SYMPLECTOMORPHISM GROUPS AND QUANTUM COHOMOLOGY

DUSA MCDUFF

to my teacher Israel Moiseevich Gelfand on the occasion of his 90th birthday

March 2004, with minor revisions June 2004 and Sep 2006

Abstract. We discuss the question of what quantum methods (J-holomorphic curves and quantum homology) can tell us about the symplectomorphism group and its compact subgroups. After describing the rather complete information we now have about the case of the product of two 2-spheres, we describe some recent results of McDuff–Tolman concerning the symplectomorphism group of toric manifolds. This leads to an interpretation of the relations in the quantum cohomology ring of a symplectic toric manifold in terms of the Seidel elements of the generating circles of the torus action.

1. Introduction

1.1. **Overview.** The group $\mathrm{Symp}(M,\omega)$ of symplectomorphisms of a symplectic manifold (M,ω) is an interesting but largely unknown group. The manifold for which we have the most information is $S^2\times S^2$ with its family of symplectic forms $\omega^\lambda := \lambda\pi_1^*(\sigma)+\pi_2^*(\sigma)$, where $\lambda\ge 1$. We begin by discussing recent results due to Abreu, McDuff, and Anjos–Granja on the homotopy type of the corresponding family of groups

$$\mathcal{G}^\lambda := \mathrm{Symp}(S^2\times S^2,\omega^\lambda).$$

In all cases that have so far been calculated, the homotopy groups of $\mathcal{G}^\lambda$ are generated by its compact subgroups. These appear as the automorphism groups of the different toric structures on $S^2\times S^2$.

As a first step towards generalizing these results, one can look at the relation of the toric automorphism group $\mathrm{Aut}(M,T)$ of a toric manifold (M,T) to $\mathrm{Symp}(M,\omega)$. Recent work by McDuff–Tolman gives examples where the inclusion of $\mathrm{Aut}(M,T)$ does not induce an isomorphism on π_1. Nevertheless, our work suggests that the map $\pi_1(\mathrm{Aut}(M,T))\to\pi_1\,\mathrm{Symp}(M,\omega)$ is noninjective only in cases where the manifold M has very special structure. After discussing such questions, we explain a new way of

1991 *Mathematics Subject Classification.* 53C15.

Key words and phrases. symplectomorphism, Seidel representation, quantum cohomology, toric manifold.

The author is partly supported by the NSF grant DMS 0305939.

understanding the corrections needed to Batyrev's formula for the quantum cohomology ring of a nonFano toric manifold. These come from the Seidel elements of the generating circles of the torus action. In the NEF case they give a new perspective on Givental's change of variable formula that relates the I- and J-functions in his proof of the mirror conjecture.

This paper is rather narrowly focussed; more general information on symplectomorphism groups may be found in the survey articles [17, 18]. For background material on symplectic topology see McDuff–Salamon [19, 20].

1.2. **Preliminaries.** Consider a closed manifold M of dimension $2n$ and its symplectomorphism group $\mathrm{Symp}(M,\omega)$, consisting of all diffeomorphisms that preserve the symplectic form. The identity component $\mathrm{Symp}_0(M,\omega)$ contains a normal subgroup, the Hamiltonian group $\mathrm{Ham}(M,\omega)$, made up of the time-1 maps of the flows $\phi_t^H, t\ge 0$, generated by time dependent Hamiltonian functions $H: M\times[0,1]\to\mathbb{R}$. If $H^1(M;\mathbb{R})=0$ then $\mathrm{Ham}(M,\omega)=\mathrm{Symp}_0(M,\omega)$; in general it is a subgroup of codimension equal to $\dim H^1(M;\mathbb{R})$. In all cases, the group $\mathcal{G}:=\mathrm{Ham}(M,\omega)$ may be considered as a Fréchét Lie group whose Lie algebra consists of all normalized Hamiltonians:

$$\mathrm{Lie}\,\mathcal{G}=\Big\{F: M\to\mathbb{R}\,|\,\int_M F\omega^n=0\Big\}.$$

As Reznikov pointed out in [23] the formula

$$\Big\langle F,G\Big\rangle := \int_M FG\omega^n$$

defines a nondegenerate form on $\mathrm{Lie}\,\mathcal{G}$ that is invariant under the adjoint action of $\mathcal{G}$, and so is analogous to a Killing form. Thus although $\mathcal{G}:=\mathrm{Ham}(M,\omega)$ is an infinite dimensional group its Lie algebra behaves like the Lie algebra of a compact Lie group, and one might hope that this is reflected in the topological properties of $\mathcal{G}$. Investigating this is one of the ideas behind this talk.

We first observe that the above invariant form may be used to define an analog of Chern–Weil theory for Hamiltonian bundles (i.e. bundles with fiber M and structural group $\mathcal{G}=\mathrm{Ham}$). Guillemin–Lerman–Sternberg pointed out in [9] that given any such bundle $M\to P\to B$ the fiberwise symplectic form ω has a closed extension Ω. One can normalize the class $[\Omega]$ by requiring that

$$\pi_!([\Omega]^{n+1})=\int_M[\Omega]^{n+1}=0\in H^2(B),$$

where $\int_M$ denotes the integral over the fiber of $\pi: P\to B$. Such a form Ω defines an Ehresmann connection on the bundle $P\to B$ whose horizontal spaces are the Ω-orthogonal complements to the fiber. It turns out that the holonomy of this connection is Hamiltonian. Moreover, given vector fields $v,w\in T_bB\mathcal{G}$ with horizontal lifts $v^\sharp, w^\sharp$, the function $\Omega(v^\sharp,w^\sharp)(x)$ restricts on each fiber $M_b:=\pi^{-1}(b)$ to an element $F(v,w)\in\mathrm{Lie}(\mathcal{G})$ that represents the curvature $\widetilde{\Omega}(v,w)$ of this connection at (v,w). By making finite dimensional approximations, one can make sense of this construction on the universal Hamiltonian bundle

$$(M,\omega) \to (M_{\mathcal{G}},\Omega) \to B\mathcal{G}.$$

Any Ad-invariant polynomial $\mathcal{I}^k : \mathrm{Lie}(\mathcal{G})^{\otimes k} \to \mathbb{R}$ therefore gives rise to a characteristic class $c_k^{\mathcal{I}}$ in $H^*(B\mathcal{G})$, namely the class represented by the closed real-valued $2k$-form $\mathcal{I}^k \circ \widetilde{\Omega}^k$. Just as in the case of $U(n)$ we may define $\mathcal{I}^k$ by using the Killing form, namely

$$\mathcal{I}^k(F_1 \otimes \cdots \otimes F_k) := \int_M F_1 \cdots F_k \, \omega^n.$$

We claim that up to a constant $c_k^{\mathcal{I}}$ equals the class defined by the fiberwise integral

$$\mu_k := \int_M [\Omega]^{n+k} \in H^{2k}(B\mathcal{G}). \tag{1.1}$$

The classes $c_k^{\mathcal{I}}$ are variants of the ones defined by Reznikov [23], while the μ_k were considered by Januszkiewicz–Kędra in [10]. The following proof is taken from Kędra–McDuff [11].

