}_{S_k}(\alpha)) = \text{tr}_{S_1 \times \dots \times S_k}(\alpha).$$

Hence $\text{tr}_{S_1 \times \dots \times S_k}(a) = 0$. This completes the proof. \square

Proposition 5.2 *Let $P \rightarrow B$ be a trivial symplectic bundle. Then any Hamiltonian automorphism $\Phi \in \text{Ham}(P, \pi)$ acts as the identity map on $H_*(P)$.*

Proof: An element $\Phi \in \text{Ham}(P, \pi)$ is a map of the form

$$\Phi : B \times M \rightarrow B \times M : (b, x) \mapsto (b, \Phi_b(x))$$

where $\Phi_b \in \text{Ham}(M)$ for all $b \in B$. Let us denote the induced map $B \times M \rightarrow M : (b, x) \mapsto \Phi_b(x)$ by α_Φ . The previous proposition implies that if B is a closed manifold of dimension > 0 , or, more generally, if it carries a fundamental cycle $[B]$ of degree > 0 ,

$$(\alpha_\Phi)_*([B] \otimes m) = \text{tr}_{[B]}(m) = 0, \quad \text{for all } m \in H_*(M).$$

We can also think of $\Phi : B \times M \rightarrow B \times M$ as the composite

$$B \times M \xrightarrow{\text{diag}_B \times \text{id}_M} B \times B \times M \xrightarrow{\text{id}_B \times \alpha_\Phi} B \times M.$$

The diagonal class in $B \times B$ can be written as $[B] \otimes [pt] + \sum_{i \in I} b_i \otimes b'_i$ where $b_i, b'_i \in H_*(B)$ with $\dim(b'_i) > 0$.⁹ Hence

$$\Phi_*([B] \otimes m) = [B] \otimes m + \sum_{i \in I} b_i \otimes \text{tr}_{b'_i}(m) = [B] \otimes m,$$

where the last equality comes from Proposition 5.1. More generally, given any class $b \in H_*(B)$, represent it by the image of the fundamental class $[X]$ of some polyhedron under a suitable map $X \rightarrow B$ and consider the pullback bundle $P_X \rightarrow X$. Since the class $\Phi_*([X] \otimes m)$ is represented by a cycle in $X \times M$ for any $m \in H_*(M)$, we can work out what it is by looking at the pullback of Φ to $X \times M$. The argument above then applies to show that $\Phi_*([X] \otimes m) = [X] \otimes m$ whenever b has degree > 0 . Thus $\Phi_* = \text{id}$ on all cycles in $H_{>0}(B) \otimes H_*(M)$. However, it clearly acts as the identity on $H_0(B) \otimes H_*(M)$ since the restriction of Φ to any fiber is isotopic to the identity. \square

A natural conjecture is that the analog of Proposition 5.2 holds for all Hamiltonian bundles. We now show that there is a close relation between this question and the problem of c -splitting of bundles. Given an automorphism Φ of a symplectic bundle $M \rightarrow P \rightarrow B$ we define $P_\Phi = (P \times [0, 1])/\Phi$ to be the corresponding bundle over $B \times S^1$. If the original bundle and the automorphism are Hamiltonian, so is $P_\Phi \rightarrow B \times S^1$, though the associated bundle $P_\Phi \rightarrow B \times S^1 \rightarrow S^1$ over S^1 will not be, except in the trivial case when Φ is in the identity component of $\text{Ham}(P, \pi)$.

