

SYMPLECTIC EMBEDDINGS OF 4-DIMENSIONAL ELLIPSOIDS

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ABSTRACT. We show how to reduce the problem of symplectically embedding one 4-dimensional rational ellipsoid into another to a problem of embedding disjoint unions of balls into appropriate blow ups of $\mathbb{C}P^2$. For example, the problem of embedding the ellipsoid $E(1, k)$ into a ball B is equivalent to that of embedding k disjoint equal balls into $\mathbb{C}P^2$, and so can be solved by the work of Gromov, McDuff–Polterovich and Biran. (Here k is the ratio of the area of the major axis to that of the minor axis.) As a consequence we show that the ball may be fully filled by the ellipsoid $E(1, k)$ for $k = 1, 4$ and all $k \geq 9$, thus answering a question raised by Hofer.

1. INTRODUCTION.

A 4-dimensional symplectic ellipsoid is a region in standard Euclidean space (\mathbb{R}^4, ω_0) described by an inequality of the form $Q(z) \leq 1$, where Q is a positive definite quadratic form and $z \in \mathbb{R}^4$. Since Q may be diagonalized by a linear change of coordinates, every ellipsoid may be written (uniquely) as $E(m, n)$ where

$$E(m, n) := \left\{ \frac{x_1^2 + y_1^2}{m} + \frac{x_2^2 + y_2^2}{n} \leq 1 \right\} \subset \mathbb{R}^4, \quad m \leq n.$$

We denote an open ellipsoid by $\overset{\circ}{E}$ and the ball $E(m, m)$ by $B(m)$. Further, λE denotes E with the rescaled form $\lambda\omega_0$. Thus $\lambda E(m, n) := E(\lambda m, \lambda n)$. Throughout this paper the word “embedding” will be used to denote a symplectic embedding. If E embeds in (M, ω) we shall write $E \hookrightarrow (M, \omega)$.

This paper is concerned with the question of when it is possible to embed one 4-dimensional ellipsoid into another. There are two known obstructions to embedding $E(m, n)$ into $E(m', n')$ when m, n, m', n' are integers; namely, if such an embedding exists, we must have

- (i) (Volume): $mn \leq m'n'$ (since $E(m, n)$ has volume $\pi^2 mn/2$); and
- (ii) (Ekeland–Hofer capacities): $N(m, n) \leq N(m', n')$.

Here $N(m, n)$ denotes the sequence obtained by arranging the numbers $km, k \geq 1$, and $\ell n, \ell \geq 1$, in nondecreasing order (with repetitions) and $N(m, n) \leq N(m', n')$ means that every number in $N(m, n)$ is no larger than the corresponding number in $N(m', n')$;

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see [3]. In particular, we must have $m \leq m'$ as follows from Gromov's nonsqueezing theorem. For example,

$$N(1, 4) = (1, 2, 3, 4, 4, 5, 6, 7, 8, 8, 9, \dots), \quad N(2, 2) = (2, 2, 4, 4, 6, 6, 8, 8, \dots),$$

so that $N(1, 4) \leq N(2, 2)$. Since the volume inequality is also satisfied in this case the question arose as to whether $E(\lambda, 4\lambda)$ embeds in the ball $B(2)$ for all $\lambda < 1$. (By Corollary 1.6 this is equivalent to asking if the interior $\mathring{E}(1, 4)$ embeds into $B(2)$.)

When considering this embedding problem it is convenient to consider the maximal packing radius

$$\lambda_{sup} := \sup \{ \lambda \mid \lambda E(m, n) \xrightarrow{s} E(m', n') \}$$

and the packing constant

$$v := \lambda_{sup}^2 \frac{mn}{m'n'} \leq 1,$$

which is the ratio of the volume of the domain to that of the target. Note that both these numbers are scale invariant, i.e. do not change if all numbers m, n, m', n' are multiplied by the same constant μ . We say that $E(m, n)$ *fully fills* $E(m', n')$ if $v = 1$. Further we denote by $\mathbb{C}P^2(\mu)$ the complex projective plane with its standard Fubini–Study form, normalized so that the area of a line is μ . It is obtained from the ball $B(\mu)$ by collapsing its boundary sphere to a line.

Theorem 1.1. *$E(1, k)$ embeds in the open ball $\mathring{B}(\mu)$ if and only if k disjoint balls $B(1)$ embed in $\mathring{B}(\mu)$.*

The “only if” part of this statement was first observed by Traynor [26]. It is very easy to prove using toric models, which make it immediately clear that $E(1, k)$ contains k disjoint open balls $\mathring{B}(1)$: see Fig. 2.4 and Lemma 2.6. However the “if” part requires more work. The main idea is to cut the ellipsoid from the ball (i.e. perform an orbifold blow up as in Godinho [5]) and then to resolve the resulting orbifold singularities by further standard blow ups. The necessary symplectic surgery techniques were developed by Symington [23] in her treatment of rational blowdowns.

Corollary 1.2. *$E(1, k)$ fully fills $B(\sqrt{k})$ if and only if $k = 1, 4$ or $k \geq 9$. Moreover for $k \leq 8$ the packing constant $v(k)$ is:*

k	1	2	3	4	5	6	7	8
$v(k)$	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{4}{5}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$

Proof of Corollary. In his foundational paper [6], Gromov showed that $v(2) \leq 1/2$ and $v(5) \leq 4/5$. When $k \leq 9$ or $k = d^2$, the rest of the above statement follows from Theorem 1.1 by McDuff–Polterovich [17]. The case $k > 9$ is due to Biran [1]. \square

For explicit realizations of the ball packings at the integers $k < 9$, see Karshon [10], Traynor [26], Schlenk [21], and Wieck [27].