Lemma 1.1. *This class $c_k^{\mathcal{I}}$ is a nonzero multiple of μ_k.*

Proof. Let $v_1,\dots,v_{2k}$ be vector fields on $B\mathcal{G}$ with horizontal lifts $v_1^\sharp,\dots,v_{2k}^\sharp$. Then, if the w_j are tangent to the fiber at $x \in M_{\mathcal{G}}$ we find

$$\begin{aligned}\Omega^{n+k}(w_1,\dots,w_{2n},v_1^\sharp,\dots,v_{2k}^\sharp)(x) \;&=\; \sum_\sigma \varepsilon(\sigma) \begin{pmatrix} n+k \\ n \end{pmatrix} \times \\ &\qquad F_{1,\sigma}(x)\cdots F_{k,\sigma}(x)\, \omega^n(w_1,\dots,w_{2n}),\end{aligned}$$

where, for each permutation σ of $\{1,\dots,2k\}$, $\varepsilon(\sigma)$ denotes its signature and

$$F_{j,\sigma}(x) := \Omega(v^\sharp_{\sigma(2j-1)}, v^\sharp_{\sigma(2j)})(x) = \widetilde{\Omega}(v_{\sigma(2j-1)}, v_{\sigma(2j)})(x).$$

Therefore $\big(\pi_!\Omega^{n+k}\big)(v_1,\dots,v_{2k}) = c\,\mathcal{I}^k \circ \widetilde{\Omega}^k(v_1,\dots,v_{2k})$ as claimed. □

2. The group of symplectomorphisms of $(S^2\times S^2,\omega^\lambda)$

Consider the group $\mathcal{G}^\lambda := \mathrm{Symp}(S^2\times S^2,\omega^\lambda)$ defined above. We need only consider the range $\lambda \ge 1$ since $\mathcal{G}^{1/\lambda}$ is isomorphic to $\mathcal{G}^\lambda$ (because $\omega^{1/\lambda}$ is a scalar multiple of ω^λ). We shall think of $S^2\times S^2$ as a trivial S^2-bundle over S^2 where the base is identified with the first (i.e. the larger) factor.

Gromov proved in [8] that

$$\mathcal{G}^1 \simeq SO(3)\times SO(3), \qquad \mathcal{G}^\lambda \not\simeq \mathcal{G}^1 \text{ if } \lambda > 1,$$

where $\simeq$ denotes homotopy equivalence. Abreu [1] calculated $H^*(\mathcal{G}^\lambda;\mathbb{Q})$ for $1<\lambda\le 2$; his calculation was completed by Abreu–McDuff [3] to all λ. The following theorem combines this with some results from McDuff [16] and very recent work by Abreu, Granja and Kitchloo [2]. [1]

[1] As pointed out in [2] the formula in (ii) below was slightly incorrect in Abreu–McDuff [3].

Theorem 2.1. (i) *The homotopy type of* $\mathcal{G}^\lambda$ *is constant on the intervals* $k < \lambda \le k+1, k \ge 1$.
(ii) $H^*(\mathcal{G}^\lambda;\mathbb{Q}) \cong \Lambda(t,x,y)\otimes\mathbb{Q}[w_k]$ *when* $k<\lambda\le k+1$. *Here* $\deg t = 1, \deg x = \deg y = 3$ *and* $\deg w_k = 4k$.
(iii) $H^*(B\mathcal{G}^\lambda;\mathbb{Q}) = \mathbb{Q}[T,X,Y]/T(X-Y+T^2)\dots(k^4X-k^2Y+T^2)$ *where* T,X,Y *are appropriate deloopings of* t,x,y *respectively.*

We now explain the relation between statements (ii) and (iii) and their connection to the different toric structures on $S^2\times S^2$. The generator t in (ii) is dual to the element $\tau\in\pi_1(\mathcal{G}^\lambda)$ represented by the circle action α on $S^2\times S^2\cong\mathbb{P}(\mathcal{O}(2)\oplus\mathbb{C})$ given by rotating the fiber of the line bundle $\mathcal{O}(2)\to\mathbb{CP}^1$. (Here we are identifying $S^2\times S^2$ with the second Hirzebruch surface $\mathbb{P}(\mathcal{O}(2)\oplus\mathbb{C})$, where $\mathcal{O}(2)$ and $\mathbb{C}$ are bundles over $S^2=\mathbb{CP}^1$. We denote by J_1 the corresponding complex structure on $S^2\times S^2$.)

The generators x,y in (ii) are dual to the 3-spheres ξ,η in $\mathcal{G}^\lambda$ given by the inclusion of each factor of $SO(3)\times SO(3)$ in $\mathcal{G}^\lambda$. Thus $x+y$ is dual to the diagonal copy $\xi+\eta$ of $SO(3)$ in $SO(3)\times SO(3)$. We claim that that this commutes with the circle action α. Indeed one can identify $S^2\times S^2$ with the toric manifold $\mathbb{P}(\mathcal{O}(2)\oplus\mathbb{C})$ in such a way that its toric (or Kähler) automorphism group $K_1:=\mathrm{Aut}(J_1)$ concides with the product $SO(3)\times S^1$ of α with the diagonal copy of $SO(3)$: see [3]. In this realization α has the formula

$$\tau_t(z,w)\mapsto(z,R_{t,z}w),$$

where $R_{t,z}:S^2\to S^2$ is the rotation through angle $2\pi t$ with axis through the point z and its antipode. Thus α fixes the points on the diagonal and antidiagonal in $S^2\times S^2$ and commutes with the diagonal $SO(3)$ action.