Proposition 5.3 *Assume that a given Hamiltonian bundle $M \rightarrow P \rightarrow B$ c -splits. Then a Hamiltonian automorphism $\Phi \in \text{Ham}(P, \pi)$ acts trivially (i.e. as the identity) on $H_*(P)$ if and only if the corresponding Hamiltonian bundle $P_\Phi \rightarrow B \times S^1$ c -splits.*

Proof: Clearly, the fibration $P \rightarrow B$ c -splits if and only if every basis of the \mathbb{Q} -vector space $H^*(M)$ can be extended to a set of classes in $H^*(P)$ that form a basis for a complement

⁹This holds because the projection onto the first factor takes the diagonal class onto the fundamental class of B . When the base is a closed manifold, the diagonal is represented by $\sum_{i \in I} (-1)^{\dim b_i} b_i \otimes b'_i$ where $\{b_i\}$ is a basis for $H_*(B)$ and $\{b'_i\}$ is its Poincaré dual.

to the kernel of the restriction map. We will call such a set of classes a *Leray–Hirsch basis*. It corresponds to a choice of splitting isomorphism $H^*(P) \cong H^*(B) \otimes H^*(M)$. Now, the only obstruction to extending a Leray–Hirsch basis from P to P' is the nontriviality of the action of Φ on $H^*(P)$. This shows the “only if” part.

Conversely, suppose that P_Φ c-splits and let $e_j, j \in J$, be a Leray–Hirsch basis for $H^*(P_\Phi)$. Then $H^*(P_\Phi)$ has a basis of the form $e_j \cup \pi^*(b_i), e_j \cup \pi^*(b_i \times [dt])$ where b_i runs through a basis for $H^*(B)$ and $[dt]$ generates $H^1(S^1)$. Identify P with $P \times \{0\}$ in P_Φ and consider some cycle $Z \in H_*(P)$. Since the cycles $\Phi_*(Z)$ and Z are homologous in P_Φ , the classes $e_j \cup \pi^*(b_i)$ have equal values on $\Phi_*(Z)$ and Z . But the restriction of these classes to P forms a basis for $H^*(P)$. It follows that $[\Phi_*(Z)] = [Z]$ in $H_*(P)$. \square

Proposition 5.4 *Let $P \rightarrow B$ be a Hamiltonian bundle. Then the group $\text{Ham}(P, \pi)$ acts trivially on $H_*(P)$ if the base*

- (i) *has dimension ≤ 2 , or*
- (ii) *is a product of spheres and projective spaces with no more than two S^1 factors, or*
- (iii) *is simply connected and has the property that all Hamiltonian bundles over B are c-split.*

Proof: In all cases, the hypotheses imply that $P \rightarrow B$ c-splits. Therefore the previous proposition applies and (i), (ii) follow immediately from Theorem 1.14. To prove (iii), suppose that B is a simply connected compact CW complex over which every Hamiltonian fiber bundle c-splits. Let $M \hookrightarrow P \rightarrow B \times S^1$ be any Hamiltonian bundle – in particular one of the form $P_\Phi \rightarrow B \times S^1$. Consider its pull-back P' by the projection map $B \times T^2 \rightarrow B \times S^1$. This is still a Hamiltonian bundle. To show that P c-splits, it is sufficient, by Lemma 4.1 (ii), to show that P' c-splits. Because $B \times T^2$ may be considered as a smooth compact Hamiltonian fibration $T^2 \hookrightarrow (B \times T^2) \rightarrow B$ with simply connected base, Lemma 4.3 applies. Thus P' c-splits since any Hamiltonian bundle over B or over T^2 c-splits. \square

Finally, we prove the statements made in §1.2 about the automorphism groups of Hamiltonian structures.

Proof of Proposition 1.8

We have to show that the following statements are equivalent for any $\Phi \in \text{Symp}_0(P, \pi)$:

- (i) Φ is isotopic to an element of $\text{Ham}(P, \pi)$;
- (ii) $\Phi^*(\{a\}) = \{a\}$ for some Hamiltonian structure $\{a\}$ on P ;
- (iii) $\Phi^*(\{a\}) = \{a\}$ for all Hamiltonian structures $\{a\}$ on P .

Recall from Lemma 2.7 that the relative homotopy groups $\pi_i(\text{Symp}(M), \text{Ham}(M))$ all vanish for $i > 1$. Using this together with the fact that $a \in H^2(P)$, we can reduce to the case when B is a closed oriented surface. The statement (i) implies (iii) then follows immediately from Proposition 5.4. Of course, (iii) implies (ii) so it remains to show that (ii) implies (i).