Remark 1.3. After this paper was written, I found out that Opshtein's paper [20] on maximal symplectic packings of $\mathbb{C}P^2$ contains a proof that $E(1, k)$ fully fills $\mathbb{C}P^2$ when $k = d^2$. Though not stated explicitly in his paper, this result follows immediately

from his Lemma 2.1. His argument has the virtue of providing a geometric recipe for constructing these packings.

In [3, Problem 15] Cieliebak, Hofer, Latschey, and Schlenk ask whether the two invariants listed above are the only obstructions to embedding one open ellipsoid into another. More formally, they asked if the Ekeland–Hofer capacities $N(m, n)$ together with the volume capacity V generate the (generalized) symplectic capacities on the space Ell^4 of open 4-dimensional ellipsoids. To understand what this means, consider the capacity given by embeddings into an open ball:

$$c^B(M, \omega) := \inf \{ \mu > 0 \mid (M, \omega) \overset{s}{\hookrightarrow} \mathring{B}(\mu) \}.$$

If this capacity were some combination of $N(m, n)$ and V for $(M, \omega) = \mathring{E}(m, n)$ then every time the inequalities given by these capacities are satisfied there would be an embedding of $\mathring{E}(m, n)$ into $\mathring{B}(\mu)$. But this is not so. For example, $c^B(\mathring{E}(1, 5)) = \frac{5}{2}$ because by Corollary 1.2 the volume of the target ball must be at least $5/4$ times the volume of $\mathring{E}(1, 5)$. On the other hand, the volume and Ekeland–Hofer capacities give no obstruction to the existence of an embedding of $\mathring{E}(1, 5)$ into $\lambda \mathring{B}(\sqrt{5})$ for all $\lambda > 1$. We conclude:

Corollary 1.4. *The Ekeland–Hofer capacities $N(m, n)$ together with the volume capacity do not generate the (generalized) symplectic capacities on the space Ell^4 of open 4-dimensional ellipsoids.*

Note that this follows from the easy (only if) part of Theorem 1.1.

Our second set of results concern the problem of embedding one ellipsoid into another. It is convenient to introduce the following terminology. Given a positive integer k and k positive numbers w_1, \dots, w_k the (*symplectic*) *packing problem for k balls with weights $\underline{w} := (w_1, \dots, w_k)$* is the question of whether the k disjoint (closed) balls $B(w_1), \dots, B(w_k)$ embed into the open ball $\mathring{B}(1)$.

Theorem 1.5. *For any positive integers m, n, m', n' and any $\lambda > 0$, there is an integer k and weights \underline{w}_λ such that the question of whether $\lambda E(m, n)$ embeds into the open ellipsoid $\mathring{E}(m', n')$ is equivalent to the symplectic packing problem for k balls with weights \underline{w}_λ .*

The following result (which was proved for balls in [14]) is an easy consequence.

Corollary 1.6. *Let a, b, a', b' be any real positive numbers. Then:*

- (i) *the space of symplectic embeddings of $E(a, b) \overset{s}{\hookrightarrow} \mathring{E}(a', b')$ is path connected whenever it is nonempty;*
- (ii) *if $\lambda E(a, b)$ embeds in $\mathring{E}(a', b')$ for all $\lambda < 1$, then $\mathring{E}(a, b)$ also embeds in $\mathring{E}(a', b')$.*

We also work out two specific examples that answer a question raised by Tolman [25, §1].

Proposition 1.7. (i) *The question of whether $\lambda E(1, 4)$ embeds in $\mathring{E}(2, 3)$ is equivalent to the packing problem with $k = 7$ and $\underline{w} = \frac{1}{3}(1, 1, 1, \lambda, \lambda, \lambda, \lambda)$. Hence the embedding exists iff $\lambda < \frac{6}{5}$.*

(ii) *The question of whether $\lambda E(1, 5)$ embeds in $\mathring{E}(2, 3)$ is equivalent to the packing problem with $k = 8$ and $\underline{w} = \frac{1}{3}(1, 1, 1, \lambda, \lambda, \lambda, \lambda, \lambda)$. Hence the embedding exists iff $\lambda < \frac{12}{11}$.*

These packing problems arose in Tolman's attempt to describe all 6-dimensional Hamiltonian S^1 -manifolds M with $H^2(M)$ of rank 1. In [16] we use the ideas of the present paper to construct the two new manifolds among her list of four possibilities, thus completing her classification.

Remark 1.8. (i) The question of which weights \underline{w} correspond to a given packing problem is intimately related to standard (rather than Hirzebruch–Jung) continued fraction expansions: see Remarks 3.9 and 3.10.

(ii) Our approach also gives a great deal of information about the function

$$c : [1, \infty) \rightarrow [1, \infty), \quad c(a) = \inf\{\mu : E(1, a) \xrightarrow{s} B(\mu)\},$$

studied in Schlenk [22] and in [3, §4]. This will be the subject of McDuff–Schlenk [19].