If $1<\lambda\le 2$ then one can show that α does not commute with the individual $SO(3)$-factors ξ,η even up to homotopy. More precisely, one can show that the remaining generator $w_1\in H^4(\mathcal{G}^\lambda)$ is dual to the element $[\xi,\tau]\in\pi_4(\mathcal{G}^\lambda)$ given by the Samelson product:

$$\begin{array}{ccc} S^3\times S^1 & \xrightarrow{(\xi,\tau)\mapsto\xi\tau\xi^{-1}\tau^{-1}} & \mathcal{G}^\lambda \\ \downarrow & & =\downarrow \\ S^4:=S^3\times S^1/S^3\vee S^1 & \xrightarrow{[\xi,\tau]} & \mathcal{G}^\lambda. \end{array}$$

Since Samelson products deloop to Whitehead products, the element w_1 is not transgressive but rather gives rise to the relation $T(X-Y+T^2)=0$ in $H^*(B\mathcal{G}^\lambda)$.

When $\lambda>2$ there is another Hirzebruch structure on $S^2\times S^2$ that supports a Kähler structure in class $[\omega^\lambda]$, namely the complex structure J_2 coming from the identification of $S^2\times S^2$ with $\mathbb{P}(\mathcal{O}(4)\oplus\mathbb{C})$. The autmorphism group $\mathrm{Aut}(J_2)$ of this structure is again isomorphic to $SO(3)\times S^1$ but its image K_2 in $\mathcal{G}^\lambda$ contains the rational homotopy classes $\xi+4\eta,\tau$. Therefore the class τ can be represented in $\mathcal{G}^\lambda$ by a circle action in K_1 that commutes with $\xi+\eta$ and by a circle action in K_2 that commutes with $\xi+4\eta$. Hence by the linearity of the Samelson product $[\xi,\tau]$ now vanishes in $\pi_*(\mathcal{G}^\lambda)$. Therefore one can define the higher product $[\xi,\xi,\tau]$ and it turns out that this does not vanish when $2 < \lambda \le 3$. Similarly, when $k < \lambda \le k+1$ there are $k+1$ toric structures on $S^2 \times S^2$ and w_k is dual to a kth order Samelson product of the form $[\xi, \dots, \xi, \tau]$.

2.1. **The integral homotopy type of $\mathcal{G}^\lambda$.** We now explain some recent work by Anjos [4] and Anjos–Granja [5] that gives a beautiful description of the full homotopy type of $\mathcal{G}^\lambda$ in the first interesting case, namely $1 < \lambda \le 2$. This description arises naturally from the geometry that is the basis of the proof of Theorem 2.1. The arguments go back, of course, to Gromov's original paper on J-holomorphic curves. Recall that an almost complex structure J on a symplectic manifold (M, ω) is said to be *tamed* by ω if $\omega(v, Jv) > 0$ for all nonzero $v \in TM$. Further a map $f : S^2 \to M$ is said to be *J-holomorphic* if $df \circ j = J \circ df$, where j is the standard complex structure on S^2. For short one sometimes calls such f a J-sphere.

Gromov looked at the contractible space

$$\mathcal{J}^\lambda$$

of all almost complex structures J on $M := S^2 \times S^2$ that are tamed by ω^λ. He proved that when $\lambda = 1$ every $J \in \mathcal{J}^1$ has the same pattern of holomophic curves as does the product structure $J_0 := j \times j$. In particular there are two foliations by J-spheres, one consisting of spheres in the class $A := [S^2 \times pt]$ and the other of spheres in the class $B := [pt \times S^2]$. For each such J one can think of these spheres as providing "coordinates" on $M = S^2 \times S^2$, i.e. there is a diffeomorphism $\psi_J : M \to M$ that takes the standard foliations of M by the spheres $S^2 \times pt, pt \times S^2$ to the two foliations by J-spheres. This diffeomorphism ψ_J is not quite symplectic, but has a canonical homotopy to an element $\psi'_J \in \mathcal{G}^1$. Moreover ψ'_J is independent of choices modulo the subgroup $SO(3) \times SO(3) = \mathrm{Aut}(J_0)$. It follows that the sequence of maps

$$\mathrm{Aut}(J_0) \longrightarrow \mathcal{G}^1 \longrightarrow \mathcal{J}$$

is a fibration up to homotopy, that is split by the map $J \mapsto \psi'_J$. Since $\mathcal{J}$ is contractible, we arrive at Gromov's result that $\mathrm{Aut}(J_0) \simeq \mathcal{G}^1$.

When $1 < \lambda \le 2$ it is no longer true that all elements in $\mathcal{J}^\lambda$ have the same pattern of J-spheres. There is now a second model, namely the pattern formed by the J_1-curves. In this case there is only one foliation, by spheres in the class $B = [pt \times S^2]$ of the smaller sphere and there is an isolated curve in the class $A - B$ of the antidiagonal (the rigid curve $\mathbb{P}(\mathcal{O}(2) \oplus 0)$ in the Hirzebruch surface). In this case there can be no A-curves by positivity of intersections: if A_1, A_2 are represented by distinct connected J-curves then one has $A_1 \cdot A_2 \ge 0$. Thus

$$\mathcal{J}^\lambda = U_0 \cup U_1,$$

where, for $i = 0, 1$, the set U_i consists of all J whose J-spheres are like those of J_i. This decomposition has the following properties.

- U_1 is a codimension 2-submanifold of $\mathcal{J}^\lambda$. In other words, there is a neighborhood NU_1 of U_1 in $\mathcal{J}^\lambda$ so that

$$NU_1 \setminus U_1 \to U_1$$

is an S^1-bundle.

- $\mathrm{Aut}(J_1) = SO(3) \times S^1$.
- There are homotopy equivalences

$$\mathcal{G}^\lambda / \mathrm{Aut}(J_0) \simeq U_0, \qquad \mathcal{G}^\lambda / \mathrm{Aut}(J_1) \simeq U_1.$$

The following theorem shows that $\mathcal{G}^\lambda$ is homotopic to an amalgamated free product of the two Lie groups $\mathrm{Aut}(J_0)$ and $\mathrm{Aut}(J_1)$.

Theorem 2.2. *$\mathcal{G}^\lambda$ is homotopy equivalent to the pushout of the diagram*

$$\begin{array}{ccc} SO(3) & \longrightarrow & \mathrm{Aut}(J_0) \\ \downarrow & & \\ \mathrm{Aut}(J_1) & & \end{array}$$

in the category of topological groups.