Let us prove this first in the case where $P \rightarrow B$ is trivial, so that Φ is a map $B \rightarrow \text{Symp}_0(M, \omega)$. Suppose that $\Phi^*(a) = a$ for some extension class a . By isotoping Φ if

necessary, we can suppose that Φ takes the base point b_0 of B to the identity map. Then, for each loop γ in B and any trivialization T_γ ,

$$\begin{aligned} 0 &= f(T_\gamma, \Phi^*(a)) - f(T_\gamma, a) = f(T_\gamma \circ \Phi_\gamma, a) - f(T_\gamma, a) \\ &= f(T_\gamma, a) \circ \text{tr}_{\Phi_\gamma} = \omega \circ \text{tr}_{\Phi_\gamma} \\ &= \text{Flux}(\Phi_\gamma) \end{aligned}$$

where Φ_γ is the loop given by restricting Φ to γ . Thus the composite

$$\pi_1(B) \xrightarrow{\Phi_*} \pi_1(\text{Symp}_0(M)) \xrightarrow{\text{Flux}} H^1(M, \mathbb{R})$$

must vanish. Thus the restriction of $\Phi : B \rightarrow \text{Symp}_0(M)$ to the 1-skeleton of B homotops into $\text{Ham}(M)$. Since the relative homotopy groups $\pi_i(\text{Symp}(M), \text{Ham}(M))$ all vanish for $i > 1$, this implies that ϕ homotops to a map in $\text{Ham}(M)$, as required.

Therefore, it remains to show that we can reduce the proof that (ii) implies (i) to the case when $P \rightarrow B$ is trivial. To this end, isotop Φ so that it is the identity map on all fibers M_b over some disc $D \subset B$. Since $P \rightarrow B$ is trivial over $X = B - D$, we can decompose $P \rightarrow B$ into the fiber connected sum of a trivial bundle P_B over B (where B is thought of as the space obtained from X by identifying its boundary to a point) and a nontrivial bundle Q over $S^2 = D/\partial D$. Further, this decomposition is compatible with Φ , which can be thought of as the fiber sum of some automorphism Φ_B of P_B together with the trivial automorphism of Q . Clearly, this reduces the proof that (ii) implies (i) to the case $\Phi_B : P_B \rightarrow P_B$ on trivial bundles, if we note that when Φ_B is the identity over some disc $D \subset B$, the isotopy between Φ and an element in $\text{Ham}(P_B)$ can be constructed so that it remains equal to the identity over D . \square

6 The cohomology of general symplectic bundles

In this section we discuss some consequences for general symplectic bundles of our results on Hamiltonian bundles. First, we prove Proposition 1.17 that states that the boundary map ∂ in the rational homology Wang sequence of a symplectic bundle over S^2 has $\partial \circ \partial = 0$.

Proof of Proposition 1.17

The map $\partial \circ \partial : H_k(M) \rightarrow H_{k+2}(M)$ is given by $a \mapsto \Psi_*([T^2] \otimes a)$ where

$$\Psi : T^2 \times M \rightarrow M : (s, t, x) \mapsto \phi_s \phi_t(x).$$

Let $b(s, t) = \phi_{s+t}^{-1} \phi_s \phi_t$. The map $(s, t) \mapsto b(s, t)$ factors through

$$f : T^2 \rightarrow S^2 = T^2 / \{s = 0\} \cup \{t = 0\}.$$

Let $Z \rightarrow M$ represent a k -cycle. We have a map

$$T^2 \times Z \xrightarrow{A_1} S^1 \times S^2 \times Z \xrightarrow{A_2} M$$

given by

$$(s, t, z) \mapsto (\phi_{s+t}, f(s, t), z) \mapsto \phi_{s+t} b(s, t) z = \phi_s \phi_t(z),$$

and want to calculate

$$\int_{T^2 \times Z} A_1^* A_2^*(\alpha) = \int_{(A_1)_*[T^2 \times Z]} A_2^*(\alpha)$$

for some $k+2$ -form α on M . But $(A_1)_*[T^2 \times Z] \in H_2(S^2) \otimes H_k(Z)$. (There is no component in $H_3(S^1 \times S^2) \otimes H_{k-1}(Z)$ since $A_1 = id$ on the Z factor.) Now observe that $A_2^*(\alpha)$ vanishes on $H_2(S^2) \otimes H_k(Z)$ by Theorem 1.16. \square