The nature of the ball packing problem. We now explain the results of [17, 15, 1, 13] that first convert the symplectic packing problem for balls into a question of understanding the symplectic cone of the k -fold blow up X_k of $\mathbb{C}P^2$, and then explain the structure of that cone. The *symplectic cone* of an oriented manifold is the set of cohomology classes of M with symplectic representatives compatible with the given orientation. By Li–Liu [13], when $M = X_k$ this is a disjoint union of (connected) subcones, each consisting of forms ω with a given canonical class $K = -c_1(M, \omega)$. Moreover, the group of diffeomorphisms of X_k acts transitively on these subcones. Fix $K := -3L + \sum E_i$, where $L := [\mathbb{C}P^1]$ and E_1, \dots, E_k are the exceptional divisors, and define $\mathcal{C}_K(X_k) \subset H^2(X_k, \mathbb{R})$ to be the set of classes represented by symplectic forms with canonical class K . Also, denote the Poincaré duals of L, E_i by ℓ, e_i .

Proposition 1.9 (McDuff–Polterovich [17]). *It is possible to embed k balls with weights \underline{w} into $\mathbb{C}P^2(1)$ or $\mathring{B}^4(1)$ if and only if the class $a_{\underline{w}} := \ell - \sum w_i e_i$ lies in $\mathcal{C}_K(X_k)$.*

Therefore the problem is equivalent to understanding $\mathcal{C}_K(X_k)$. Though not formulated in these terms, Lemma 1.1 of [15] in essence describes the closure of $\mathcal{C}_K(X_k)$. (The argument is sketched below.) That paper solved the uniqueness question for symplectic packings. In [1, 2], Biran recast this lemma in terms of the symplectic cone, using it to solve the existence question for packings by equal balls. However the cone itself was first described in Li–Liu [13]. They consider the set $\mathcal{E}_K(X_k) \subset H_2(X_k; \mathbb{Z})$ of classes E with $K \cdot E = 1, E^2 = -1$ that can be represented by smoothly embedded -1 spheres. By Theorems B and C in Li–Li [12], given any symplectic form ω with canonical class K , each $E \in \mathcal{E}_K(X_k)$ can be represented by an ω -symplectically embedded -1 sphere. Thus the above set $\mathcal{E}_K(X_k)$ is the same as that used in [17, 15, 1].

Proposition 1.10 (Li-Liu, [13]).

$$\mathcal{C}_K(X_k) = \{a \in H^2(X_k) : a^2 > 0, a(E) > 0 \text{ for all } E \in \mathcal{E}_K(X_k)\}.$$

Sketch of proof. Let ω be any symplectic form on X_k with canonical class K . It follows from the work of Kronheimer and Mrowka [11] on wall crossing in Seiberg–Witten theory that for all $a \in H^2(X_k; \mathbb{Q})$ with $a^2 > 0$ the class qa has nontrivial Seiberg–Witten invariant for all sufficiently large integers q . Therefore, by work of Taubes [24], the Poincaré dual $PD(qa)$ has a J -holomorphic representative for every ω -tame J . Moreover, if $a(E) \geq 0$ for all $E \in \mathcal{E}$ and J is generic this representative is a connected and embedded submanifold. By inflating along this submanifold one can construct a family of symplectic forms ω_t with $\omega_0 = \omega$ and such that $[\omega_t]$ converges to a as $t \rightarrow \infty$. Therefore, the class a is in the closure of $\mathcal{C}_K(X_k)$; see [15, Lemma 1.1]. A more careful version of this argument shows that a must in fact be in $\mathcal{C}_K(X_k)$ provided that $a(E) > 0$ for all E ; see [13, §4]. \square

The arguments in §2 below show explicitly how to use these ideas to construct full packings by the ellipsoids $E(1, k)$ for $k \in \mathcal{N}$ and hence by equal balls. The general existence question is fully understood when $k < 9$ since in that case \mathcal{E}_K is finite and is easily enumerated. However, it is not always so easy to answer specific problems when $k \geq 9$ since \mathcal{E}_K is more complicated. We shall return to this question in [19].

Remark 1.11. The above results imply there are two obstructions to the existence of an embedding from one ellipsoid to another. If $a_{\underline{w}}$ is the class in X_k of the corresponding packing problem, one needs:

- (i) $a_{\underline{w}}^2 > 0$, and
- (ii) $a_{\underline{w}}(E) > 0$ for all $E \in \mathcal{E}_K(X_k)$.

The first condition is equivalent to the volume obstruction, while the second is a generalization of the condition used by Gromov [6] to find a packing obstruction when $k = 2, 5$. It is given by the rigid J -holomorphic spheres in X_k , and hence should appear as a genus zero holomorphic trajectory in other contexts. However, note that although the classes $E \in \mathcal{E}_K$ are rigid in the blow up, they correspond in $\mathbb{C}P^2$ to curves that satisfy some constraints, e.g. they might have to go through a certain number of points with given multiplicities. Moreover, in general these constraints may not be purely homological, but may involve descendents: cf. the blow down formulas in [9]. Thus, for example, if one tried to understand the obstructions to embedding $E(m, n)$ into $E(m', n')$ by looking at the properties of the induced cobordism between their boundaries then these curves should appear in one of the higher dimensional (but genus zero) SFT moduli spaces and hence should be visible, provided that one works in a context that takes these higher dimensional spaces into account.