To prove this it suffices to show that the pushout of the quotients

$$\begin{array}{ccc} \mathcal{G}^\lambda / SO(3) & \longrightarrow & \mathcal{G}^\lambda / \mathrm{Aut}(J_0) \\ \downarrow & & \\ \mathcal{G}^\lambda / \mathrm{Aut}(J_1) & & \end{array}$$

in the homotopy category is contractible. But the above remarks show that this is equivalent to the pushout of the diagram

$$\begin{array}{ccc} NU_1 \setminus U_1 & \longrightarrow & U_0 \\ \downarrow & & \\ U_1. & & \end{array}$$

Since all the maps here are cofibrations (inclusions), the pushout is simply the contractible set $U_0 \cup U_1 = \mathcal{J}^\lambda$.

It is very likely that this work can be extended to all $\lambda > 1$, though the resulting pushout diagrams will be more complicated. As already mentioned when $k < \lambda \le k+1$, there are $k+1$ different integrable complex structures $J_0, \dots, J_k$ that are tamed by ω^λ, and the corresponding Lie groups $\mathrm{Aut}(J_i)$ define the structure of the generators and relations for the rational homotopy type of $\mathcal{G}^\lambda$ and $B\mathcal{G}^\lambda$. One can show (using a gluing argument) that there is a corresponding stratification of $\mathcal{J}^\lambda$. Therefore all the ingredients are in place except that one has to find a pushout diagram (or other categorical construction) that corresponds to a higher order Whitehead product.

Similar results are true for the nontrivial S^2-bundle over S^2, i.e. the one point blow up of $\mathbb{CP}^2$, and also, by recent work of Lalonde–Pinsonnault [14], for the one point blow up of $S^2 \times S^2$ in the range $1 < \lambda \le 2$. Here are two open problems:

- what happens with the many point blow up of $\mathbb{CP}^2$?
- what is the homotopy type of $\mathrm{Symp}(T^2 \times S^2, \lambda\sigma_0 + \sigma_1)$ for $\lambda > 0$?

Lalonde and Pinsonnault are working on the first of these problems. Some information on the second may be found in McDuff [16] and Buse [7]. In particular the homotopy type of the group is constant for λ in the range $0 < \lambda \le 1$. However, it is not yet known what this group is, even rationally.

3. Higher dimensional toric manifolds

A $2n$-dimensional symplectic manifold (M,ω) is said to be toric if it admits a Hamiltonian action of an n-torus $T := T^n$. Such a manifold (M,T) can always be realised as the symplectic reduction $M := \mathbb{C}^N /\!\!/ T'$ at some element $\nu \in (\mathfrak{t}')^* \cong \mathbb{R}^{N-n}$ of a high dimensional Euclidean space $\mathbb{C}^N$ by the action of a $(N-n)$-dimensional subtorus T' of the standard N-torus $T^N \subset U(N)$. Further T can be identified with the quotient T^N/T' and the toric autmorphism group $\mathrm{Aut}(M,T)$ is the quotient by T' of the centralizer of T' in $U(N)$:

$$\mathrm{Aut}(M,T) := \mathrm{Cent}(T')/T'.$$

Here is an example. Let M be the symplectic quotient $\mathbb{C}^5 /\!\!/ T'$ where T' is a 2-torus whose generators ξ_1, ξ_2 act with the weights $(1,1,1,0,0)$ and $(5,1,0,1,1)$ respectively. Quotienting out by ξ_2 gives the vector bundle $\mathcal{O}_5 \oplus \mathcal{O}_1 \oplus \mathbb{C} \to \mathbb{CP}^1$, and quotienting that by ξ_1 gives the projectivization. Thus M is a bundle over $\mathbb{CP}_1$ with fiber $\mathbb{CP}^2$. M can also be written as the (ordinary) quotient $S^3 \times_{S^1} \mathbb{CP}^2$, where S^1 acts on $\mathbb{CP}^2$ via the circle $\mathrm{diag}(\lambda^3, \lambda^{-1}, \lambda^{-2}), \lambda \in S^1$, in $PSU(3)$. Since this circle contracts in $PSU(3)$, M is diffeomorphic to the product, and it is shown in McDuff–Tolman [22] that if we choose any of its toric Kähler forms, it is symplectomorphic to a product $(\mathbb{CP}^1 \times \mathbb{CP}^2, \omega^\mu)$ where

$$\omega^\mu := \mu\sigma_{\mathbb{CP}^1} + \sigma_{\mathbb{CP}^2}, \qquad \mu > 3.$$

(Here we assume the forms $\sigma_{\mathbb{CP}^i}$ are normalized to have integral 1 over the projective line.)

Observe that the centralizer of T' is isomorphic to $S^1 \times S^1 \times S^1 \times U(2)$ so that $\mathrm{Aut}(M,T) = T^3 \times U(2)/T'$. Thus $\pi_1(\mathrm{Aut}(M,T))$ has rank 2.

The product $(\mathbb{CP}^1 \times \mathbb{CP}^2, \omega^\mu)$ has many toric structures, denoted $(M_{a,b}, T_{a,b})$, that are obtained similarly but with ξ_2 allowed to have any weights $(a,b,0,1,1)$ such that

$$a \ge b \ge 0, \qquad a+b = 3k, \qquad 3\mu > 2a - b.$$

Thus for $1 < \mu \le 2$ the possibilities for (a,b) are $(0,0), (2,1)$ and $(3,3)$, while for $2 < \mu \le 3$ one must add to these the pairs $(3,0), (4,2), (5,4)$ and $(6,6)$.

It would be very interesting to understand the relation of the different Lie subgroups $\mathrm{Aut}(M_{a,b}, T_{a,b})$ to the homotopy groups of $\mathcal{G}^\mu := \mathrm{Symp}(M, \omega^\mu)$. One cannot hope for such complete information as in the 4-dimensional case since the analysis there was based on our exhaustive knowledge of the J-curves. Nevertheless, the more elementary aspects of the 4-dimensional case do generalize. For example, the group $\pi_1(\mathcal{G}^\mu)$ for $\mu > 1$ still contains an element of infinite order that does not appear in $\mathcal{G}^1$. This was discovered by Seidel [25]. This element is realised as the circle action corresponding to a suitable facet of the moment polytope of $(M_{2,1}, T_{2,1})$ and hence appears in $\mathrm{Aut}\big(M_{2,1}, T_{2,1}\big)$.