The previous lemma is trivially true for any smooth (not necessarily symplectic) bundle over S^2 that extends to $\mathbb{C}P^2$. For the differential d_2 in the Leray cohomology spectral sequence can be written as

$$d_2(a) = \partial(a) \cup u \in E_2^{2,q-1},$$

where $a \in H^q(M) \equiv E_2^{0,q}$ and u generates $H^2(\mathbb{C}P^2) \equiv E_2^{2,0}$. Hence

$$0 = d_2(d_2(a)) = d_2(\partial(a) \cup u) = d_2(\partial(a)) \cup u = \partial(\partial(a)) \otimes u^2.$$

Lemma 6.1 *If $\pi : P \rightarrow B$ is any symplectic bundle over a simply connected base, then $d_3 \equiv 0$.*

Proof: As in the proof of Lemma 2.3 we can reduce to the case when B is a wedge of S^2 s and S^3 s. The differential d_3 is then given by restricting to the bundle over $\vee S^3$. Since this is Hamiltonian, $d_3 \equiv 0$ by Theorem 1.14. \square

The next lemma describes the Wang differential $\partial = \partial_\phi$ in the case of a symplectic loop ϕ with nontrivial image in $H_1(M)$.

Lemma 6.2 *Suppose that ϕ is a symplectic loop such that $[\phi_t(x)] \neq 0$ in $H_1(M)$. Then $\ker \partial = \text{im } \partial$, where $\partial = \partial_\phi : H^k(M) \rightarrow H^{k-1}(M)$ is the corresponding Wang differential.*

Proof: Let $\alpha \in H^1(M)$ be such that $\alpha([\phi_t(x)]) = 1$. So $\partial \alpha = 1$. Then, for every $\beta \in \ker \partial$, $\partial(\alpha \cup \beta) = \beta$. This means that $\ker \partial \subset \text{im } \partial$ and so $\ker \partial = \text{im } \partial$ (using the fact that $\partial \circ \partial = 0$). Moreover the map

$$\alpha \cup : H^k(M) \rightarrow H^{k+1}(M)$$

is injective on $\ker \partial$ and $H^*(M)$ decomposes as the direct sum $\ker \partial \oplus (\alpha \cup \ker \partial)$. \square

Proof of Corollary 1.18

This claims that for a symplectic loop ϕ , $\ker \partial = \text{im } \partial$ if and only if $[\phi_t(x)] \neq 0$ in $H_1(M)$. The above lemma proves the ‘‘if’’ statement. But the ‘‘only if’’ statement is easy. Since $1 \in H^0(M)$ is in $\ker \partial$ it must equal $\partial(\alpha)$ for some $\alpha \in H^1(M)$. This means that $\alpha([\phi_t(x)]) \neq 0$ so that $[\phi_t(x)] \neq 0$. \square

Remark 6.3 The only place that the symplectic condition enters in the proof of Lemma 6.2 is in the claim that $\partial \circ \partial = 0$. Since this is always true when the loop comes from a circle action, this lemma holds for all, not necessarily symplectic, circle actions. In this case, we can interpret the result topologically. For the hypothesis $[\phi_t(x)] \neq 0$ in $H_1(X)$ implies that the action has no fixed points, so that the quotient M/S^1 is an orbifold with cohomology isomorphic to $\ker \partial$. Thus, the argument shows that M has the same cohomology as the product $(M/S^1) \times S^1$.