Organization of the paper. §2 considers the problem of embedding ellipsoids of the form $E(1, k)$ into balls, where $k \in \mathbb{N}$. The case $k = d^2$ is particularly simple and is treated first. Theorem 1.1 is proved in §2.2. In order to deal with general integral ellipsoids one must understand exactly how to approximate them by chains of spheres.

This is the subject of §3.1. Theorem 1.5, Corollary 1.6 and Proposition 1.7 are proved in §3.2.

Acknowledgements. I was inspired to think about this problem by discussions with Hofer and Guth, who in [7] recently solved (in the negative) Hofer’s question about the existence of higher dimensional capacities. I also thank Tolman for giving me an advance copy of her paper [25], and Schlenk for his encouragement and useful comments.

2. EMBEDDING $E(1, k)$ INTO A BALL.

We first give a direct geometric construction for embedding $E(1, d^2)$ into a ball. We then prove Theorem 1.1.

2.1. **The case $k = d^2$.** This section proves the following result.

Proposition 2.1. $E(1, d^2)$ fully fills $B(d)$.

Denote by $\Delta(m, n)$ the triangle with outward conormals $(-1, 0)$, $(0, -1)$, (m, n) and vertices at $(0, 0)$, $(n, 0)$, $(0, m)$; see Figure 2.1. If m, n are integers then the open ellipsoid $\mathring{E}(m, n) \subset \mathbb{C}^2$ is invariant under the obvious T^2 action and is taken by the moment map onto the “interior” $\mathring{\Delta}(m, n)$ of $\Delta(m, n)$, which we define to be the complement of its slanted edge. The closed triangle $\Delta(m, n)$ is the moment polytope of a weighted projective space, but we sometimes think of it as the moment polytope of $E(m, n)$ since it is the image of $E(m, n)$ under the moment map.

Recall that if (m_1, n_1) , (m_2, n_2) are the outward conormals to two successive edges (ordered anticlockwise) of a moment polytope in \mathbb{R}^2 then their intersection is the image of an orbifold point of order k iff

$$(2.1) \quad \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} = k.$$

In particular, it is smooth iff $k = 1$. We shall call a moment polytope *smooth* if all its vertices are smooth. (For basic information on toric geometry in the present context see Symington [23] and Traynor [26].)

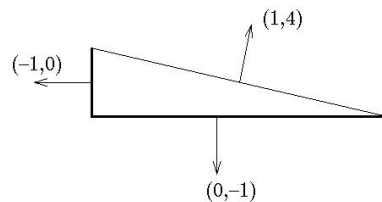


FIGURE 2.1. A toric model of the open ellipsoid $\mathring{E}(1, 4)$; it does not include the slanted edge.

Just as in the case of balls, if $\lambda E(m, n)$ embeds in (M, ω) then one can cut it out and collapse the boundary along the characteristic flow to obtain an orbifold \overline{M} with an

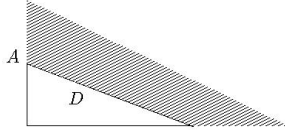


FIGURE 2.2. Toric model of an ellipsoidal blow up

exceptional divisor D (the heavy line in Figure 2.2). In general, the two new vertices are singular points of \overline{M} . However, in the case of $\lambda E(1, k)$ there is just one singular point on D at A . For more details, see [5].

The main idea of the current note is that instead of working with the orbifold \overline{M} we can cut away more (i.e. blow up further) in order to get a smooth manifold. The case $(m, n) = (1, k)$ is particularly simple; we simply need to blow up $k - 1$ more times. In the toric picture this amounts to cutting the polytope along lines with conormals $(-1, -1), (-1, -2), (-1, -3), \dots, (-1, -k)$. The curve D is then transformed into an exceptional curve in a smooth manifold. As we see below, it is possible to carry out this process omitting all mention of orbifolds. We explain it first in the case $k = 4$.

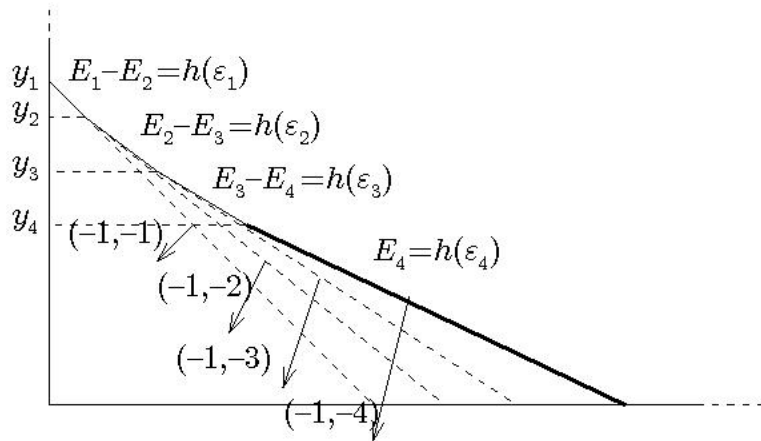


FIGURE 2.3. The three blow ups needed to resolve A in the case $E(1, 4)$. The four edges are named ϵ_i and their homology classes $h(\epsilon_i)$.