In general, given a symplectic toric manifold (M, T, ω^μ), one can try to understand the induced map

$$\pi_* \mathrm{Aut}(M,T) \to \pi_*(\mathcal{G}^\mu)$$

where $\mathcal{G}^\mu := \mathrm{Symp}_0(M, \omega^\mu)$? For example, is it always injective, at least rationally? How does it vary with the cohomology class of $[\omega^\mu]$?

The following result from [22] applies to generic low dimensional toric manifolds.

Proposition 3.1. *If $\dim M \le 6$ and $\mathrm{Aut}(M,T) = T$ then $\pi_1 T$ injects into $\pi_1(\mathcal{G}^\mu)$ for all forms ω^μ.*

The proof is elementary, using only the geometry of the moment polytope. One might hope that this result would generalize to all dimensions, but that is not so; it fails already in dimension 8. One important reason for this lack of injectivity is demonstrated in the example $(M_{a,b}, T_{a,b})$ discussed above. We saw earlier that $\pi_1(\mathrm{Aut}(M,T))$ has rank 2 when $a > b > 0$. On the other hand this automorphism group is contained in the group $\mathcal{G}_{\mathrm{fib}}$ of diffeomorphisms of M that commute with the projection to $\mathbb{CP}^1$ and restrict to symplectomorphisms on each fiber $pt \times \mathbb{CP}^2$. Since $\mathrm{Symp}_0(\mathbb{CP}^2) \simeq PSU(3)$ (by Gromov), $\mathcal{G}_{\mathrm{fib}}$ deformation retracts to the product of the group of orientation preserving diffeomorphisms of the base with the group of fiberwise diffeomorphisms that fix each fiber:

$$\mathcal{G}_{\mathrm{fib}} \simeq SO(3) \times \mathrm{Map}(S^2, PSU(3))$$

Hence $\pi_1(\mathcal{G}_{\mathrm{fib}})$ has rank 1. Although the elements of $\mathcal{G}_{\mathrm{fib}}$ are not symplectomorphisms, a standard Moser-type argument shows that any compact subset of $\mathcal{G}_{\mathrm{fib}}$ can be homotoped into $\mathcal{G}^\mu$ for sufficiently large μ. It follows that

$$\pi_1\Big(\mathrm{Aut}(M_{2,1}, T_{2,1})\Big) \to \pi_1 \mathcal{G}^\mu$$

is not injective for sufficiently large μ. By directly constructing a homotopy, one can show that in fact it is not injective for all $\mu > 1$: see [22].

Note that $\mathrm{Aut}(M,T)$ is always a maximal connected compact subgroup of $\mathrm{Symp}_0(M)$ because its subgroup T is a maximal connected abelian subgroup of $\mathrm{Symp}_0(M)$. Therefore by analogy with what happens in the finite dimensional case (where a simple G deformation retracts to its maximal compact subgroup) one would expect that at the very least the homotopy carried by $\mathrm{Aut}(M,T)$ would not completely disappear in $\mathrm{Ham}(M,\omega)$. The next result from Kędra–McDuff [11] gives some supporting evidence.

Theorem 3.2. *Let (M,ω) be a symplectic manifold of dimension $2n$ and set $\mathcal{G} := \mathrm{Ham}(M,\omega)$. Suppose given a nonconstant homomorphism $\alpha : S^1 \to \mathcal{G}$ that represents the zero element in $\pi_1(\mathcal{G})$ and so extends to a map $\widetilde{\alpha} : D^2 \to \mathcal{G}$. Define $\rho \in \pi_3(\mathcal{G}) \otimes \mathbb{Q}$ by*

$$S^3 := \big(D^2 \times S^1\big)/\big((D^2 \times \{1\}) \vee (\partial D^2 \times S^1)\big) \to \mathcal{G}, \qquad (3.1)$$
$$(z,t) \mapsto [\widetilde{\alpha}(z), \alpha(t)],$$

where the bracket $[\phi, \psi]$ denotes the commutator $\phi\psi\phi^{-1}\psi^{-1}$. Then $\rho \neq 0$ and is independent of the choice of extension $\widetilde{\alpha}$. Moreover, ρ transgresses to an element $\overline{\rho} \in \pi_4(B\mathcal{G}) \otimes \mathbb{Q}$ with nonzero image in $H_4(B\mathcal{G})$.

This is proved by showing that the characteristic class μ_2 defined in (1.1) is nontrivial on $\overline{\rho}$. As an example, if α is a nonzero element in the kernel of the map

$\pi_1\Big(\mathrm{Aut}(M_{2,1}, T_{2,1})\Big) \to \pi_1 \mathcal{G}^\mu$ then ρ is represented in the $PSU(3)$-factor of

$$\mathrm{Aut}(M_{0,0}, T_{0,0}) \cong PSU(3) \times SO(2).$$

We conclude with several remarks.

• Because the element ρ above is detected by a characteristic class it is very robust and persists under small variations of the class $[\omega]$ of the form. This should be contrasted with the elements w_k of Theorem 2.1 that disappear under appropriate perturbations of $[\omega]$. Thus these elements are quite different in nature even though they are constructed in similar ways, i.e. via commutators.

• In the case of the toric manifolds $(M_{a,b}, T_{a,b}, \omega^\mu)$ it would be interesting to work out the relation between the groups $\mathcal{G}^\mu$ and the fiberwise diffeomorphism group $\mathcal{G}_{\mathrm{fib}}$. There is an analogous question in the case $(S^2 \times S^2, \omega^\lambda)$. But here one can use the existence of J-spheres in the fiber class B to define natural maps $\mathcal{G}^\lambda \to \mathcal{G}^{\lambda'}$ whenever $\lambda < \lambda'$ and can show that

$$\lim_{\lambda\to\infty} \mathcal{G}^\lambda \simeq \mathcal{G}_{\mathrm{fib}}.$$

In the 6-dimensional case such maps do not seem to exist. Nevertheless, one still should be able to make sense of the limit as $\mu \to \infty$ and to investigate its relation to the group $\mathcal{G}_{\mathrm{fib}}$. In this case, of course, there is no symmetry between μ and $1/\mu$ and so one could similarly consider the limit as $\mu \to 0$. This should be related to the group of fiberwise diffeomorphisms of the fibration $\mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^2$.