A More on Hamiltonian structures

Another approach to characterizing a Hamiltonian structure is to define it in terms of a structure on the fiber that is preserved by elements of the Hamiltonian group. This section developed via discussions with Polterovich.

Definition A.1 A *marked symplectic manifold* $(M, \omega, [L])$ is a pair consisting of a closed symplectic manifold (M, ω) together with a marking $[L]$. Here L is a collection $\{\ell_1, \dots, \ell_k\}$ of loops $\ell_i : S^1 \rightarrow M$ in M that projects to a minimal generating set $\mathcal{G}_L = \{[\ell_1], \dots, [\ell_k]\}$ for $H_1(M, \mathbb{Z})/\text{torsion}$. A marking $[L]$ is an equivalence class of generating loops L , where $L \sim L'$ if for each i there is a singular integral 2-chain c_i whose boundary loops modulo torsion is $\ell'_i - \ell_i$ such that $\int_{c_i} \omega = 0$.

The symplectomorphism group acts on the space \mathcal{L} of markings. Moreover, it is easy to check that if a symplectomorphism ϕ fixes one marking $[L]$ it fixes them all. Hence the group

$$\text{LHam}(M, \omega) = \text{LHam}(M, \omega, [L]) = \{\phi \in \text{Symp}(M, \omega) : \phi_*[L] = [L]\}$$

independent of the choice of $[L]$. Its identity component is $\text{Ham}(M, \omega)$.

There is a forgetful map $[L] \rightarrow \mathcal{G}_L$ from the space \mathcal{L} of markings to the space of minimal generating sets for the group $H_1(M, \mathbb{Z})$, and it is not hard to check that its fiber is $(\mathbb{R}/\mathcal{P})^k$, where \mathcal{P} is the image of the period homomorphism

$$I_{[\omega]} : H_2(M, \mathbb{Z}) \rightarrow \mathbb{R}.$$

If \mathcal{P} is not discrete, there is no nice topology one can put on \mathcal{L} . However, it has a pseudotopology, i.e. one can specify which maps of finite polyhedra X into \mathcal{L} are continuous, namely: $f : X \rightarrow \mathcal{L}$ is continuous if and only if every $x \in X$ has a neighborhood U_x such that $f : U_x \rightarrow \mathcal{L}$ lifts to a continuous map into the space of generating loops L .

Here is an alternative formulation of Theorem 1.6 in the language of markings.

Proposition A.2 Fix a marking $[L]$ on (M, ω) . A Hamiltonian structure on a symplectic bundle $\pi : P \rightarrow B$ is an isotopy class of symplectomorphisms $(M, \omega, [L]) \rightarrow (M_{b_0}, \omega_{b_0}, [L_0])$ together with a continuous choice of marking $[L_b]$ on each fiber (M_b, ω_b) that is trivial over the 1-skeleton B_1 of B in the sense that there is a symplectic trivialization

$$\Phi : \pi^{-1}(B_1) \rightarrow B_1 \times (M, \omega, [L])$$

that respects the markings on each fiber.

Here is another way of thinking of a Hamiltonian structure due to Polterovich.¹⁰ He observed that there is an exact sequence

$$0 \rightarrow \mathbb{R}/\mathcal{P} \rightarrow SH_1(M, \omega) \rightarrow H_1(M, \mathbb{Z}) \rightarrow 0,$$

where $SH_1(M, \omega)$ is the “strange homology group” formed by quotienting the space of integral 1-cycles by the image under d of the space of integral 2-chains with zero symplectic area. The group $\text{Symp}(M, \omega)$ acts on $SH_1(M, \omega)$. Moreover, if $\phi \in \text{Symp}_0(M)$ and

¹⁰Private communication

$\tilde{a} \in SH_1(M)$ projects to $a \in H_1(M)$, then $\phi_*(\tilde{a}) - \tilde{a} \in \mathbb{R}/\mathcal{P}$ can be thought of as the value of the class $\text{Flux}(\phi) \in H^1(M, \mathbb{R})/\Gamma_\omega$ on a . It is easy to see that $\text{LHam}(M, \omega)$ is the subgroup of $\text{Symp}(M, \omega)$ that acts trivially on $SH_1(M, \mathbb{Z})$. Further, a marking on (M, ω) is a pair consisting of a splitting of the above sequence together with a generating set \mathcal{G}_L for $H_1(M, \mathbb{Z})/\text{torsion}$.