When $k = 4$, this blow up process gives a configuration of 4 spheres intersecting transversally, the -1 -sphere C_4 in class E_4 , and three -2 -spheres $C_i, i = 1, 2, 3$, each in class $E_i - E_{i+1}$: see Figure 2.3. If we start off in a large ball or in a large $\mathbb{C}P^2$, we can construct such a configuration. Namely, start with the first quadrant; form E_1 by a cut with conormal $(-1, -1)$ and size $\lambda + \delta_1$. (The size of the cut is given by the affine length¹ of the resulting exceptional divisor, and hence in this case is given by y_1 , the

¹The *affine length* $\alpha(\epsilon)$ of an edge ϵ of a moment polytope can be measured as follows. Take any affine transformation Φ of \mathbb{R}^2 that preserves the integer lattice and is such that $\Phi(\epsilon)$ lies along one of

height of the point where the cut meets the y axis.) Then cut off most of E_1 by a cut with conormal $(-1, -2)$ and size $y_2 = \lambda + \delta_2 < y_1$. Make one more cut along E_3 of size $y_3 = \lambda + \delta_3 < y_2$. Then E_4 is what is left of D . It is easy to check that it has size $y_4 = 4\lambda - (\lambda + \delta_1) - (\lambda + \delta_2) - (\lambda + \delta_3) =: \lambda - \delta_4$, where $\delta_4 = \sum_{i \leq 3} \delta_i$.

Let us denote by $\mathcal{N}(\widehat{\mathcal{C}}_4(\lambda, \delta))$ a small open neighborhood in the toric model of the final configuration $\widehat{\mathcal{C}}_4(\lambda, \delta)$ of 4 spheres. Thus $\widehat{\mathcal{C}}_4(\lambda, \delta)$ is a configuration of 4 symplectic spheres C_1, \dots, C_4 where C_i intersects C_{i+1} transversally for $i = 1, 2, 3$ (and there are no other intersection points), where C_4 has self-intersection -1 and size $\lambda - \delta_4$ and where the other C_i have self-intersection -2 and (positive) sizes $\delta_1 - \delta_2, \delta_2 - \delta_3$ and $\delta_3 + \delta_4$. In the following, we assume that $\delta_1 > \delta_2 > \delta_3 > 0$ and that $\delta_4 = \delta_1 + \delta_2 + \delta_3$.

Lemma 2.2. *Any embedding of $\widehat{\mathcal{C}}_4(\lambda, \delta)$ into (M, ω) is isotopic to one that extends to an embedding of $\mathcal{N}(\widehat{\mathcal{C}}_4(\lambda, \delta))$.*

Proof. Note that in any smooth toric manifold the spheres represented by 2 edges meet orthogonally since we can put them on the axes by an affine transformation. This may not be the case for the given embedding of $\widehat{\mathcal{C}}_4(\lambda, \delta)$. However, one can slightly perturb this embedding so that the different spheres do meet orthogonally. (A similar point occurs in the proof of [18, Thm 9.4.7]. See also [18, Ex. 9.4.8].) It then follows from the symplectic neighborhood theorem that the embedding extends to $\mathcal{N}(\widehat{\mathcal{C}}_4(\lambda, \delta))$. \square

Denote by $X_4(\mu; \lambda, \delta)$ the 4-point blow up of the projective plane in which the line has symplectic area μ and we have blown up 4 times by the amounts $\lambda + \delta_i, i = 1, 2, 3$ and $\lambda - \sum \delta_i$. If $\lambda < 1$ we can choose the δ_i so that each $\lambda + \delta_i < 1$. Because $\mathbb{C}P^2$ can be fully filled by 4 balls of equal size, we can therefore construct $X_4(2; \lambda, \delta)$ for any $\lambda < 1$ and sufficiently small $\delta := (\delta_1, \delta_2, \delta_3)$ as above.

Lemma 2.3. *$\lambda E(1, 4)$ embeds into the interior of $B(2)$ iff $\widehat{\mathcal{C}}_4(\lambda, \delta)$ embeds into the complement of a line in $X_4(2; \lambda, \delta)$ for some small δ .*

Proof. If $\lambda E(1, 4)$ embeds then this embedding extends to $(\lambda + \kappa)E(1, 4)$ for some small $\kappa > 0$. Hence $\lambda E(1, 4)$ has a standard neighborhood with a toric structure as above. Complete $B(2)$ to $\mathbb{C}P^2(2)$ and then blow up as explained above to get an embedding of $\widehat{\mathcal{C}}_4(\lambda, \delta)$ into the complement of a line in $X_4(2; \lambda, \delta)$.

Conversely, suppose that $\widehat{\mathcal{C}}_4(\lambda, \delta)$ embeds into the complement of a line in $X_4(2; \lambda, \delta)$. Then, by Lemma 2.2 we may suppose that some standard neighborhood $\mathcal{N}(\widehat{\mathcal{C}}_4(\lambda, \delta))$ also embeds. By Symington's discussion in [23] of toric models for the rational blow down, we may then "blow down" $\widehat{\mathcal{C}}_4(\lambda, \delta)$, i.e. perform a symplectic surgery along this nonsmooth divisor that adds the singular piece that was cut out when resolving the singularity. The resulting blow down manifold (M, ω) contains a symplectically embedded copy of $\mathbb{C}P^1$ of size 2 and a disjoint copy of $\lambda E(1, 4)$. Moreover, it is diffeomorphic to $\mathbb{C}P^2$. Therefore, by Gromov's uniqueness result for symplectic forms

the axes, and then measure the Euclidean length of $\Phi(\epsilon)$. Thus if ϵ has rational slope and endpoints on the integer lattice, $\alpha(\epsilon) = k + 1$ where k is the number of points of the integer lattice in the interior of ϵ . Note also that the divisor that is taken to ϵ by the moment map has symplectic area $\alpha(\epsilon)$.

on $\mathbb{C}P^2$ (cf [18, Ch. 9]), (M, ω) can be identified with $\mathbb{C}P^2(2)$, so that $M \setminus \mathbb{C}P^1$ is the interior of the standard ball $B(2)$. This completes the proof. \square

Remark 2.4. There is an obvious analog of this result for any k . An appropriate definition of $\widehat{\mathcal{C}}_k(\lambda, \delta)$ is explained in the proof of Theorem 1.1 at the end of this section.