• An interesting question in the 6-dimensional case is the extent to which J-spheres can give useful information. One can no longer use their geometry; for example spheres in the class $[\mathbb{CP}^1 \times pt]$ need not form the leaves of a foliation. Nevertheless, one can get information from quantum cohomology as in Seidel [25]. Such methods allow one to show that the elements in $\pi_*(\mathcal{G}_{\mathrm{fib}}) \otimes \mathbb{Q}$ coming from $\pi_*\big(\Omega^2(PSU(3))\big) \otimes \mathbb{Q}$ do not appear in $\mathcal{G}^\mu$ for $\mu \le 1$ though they are there when $\mu > 1$. Buse [7] has a different approach that uses equivariant Gromov–Witten invariants.

4. Quantum cohomology of toric manifolds

Consider a compact symplectic manifold (M, ω) and its Hamiltonian group $\mathcal{G} := \mathrm{Ham}(M, \omega)$. One very useful tool in understanding $\pi_1(\mathcal{G})$ is Seidel's representation

$$\mathcal{S} : \pi_1(\mathcal{G}) \to \mathrm{Units}\big(\mathrm{QH}^*(M)\big)$$

of $\pi_1(\mathcal{G})$ in the group of multiplicative units in the quantum cohomology of M. Here we use the coefficients $\Lambda := \Lambda^{\mathrm{univ}}[q, q^{-1}]$ for quantum cohomology $\mathrm{QH}^*(M) := H^*(M) \otimes \Lambda$, where where q is a variable of degree 2 and Λ^{univ} is a generalized Laurent series ring in a variable t of degree 0:

$$\Lambda^{\mathrm{univ}} := \Big\{ \sum_{\kappa\in\mathbb{R}} r_\kappa t^\kappa \;\Big|\; r_\kappa \in \mathbb{Q},\ \ \#\{\kappa < c \,|\, r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \Big\}.$$

Thus typical elements in $\mathrm{QH}^*(M)$ have the form $\sum_{d,\kappa} a_{d,\kappa} q^d t^\kappa$ where $a_{d,\kappa} \in H^*(M)$ satisfy the same finiteness condition as do the r_κ.

Given an element $\phi \in \pi_1(\mathcal{G})$, the element $\mathcal{S}(\phi)$ is constructed as follows. First consider the Hamiltonian fibration

$$(M,\omega) \longrightarrow (P_\phi,\Omega) \stackrel{\pi}{\longrightarrow} S^2$$

whose clutching function is a representing loop $\{\phi_t\}$ for ϕ. Thus P_ϕ is the union of two copies of $D^2 \times M$ whose boundaries identified via $\{\phi_t\}$. It carries two canonical cohomology classes, $u \in H^2(P_\phi)$ which is the class of the coupling form and $c_1^{\mathrm{Vert}} \in H^2(P_\phi)$, the first Chern class of the vertical tangent bundle.

As mentioned in §1.2, the fiberwise form ω has a closed extension Ω that we may assume to be symplectic by adding to it the pullback of a suitable area form on the base. Choose an Ω-tame almost complex structure $\widetilde{J}$ on P_ϕ that preserves the tangent bundle to the fiber and projects under π to the standard complex structure on S^2. Then, because the fibers of π are $\widetilde{J}$-holomorphic, every $\widetilde{J}$-sphere $f: S^2 \to P_\phi$ that represents a section class $\widetilde{A}$ in P_ϕ (i.e. class such that $\widetilde{A} \cap [M] = 1$) may be parametrized as a section. Denote by $\mathcal{M}(\widetilde{A},\widetilde{J})$ the space of all such sections. In good cases this is a manifold with boundary of codimension ≥ 2, so that its intersection with a fixed fiber $[M]$ represents a homology class in M that we denote $\alpha_{\widetilde{A}} := [\mathcal{M}(\widetilde{A},\widetilde{J})] \cap [M]$. In general, this homology class is defined using Gromov–Witten invariants in P:

$$\alpha_{\widetilde{A}} \cdot_M \beta = \mathrm{GW}^P_{\widetilde{A},3}([M],[M],i_*(\beta)), \quad \forall \beta \in H_*(M).$$

Here $[M] \in H_*(P)$ denotes the homology class of a fiber and $i: M \to P$ is the inclusion of a fiber. Then we define

$$\mathcal{S}(\phi) := \sum_{\widetilde{A}} \mathrm{PD}(a_{\widetilde{A}}) \otimes q^{c_1^{\mathrm{Vert}}(\widetilde{A})} t^{u(\widetilde{A})}.$$

One can show that this sum satisfies the requisite finiteness condition and so represents an element in $\mathrm{QH}^*(M) = H^*(M) \otimes \Lambda$ of degree 0. Further, the image $\mathcal{S}(0)$ of the constant loop is the multiplicative unit $1 \in H^*(M) \subset \mathrm{QH}^*(M)$, and a gluing argument shows that

$$\mathcal{S}(\phi+\psi) = \mathcal{S}(\phi) * \mathcal{S}(\psi),$$

where $*$ denotes quantum multiplication. (Proofs in various contexts may be found in Seidel [24], Lalonde–McDuff–Polterovich [13], McDuff [15] and McDuff–Salamon [20]. We denote the group operation in $\pi_1(\mathcal{G})$ by $+$ since this is an abelian group.)

In general it is not easy to calculate $\mathcal{S}(\phi)$. The following result is proved in McDuff–Tolman [21]. It applies when ϕ is represented by a circle action $t \mapsto \phi_t$. Denote by $K: M \to \mathbb{R}$ the normalized moment map of this action (i.e. $\int_M K\omega^n = 0$), and by

$$X_{\max} := K^{-1}(\max K)$$

the maximal fixed point set. Further if J is an S^1-invariant and ω-tame almost complex structure then we say that (M,J) is Fano (resp. NEF) if $c_1(TM,J)$ is positive (resp. nonnegative) on every J-sphere.

Theorem 4.1. *Suppose that ϕ_K is represented by a circle action with normalized moment map K. Suppose further that the weights of the linearized action at the maximal fixed point component $X_{\max}$ are 0 or -1. Then*

$$\mathcal{S}(\phi_K) = \mathrm{PD}[X_{\max}] \otimes q^{-1}\, t^{-\max K} + \sum_{\kappa > -\max K, d} a_{d,\kappa} \otimes q^d t^\kappa.$$

If (M, J) is Fano all the higher order terms $a_{d,\kappa} \in H^*(M)$ vanish, while if (M, J) is NEF they vanish unless $\deg(a_{d,\kappa}) \leq 2$.