Given any symplectic bundle $P \rightarrow B$ there is an associated bundle of abelian groups with fiber $SH_1(M, \omega)$. A Hamiltonian structure on $P \rightarrow B$ is a flat connection on this bundle that is trivial over the 1-skeleton B_1 , under an appropriate equivalence relation.

These ideas can obviously be generalized to bundles that are not trivial over the 1-skeleton. Equivalently, one can consider bundles with disconnected structural group. This group could be the whole of $\text{LHam}(M, \omega)$. One could also restrict to elements acting trivially on $H^*(M)$ and/or to those that act trivially on the groups

$$SH_{2k-1}(M, \omega) = \frac{\text{integral } (2k-1)\text{-cycles}}{d(2k\text{-chains in the kernel of } \omega^k)}.$$

These generalizations of $SH_1(M, \omega)$ are closely connected to Reznikov's Futaki type characters: see [24] §4. It is not yet clear what is the most natural disconnected extension of $\text{Ham}(M, \omega)$.

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B Erratum

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The statements in this paper that characterize Hamiltonian bundles $(M, \omega) \rightarrow P \rightarrow B$ are not correct when $H_1(B; \mathbb{Z})$ has torsion. The affected results are Theorem 1.1, Proposition 1.2 and Lemma 1.4. The problem is that the proof of Lemma 2.5 works only if $H_1(B; \mathbb{Z})$ is a free group. Hence the arguments in this lemma prove the following weaker version of Theorem 1.1.

Proposition B.1 *Let $\pi : P \rightarrow B$ be a smooth symplectic fiber bundle with fiber (M, ω) that is symplectically trivializable over the 1-skeleton. Then the following conditions are equivalent:*

- (i) *there is a cohomology class $a \in H^2(P, \mathbb{R})$ that restricts to $[\omega]$ on the fiber M ;*
- (ii) *the pullback of $P \rightarrow B$ over a suitable finite cover $\tilde{B} \rightarrow B$ has a Hamiltonian structure.*

The following example (due to Dietmar Salamon) shows that it can be necessary to pass to the finite cover in (ii). Consider the quotient

$$P := \frac{S^2 \times \mathbb{T}^2}{\mathbb{Z}_2}$$

where we think of $S^2 \subset \mathbb{R}^3$ as the unit sphere and of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as the standard torus; the nontrivial element of \mathbb{Z}_2 acts by the involution

$$(x, y) \mapsto (-x, y + (1/2, 0)), \quad x \in S^2, y \in \mathbb{T}^2.$$

The closed 2-form $\tau = dy_1 \wedge dy_2 \in \Omega^2(S^2 \times \mathbb{T}^2)$ descends to a closed connection 2-form on P ; its holonomy around each contractible loop in $\mathbb{R}P^2$ is the identity and around each noncontractible loop is the symplectomorphism $(y_1, y_2) \mapsto (y_1 + 1/2, y_2)$. Thus the bundle $\pi : P \rightarrow \mathbb{R}P^2$ satisfies the hypothesis of the above proposition as well as conditions (i) and (ii). But it does not have a Hamiltonian structure because the classifying map $\mathbb{R}P^2 \rightarrow BSymp_0(\mathbb{T}^2)$ is not null homotopic, while $\text{Ham}(\mathbb{T}^2)$ is contractible.