Therefore we just need to embed $\widehat{\mathcal{C}}_4(\lambda, \delta)$ into the complement of a line in $X_4(2; \lambda, \delta)$. This is possible for small λ . We then will use the inflation process to increase λ . In this case, the construction can be done entirely explicitly: there is no need to use Seiberg–Witten theory.

Proof of Proposition 2.1.

Step 1: *Explicit embedding of $\widehat{\mathcal{C}}_4(\lambda, \delta)$.* Start with $\mathbb{C}P^2(2)$. Let $Q \subset \mathbb{C}P^2$ be a smooth conic, L a line and p_1 a point on Q but not L . Blow up at p_1 with size $\lambda + \delta_1$ where λ is small and δ_1 is tiny. Let p_2 be the intersection of the exceptional divisor E_1 with the proper transform Q_1 of Q and blow up at p_2 with size $\lambda + \delta_2$ to get an exceptional divisor E_2 . Now repeat this twice more, blowing up at

$$p_2 \in E_2 \cap Q_2, \quad Q_2 := \text{proper transform of } Q_1$$

by $\lambda + \delta_3$ to get exceptional divisor E_3 and finally blowing up at

$$p_3 \in E_3 \cap Q_3, \quad Q_3 := \text{proper transform of } Q_2$$

by $\lambda - \delta_4$ (where $\delta_4 := \sum_{i=1}^3 \delta_i$) to get C_4 in the 4-fold blow up X_4 . For $i < 4$ denote by C_i the proper transform of E_i in the next blow up. Then $[C_i] = E_i - E_{i+1}$ for $i \leq 3$ and $[C_4] = E_4$. Thus we have constructed a copy of the configuration $\widehat{\mathcal{C}}_4(\lambda, \delta)$ in $X_4(2; \lambda, \delta)$. Denote the symplectic form on $X := X_4(2; \lambda, \delta)$ by ω_0 .

Note that if λ is sufficiently small we may assume that none of these blowups affect L . The conic Q becomes a curve Q_0 in class $2L - E_1 - E_2 - E_3 - E_4$ and so has area $4 - 4\lambda$. By construction Q_0 meets C_4 once but not $C_i, i < 4$. Moreover,

$$\begin{aligned} \int_{C_1} \omega_0 &= \delta_1 - \delta_2, & \int_{C_2} \omega_0 &= \delta_2 - \delta_3, & \int_{C_3} \omega_0 &= \delta_3 + \delta_4 \\ \int_{C_4} \omega_0 &= \lambda - \delta_4, & \int_{Q_0} \omega_0 &= 4 - 4\lambda, & \int_L \omega_0 &= 2. \end{aligned}$$

Step 2: *The inflation process.* We will inflate (X, ω_0) along Q_0 . Note that $Q_0 \cdot Q_0 = 0$. Therefore we may identify a neighborhood $\mathcal{N}(Q_0)$ with the product $(Q_0 \times D^2, \omega_Q \times \alpha)$ where α is some area form on D^2 . By perturbing C_4 and the line L and then shrinking $\mathcal{N}(Q_0)$ if necessary, we may assume that

- $C_i \cap \mathcal{N}(Q_0) = \emptyset$ for $i \leq 3$;
- $C_4 \cap \mathcal{N}(Q_0)$ is a flat disc $pt \times D^2$;
- $L \cap \mathcal{N}(Q_0)$ is the union of two disjoint flat discs $pt \times D^2$.

Let β be a nonnegative form on the two disc D^2 with support in its interior and $\int_{D^2} \beta = 1$. Define $\omega_t := \omega_Q \times (\alpha + t\beta)$ in $\mathcal{N}(Q_0)$ and equal to the original symplectic

form ω_0 outside $\mathcal{N}(Q_0)$. This is clearly symplectic everywhere. Then the integral of ω_t over Q_0 and the $C_i, i \leq 3$, is constant while

$$\int_{C_4} \omega_t = \lambda - \delta_4 + t, \quad \int_L \omega_t = 2 + 2t.$$

Given any $\lambda_0 < 1$, choose T so that $\lambda' := \frac{\lambda - \delta_4 + T}{1 + T} \geq \lambda_0$ and set $\tau := \omega_T / (1 + T)$. Define $\delta'_i = \delta_i / (1 + T)$. By the uniqueness of symplectic forms on the blow ups of $\mathbb{C}P^2$ (cf. [15]) we may identify (X, τ) with $X_4(2; \lambda', \delta')$, and may also identify the configuration with $\widehat{C}_4(\lambda', \delta')$. Since the configuration is disjoint from a line by construction, we can now deduce the proposition from Lemma 2.3.