Again following [21], we now explain what this theorem tells us about the quantum cohomology of a symplectic toric manifold (M, T). We begin by reviewing the structure of the usual cohomology ring. Denote the Lie algebra of T by $\mathfrak{t}$ and its dual by $\mathfrak{t}^*$. Let $\Phi : M \to \mathfrak{t}^*$ be the normalized moment map for the T-action, i.e. each of its components is mean normalized. The image of Φ is a convex polytope $\Delta \subset \mathfrak{t}^*$, and we denote its facets (the codimension one faces) by $D_1, \ldots, D_N$ and the outward primitive integral normal vectors by $\eta_1, \ldots \eta_N \in \mathfrak{t}$. Let Σ be the set of subsets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$ for which $D_{i_1} \cap \cdots \cap D_{i_k} \neq \emptyset$. Define two ideals in $\mathbb{Q}[x_1, \ldots, x_N]$:

$$P(\Delta) = \Big\langle \sum (\xi, \eta_i)\, x_i \Big| \, \xi \in \ell^* \Big\rangle, \quad \text{and} \quad SR(\Delta) = \Big\langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \notin \Sigma \Big\rangle.$$

A subset $I \subseteq \{1, \ldots, N\}$ is called **primitive** if I is not in Σ but every proper subset is. Clearly,

$$SR(\Delta) = \Big\langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\} \text{ is primitive} \Big\rangle.$$

The following result is well known.

Proposition 4.1. *The map that sends x_i to the Poincaré dual of $\Phi^{-1}(D_i)$ (which we shall also denote by $x_i \in H^2(M)$) induces an isomorphism*

$$\mathbb{Q}[x_1, \ldots, x_N]/(P(\Delta) + SR(\Delta)) \cong H^*(M, \mathbb{Q}). \tag{4.1}$$

Moreover, there is a natural isomorphism between $H_2(M; \mathbb{Z})$ and the set of tuples $(a_1, \ldots, a_N) \in \mathbb{Z}^N$ such that $\sum a_i \eta_i = 0$, under which the pairing between such an element of $H_2(M, \mathbb{Z})$ and x_i is a_i.

The linear functional η_i is constant on D_i; let $\eta_i(D_i)$ denote its value. Under the isomorphism of (4.1) (extended to real coefficients)

$$[\omega] = \sum_i \eta_i(D_i) x_i, \quad \text{and} \quad c_1(M) = \sum_i x_i. \tag{4.2}$$

Note also that each element η_i lies in the integer lattice of $\mathfrak{t}$ and so corresponds to a circle action λ_i on M. By Theorem 4.1

$$\mathcal{S}(\lambda_i) = y_i \otimes q^{-1} t^{-\eta_i(D_i)} \in \mathrm{Units}\big(\mathrm{QH}^*(M)\big),$$

where the element y_i has the form $x_i + \sum_{\kappa > 0} a_{d,\kappa} q^d t^\kappa$.

We are now ready to examine the quantum cohomology of a toric variety. Given any face of Δ, let $D_{j_1}, \ldots, D_{j_\ell}$ be the facets that intersect to form this face. The **dual**

cone is the set of elements in $\mathfrak{t}$ which can be written as a positive linear combination of $\eta_{j_1}, \ldots, \eta_{j_\ell}$. Every vector in $\mathfrak{t}$ lies in the dual cone of a unique face of Δ. Therefore, given any subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$ there is a unique face of Δ so that $\eta_{i_1} + \cdots + \eta_{i_k}$ lies in its dual cone. Let $D_{j_1}, \ldots, D_{j_\ell}$ be the facets that intersect to form this unique face. Then there exist unique positive integers $m_1, \ldots, m_\ell$ so that

$$\eta_{i_1} + \cdots + \eta_{i_k} - m_1\eta_{j_1} - \cdots - m_\ell\eta_{j_\ell} = 0.$$

Batyrev showed that if I is primitive the sets I and $J = \{j_1, \ldots, j_\ell\}$ are disjoint. Let $\beta_I \in H_2(M, \mathbb{Z})$ be the class corresponding to the above relation. By (4.2), we see that

$$\begin{aligned} c_1(\beta_I) &= k - m_1 - \cdots - m_\ell, \quad \text{and} \\ \omega(\beta_I) &= \eta_{i_1}(D_{i_1}) + \cdots + \eta_{i_k}(D_{i_k}) - m_1\eta_{j_1}(D_{j_1}) - \cdots - m_\ell\eta_{j_\ell}(D_{j_\ell}). \end{aligned}$$

Since $\eta_{i_1} + \cdots + \eta_{i_k} = m_1\eta_{j_1} + \cdots + m_\ell\eta_{j_\ell}$, the corresponding circle actions are also equal. Using the fact that the Seidel representation is a homomorphism, we find

$$\begin{aligned} y_{i_1} * \cdots * y_{i_k} \otimes q^{-k} t^{-\eta_{i_1}(D_{i_1}) - \cdots - \eta_{i_k}(D_{i_k})} &= \\ {y_{j_1}}^{m_1} * \cdots * {y_{j_\ell}}^{m_\ell} \otimes q^{-m_1 - \ldots - m_\ell}\, t^{-m_1\eta_{j_1}(D_{j_1}) - \cdots - m_\ell\eta_{j_\ell}(D_{j_\ell})}&. \end{aligned}$$

Therefore

$$y_{i_1} * \cdots * y_{i_k} - {y_{j_1}}^{m_1} * \cdots * {y_{j_\ell}}^{m_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} = 0.$$

Since $x_1, \ldots, x_N$ generate $H^*(M)$, the natural homomorphism

$$\Theta : \mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda \to \mathrm{QH}^*(M)$$

which takes x_i to the Poincaré dual of $\Phi^{-1}(D_i)$ is surjective. To compute $\mathrm{QH}^*(M)$, we need to find the kernel of Θ. It is not hard to check that there is

$$Y_i = x_i + \text{ higher order terms } \in \mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda \tag{4.3}$$

such that $\Theta(Y_i) = y_i$. Define an ideal $SR_Y(\Delta) \subset \mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda$ by

$$SR_Y(\Delta) = \left\langle Y_{i_1} \cdots Y_{i_k} - {Y_{j_1}}^{m_1} \cdots {Y_{j_\ell}}^{m_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} \,\middle|\, I = \{i_1, \ldots, i_k\} \text{ is primitive}\right\rangle, \tag{4.4}$$

where the Y_i are as in (4.3). Note that $SR_Y(\Delta)$ depends on the Y_i. Additionally, even if y_i is known, it is not in general possible to describe Y_i without prior knowledge of the ring structure on $\mathrm{QH}^*(M)$. On the other hand, SR_Y is clearly contained in the kernel of Θ. Moreover, Batyrev shows that $\omega(\beta_I) > 0$ for all primitive I. Hence, we conclude:

Proposition 4.2. *Let* $\mathrm{QH}^*(M)$ *denote the small quantum cohomology of the toric manifold* (M, ω) *with coefficients* $\Lambda = \Lambda^{\mathrm{univ}}[q, q^{-1}]$. *The map which sends* x_i *to the Poincaré dual of* $\Phi^{-1}(D_i)$ *induces an isomorphism*

$$\mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda / \big\langle P(\Delta) + SR_Y(\Delta) \big\rangle \cong \mathrm{QH}^*(M).$$

In the Fano case, Theorem 4.1 states that the higher order terms in Y_i vanish. Therefore we recover the formula for the small quantum cohomology of a Fano toric variety given by Batyrev and proved by Givental.

In the NEF case there may be higher order terms in the Seidel elements y_i. However, these terms only involve cohomology classes $a_{d,\kappa}$ of degree ≤ 2. Therefore $a_{d,\kappa}$ either lifts to the unit 1 in $\mathbb{Q}[x_1, \dots, x_N] \otimes \Lambda$ or to some linear combination of the x_i that is unique modulo the additive relations $P(\Delta)$. Hence we do not need to know the quantum multiplication in M in order to define the Y_i. The rest of the information needed to define the relations $P(\Delta)$ and $SR_Y(\Delta)$ is contained explicitly in Δ.

Thus, in the NEF case, once one knows the Seidel elements $\Psi(\Lambda_i), i = 1, \dots, N$, there is an easy formula for the quantum cohomology ring based on the combinatorics of the moment polytope Δ. This substitution of the Y_i for the x_i in the Stanley–Reisner ring SR_Y should be related to Givental's change of variable formulae as discussed in [6, 11.2.5.2].

It should also be possible to calculate the Y_i by an explicit formula from the polytope. In [21] we show that its terms are generated by certain chains of edges in the polytope, and that the coefficient of each such term is determined locally, i.e. by a neighborhood of the chain. However we do not attempt to calculate these coefficients. Even in NEF examples in 4-dimensions, the chains can be quite complicated.

Observe finally that here we are working with the stripped down coefficient ring $\Lambda^{\text{univ}}[q, q^{-1}]$. However, as described in McDuff–Salamon[20, Ch 11.4], it is possible to obtain a similar description of the quantum cohomology for (M, T) with coefficients in the usual Novikov ring (the completed group ring of $H_2(M; \mathbb{Z})/\text{tor}$) by varying the cohomology class of $[\omega]$.

References

[1] M. Abreu, Topology of symplectomorphism groups of $S^2 \times S^2$, *Inv. Math.*, **131** (1998), 1-23.

[2] M. Abreu, G. Granja and N. Kitchloo, Moment maps, symplectomorphism groups and compatible complex structures, SG/0507382 *Jurn. Symp. Geom.*, (2006).

[3] M. Abreu and D. McDuff, Topology of symplectomorphism groups of rational ruled surfaces, SG/9910057, *Journ. of Amer. Math. Soc.* **13**, (2000) 971–1009.

[4] S. Anjos, The homotopy type of symplectomorphism groups of $S^2 \times S^2$, *Geometry and Topology*, **6** (2002), 195–218.

[5] S. Anjos and G. Granja, Homotopy decomposition of a group of symplectomorphisms of $S^2 \times S^2$, AT/0303091 preprint (2003)

[6] D. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical Surveys bf 68, Amer. Math. Soc (1999).

[7] O. Buse, in preparation.

[8] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Inventiones Mathematicae*, **82** (1985), 307–47.

[9] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge University Press (1994).

[10] T. Januszkiewicz and J. Kędra, Characteristic classes of smooth fibrations, SG/0209288.

[11] J. Kędra and D. McDuff, Homotopy properties of Hamiltonian group actions, SG/0404539.

[12] F. Lalonde and D. McDuff, Symplectic structures on fiber bundles, SG/0010275, *Topology* **42** (2003), 309–347.

[13] L. Lalonde, D. McDuff and L. Polterovich, Topological regidity of Hamiltoian loops and quantum homology, *Invent. Math.* **135** (1999), 69–85.

[14] F. Lalonde and M. Pinsonnault, The topology of the space of symplectic balls in rational 4-manifolds, SG/0207096. to appear in *Duke Math J.*

[15] D. McDuff, Quantum homology of Fibrations over S^2, *International Journal of Mathematics*, **11**, (2000), 665–721.

[16] D. McDuff, Symplectomorphism Groups and Almost Complex Structures, SG/0010274, *Enseignment Math.* **38** (2001), 1–30.

[17] D. McDuff, Lectures on groups of symplectomorphisms, SG/0201032 *Rendiconti di Circolo di Mat, Palermo* (2004).

[18] D. McDuff, A survey of topological properties of groups of symplectomorphisms, SG/0404340 in *Topology, Geometry and Quantum Field Theory*, Proceedings of 2002 Symposium in honor of G.B. Segal, ed U.L. Tillmann, Cambridge Univ Press (2004).

[19] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd edition (1998) OUP, Oxford, UK

[20] D. McDuff and D. Salamon, *J-holomorphic curves and Symplectic Topology*, Amer. Math. Soc. Colloquium Publications **52** (2004)

[21] D. McDuff and S. Tolman, Topological properties of Hamiltonian circle actions, SG/0404338 (2004).

[22] D. McDuff and S. Tolman, The symplectomorphism group of toric manifolds, in preparation.

[23] A. G. Reznikov, Characteristic classes in symplectic topology, *Selecta Math,* **3** (1997), 601–642.

[24] P. Seidel, π_1 of symplectic automorphism groups and invertibles in quantum cohomology rings, *Geom. and Funct. Anal.* **7** (1997), 1046 -1095.

[25] P. Seidel, On the group of symplectic automorphisms of $\mathbb{C}P^m \times \mathbb{C}P^n$, *Amer. Math. Soc. Transl.* (2) **196** (1999), 237–250.

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA

E-mail address: `dusa@math.sunysb.edu`

URL: `http://www.math.sunysb.edu/ dusa`