The mistake in the proof of Lemma 2.5 was the tacit assumption that the flux class $[f(T_\gamma, a)] \in H^1(M, \mathbb{R})/\Gamma_\omega$ vanishes when $\gamma \in \pi_1(B)$ has finite order in $H_1(B; \mathbb{Z})$. If this condition holds, there is a lifted homomorphism $\tilde{f}_a : \pi_1(B) \rightarrow H^1(M, \mathbb{R})$ such that $pr \circ \tilde{f}_a([\gamma]) = [f(T_\gamma, a)]$, where pr denotes the projection, and the proof of Lemma 2.5 goes through.

We claim that the existence of the lift \tilde{f}_a does not depend on the choice of extension a . To see this, choose a symplectic trivialization T over the 1-skeleton B_1 and a closed

connection form τ in class a . Then τ -parallel translation around a loop γ in B gives rise to a path $g_t, t \in [0, 1]$, in $\text{Symp}(M)$ that starts at the identity, and by definition

$$f(T_\gamma, a) := \text{Flux}(\{g_t\}) \in H^1(M; \mathbb{R}).$$

Note that the image $[f(T_\gamma, a)]$ of $f(T_\gamma, a)$ in $H^1(F; \mathbb{R})/\Gamma_\omega$ is independent of the choice of trivialization. The lift \tilde{f}_a exists if and only if $f(T_\gamma, a)$ belongs to Γ_ω for all loops γ with finite order in $H_1(B; \mathbb{Z})$. Given such a loop γ (which we can assume to be embedded) and a 1-cycle δ in M , denote by $C(\gamma, \delta)$ the 2-cycle in P that equals $\gamma \times \delta$ under the identification of $\pi^{-1}(\gamma)$ with $\gamma \times M$ given by T . Then

$$f(T_\gamma, a)([\delta]) = \int_{C(\gamma, \delta)} \tau.$$

But if γ has order k in $H_1(B; \mathbb{Z})$ the cycle $kC(\gamma, \delta)$ is homologous to a cycle in the fiber M . Hence the integral of τ over $C(\gamma, \delta)$ is determined by $[\omega]$. Thus, when γ is homologically torsion, the class $f(T_\gamma, a)$ does not depend on the choice of a . Moreover, if $f(T_\gamma, a)$ belongs to Γ_ω , one can change the trivialization T over γ to a trivialization T' such that $f(T'_\gamma, a) = 0$. If $f(T'_\gamma, a) = 0$ for all loops γ that represent a torsion class in $H_1(B; \mathbb{Z})$ we shall say that **the flux of T' vanishes on torsion loops**.

Here are corrected versions of Theorem 1.1 and Proposition 1.2.

Theorem B.2 *A symplectic bundle $\pi : P \rightarrow B$ is Hamiltonian if and only if the following conditions hold:*

- (i) *the restriction of π to the 1-skeleton B_1 of B has a symplectic trivialization whose flux vanishes on torsion loops, and*
- (ii) *there is a cohomology class $a \in H^2(P, \mathbb{R})$ that restricts to $[\omega]$ on the fiber M .*

Proposition B.3 *A symplectic bundle $\pi : P \rightarrow B$ is Hamiltonian if and only if the forms ω_b on the fibers have a closed extension τ such that the holonomy of the corresponding connection ∇_τ around any loop γ in B lies in the identity component $\text{Symp}_0(M)$ of $\text{Symp}(M)$ and moreover lies in $\text{Ham}(M)$ whenever γ has finite order in $H_1(M; \mathbb{Z})$.*

Lemma 1.4 is correct if one understands it to refer to the corrected version of Theorem 1.1. The only other argument that requires comment is the proof of Hamiltonian stability. Note first that Corollary 3.2 needs an extra hypothesis to ensure that the transition functions ϕ_{ij} of $P \rightarrow B$ preserve the cohomology class $[\omega']$ of the perturbed form. This hypothesis is satisfied in the Hamiltonian case, since the ϕ_{ij} may be assumed to be isotopic to the identity. The next problem is that the proof of Lemma 3.4 uses the incorrect version of Theorem 1.1. However, we can first reduce to the case when $\pi_1(B) = 0$ by using Corollary 2.6, and then the two versions of Theorem 1.1 coincide.