Step 3: Completion of the proof. This argument extends immediately to the case $d > 2$: simply replace Q in the above construction by a degree d curve in $\mathbb{C}P^2(d)$, and blow up d^2 times. \square

Remark 2.5. Because the above argument uses symplectic inflation, it does not give a completely explicit geometric construction of the embedding. It is not clear whether such a construction exists. (The construction in Opshtein [20] gives an embedding into $\mathbb{C}P^2$.) The best results obtained so far by explicit construction are those of Schlenk, who used a multiple symplectic folding technique to show that $E(1, 4)$ embeds in $B(\mu)$ for μ approximately equal to 2.7, see [22, Ch. 3.3].

2.2. The general case. We now prove Theorem 1.1.



FIGURE 2.4. Three copies of $\mathring{B}(1)$ embed in $E(1, 3)$.

Lemma 2.6. $E(1, k)$ contains the disjoint union of k open balls $\mathring{B}(1)$.

Proof. Denote by $\check{\Delta} \subset \mathbb{R}^2$ the open triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. As is clear from Figure 2.4, there are integral affine transformations $A_1 = id, A_2, \dots, A_k$ of \mathbb{R}^2 such that the k triangles $A_1(\check{\Delta}), \dots, A_k(\check{\Delta})$ embed in the moment polytope of $E(1, k)$. On the other hand $\check{\Delta}$ is the moment polytope of the open subset

$$U = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_i \neq 0, |z_1|^2 + |z_2|^2 < 1\} \subset B(1),$$

which contains an embedded image of the open ball $\mathring{B}(1)$ by Traynor [26]. \square

Proof of Theorem 1.1. If $E(1, k)$ embeds in $\mathring{B}(\mu)$ then $(1 + \kappa)E(1, k)$ also embeds for some $\kappa > 0$. But Lemma 2.6 implies that $(1 + \kappa)E(1, k)$ contains k disjoint closed balls $B(1)$. Hence so does $\mathring{B}(\mu)$.

Conversely, if k copies of $B(1)$ embed in $\mathring{B}(\mu)$, choose $\mu_0 < \mu$ so that their image is contained in $\mathring{B}(\mu_0)$. By Proposition 1.9 the class $a := \mu_0 \ell - \sum_{i=1}^k e_i \in H^2(X_k)$ has

a symplectic representative with canonical class K and so lies in $\mathcal{C}_K(X_k)$. Hence, as in Proposition 1.10, $a(E) > 0$ for all $E \in \mathcal{E}(X_k)$. Without loss of generality, we may suppose that $\mu_0 \in \mathbb{Q}$ and then choose q so large that $qa \in H^2(X_k; \mathbb{Z})$ and $PD(qa)$ has nontrivial Gromov invariant.

Now let $\widehat{\mathcal{C}}_k(\lambda, \delta)$ be a configuration of k symplectic spheres C_1, \dots, C_k such that

- (a) $C_i \cap C_j \neq \emptyset$ iff $|i - j| < 2$,
- (b) $C_i^2 = -2, i < k$, and $C_k^2 = -1$, and
- (c) $\text{area}(C_i) = \delta_i - \delta_{i+1}$ if $1 \leq i < k-1$, $\text{area}(C_{k-1}) = \delta_{k-1} + \delta_k$, and $\text{area}(C_k) = \lambda - \delta_k$, where $\delta_1 > \dots > \delta_{k-1} > 0$ and $\delta_k = \sum_{i < k} \delta_i$.

If λ and the δ_i are sufficiently small, the blow up construction in §2.1 shows that there is a symplectic form ω_0 on X_k with $\int_L \omega_0 = 1$ and such that the configuration $\widehat{\mathcal{C}}_k(\lambda, \delta)$ embeds in the complement of a line L in (X_k, ω_0) in such a way that C_i lies in class $E_i - E_{i+1}$ for $i < k$ and $[C_k] = E_k$. (Observe here that the class $[\omega_0]$ is determined by λ and the δ_i : it will not equal a .) Moreover, $\widehat{\mathcal{C}}_k(\lambda, \delta)$ has a toric neighborhood $\mathcal{N}(\widehat{\mathcal{C}}_k(\lambda, \delta))$ constructed just as in the case $k = 4$, and by Lemma 2.2 we may suppose that $\mathcal{N}(\widehat{\mathcal{C}}_k(\lambda, \delta))$ also embeds disjointly from L . For simplicity, we shall identify $\mathcal{N}(\widehat{\mathcal{C}}_k(\lambda, \delta))$ with its image in (X_k, ω_0) .

Denote by $\mathcal{J}_{\mathcal{N}}$ the set of ω -tame J for which $\widehat{\mathcal{C}}_k(\lambda, \delta)$ and the line L are holomorphic. Because $PD(qa)$ has nontrivial Gromov invariant and $qa(E) > 0$ for all $E \in \mathcal{E}_K$, $PD(qa)$ is represented by an embedded J -holomorphic curve Q for every ω -tame J that is sufficiently generic. Since no smooth curve in class $PD(qa)$ is represented entirely in $\widehat{\mathcal{C}}_k(\lambda, \delta) \cup L$, it follows from [18, Ch 3] that we can take J to be a generic element of $\mathcal{J}_{\mathcal{N}}$. Then, by positivity of intersections, the fact that $a(C_i) = 0, i < k$, implies that $Q \cap C_i = 0$. Further, we may perturb Q so that it intersects C_k transversally q times and L transversally $q\mu_0$ times.

Now inflate along Q . This construction gives a family $\omega_t, t \geq 0$, of symplectic forms on X_k lying in class $[\omega_0] + tqa$ that equal ω_0 outside a small neighborhood of Q and restrict on C_k (resp. L) to a symplectic form of area $\omega_0(C_k) + qt = \lambda - \delta_k + qt$ (resp. $1 + qt\mu_0$). Thus $\mathcal{N}(\widehat{\mathcal{C}}_k(\lambda - \delta_k + qt, \delta))$ embeds in (X_k, ω_t) . Therefore, by Lemma 2.3 (see also Remark 2.4), $(\lambda + qt)E(1, k)$ embeds in $\mathring{B}(1 + qt\mu_0)$ for all $t > 0$. Hence

$$E(1, k) \xrightarrow{s} \frac{1}{\lambda + qt} \mathring{B}(1 + qt\mu_0).$$

Since $\mu_0 < \mu$, $\frac{1 + qt\mu_0}{\lambda + qt} < \mu$ for large t . Hence the result. \square

3. EMBEDDING ELLIPSOIDS INTO ELLIPSOIDS.

We first show how to find the weights of the ball embedding problem that is equivalent to a given ellipsoidal embedding problem. Theorem 1.5 is proved in §3.2.

3.1. Toric approximations to ellipsoids. We begin by discussing inner and outer approximations. Throughout, (m, n) are mutually prime and $0 < m \leq n$. As in the case $(m, n) = (1, k)$, it is possible to approximate the moment polytope $\Delta(m, n)$ of $E(m, n)$ by a smooth polytope Δ' by blowing it up appropriately. Since every moment polytope is a blow up of $\Delta(1, 1)$ (up to scaling and integral affine transformation), one can equivalently start with $\Delta(n, n)$ and by a sequence of blow ups arrive at a polytope Δ' lying inside $\Delta(m, n)$ and with one conormal equal to (m, n) . One can then adjust the side lengths of this polytope to make its edge ϵ_N with conormal (m, n) coincide with part of the corresponding edge of $\Delta(m, n)$ and also the region $\Delta(m, n) \setminus \Delta'$ arbitrarily small (in area). We will call such Δ' an *inner approximation* to $\Delta(m, n)$; see Figure 3.1(ii). This is the relevant approximation when $E(m, n)$ is the target of the embedding. If $E(m, n)$ is the source, then as in §2 one should look for *outer approximations* $\Delta'' \supset \Delta(m, n)$ such that $\Delta'' \setminus \Delta(m, n)$ is small. These are essentially the same as inner approximations to $\Delta(n', n') \setminus \Delta(m, n)$ for $n' > n$: see Remark 3.9 (i).

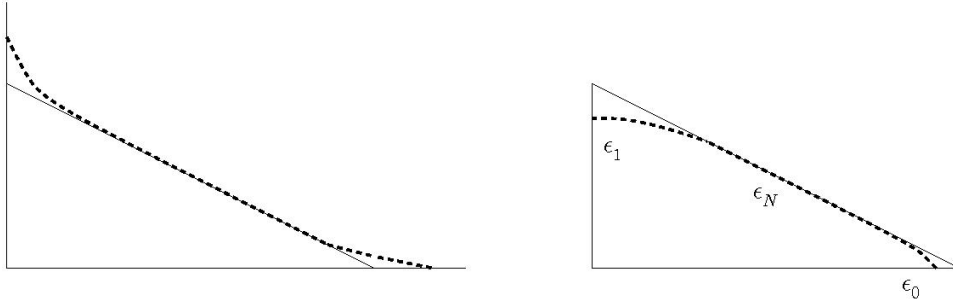


FIGURE 3.1. (i) is an outer approximation to $\Delta(m, n)$ while (ii) is an inner approximation

For clarity, let us first concentrate on inner approximations. There are many possible choices of approximation. However, as we show below, there is a unique minimal sequence of blow ups of $\Delta(n, n)$ with exceptional divisors E_1, E_2, \dots, E_N such that the conormal to the edge ϵ_N given by the last blow up is (m, n) and so that none of the other new edges have self intersection -1 .² Let us denote by ϵ_0 the edge with conormal $(1, 1)$ and by ϵ_i the edge created by the i th blow up. Further, denote by

- $h(\epsilon_i)$ the homology class of the edge in the N -fold blow up X_N ;
- $\alpha(\epsilon_i)$ the affine length of ϵ_i ;
- $\nu(\epsilon_i)$ the conormal of ϵ_i .

Then our conditions imply:

- $h(\epsilon_N) = E_N$;
- for $1 \leq i < N$, $h(\epsilon_i) = E_i - E_{i_1} - \dots - E_{i_k}$ for suitable $i < i_1 < \dots < i_k$ and $k > 0$.

²If one of the other edges did have self-intersection -1 , it would have to be disjoint from ϵ_N and hence could be blown down.

- $h(\epsilon_0) = L - E_{0_1} - \cdots - E_{0_k}$ for suitable indices $0_1 < \cdots < 0_k$, where L is the class of the line in $\mathbb{C}P^2$.

We shall first discuss the blow up process outlined above and then consider how to choose the lengths $\alpha(\epsilon_i)$.

Construction of the minimal blow up sequence for (m, n) . The blow up process replaces the intersection of two adjacent edges with conormals $(p, q), (p', q')$ by a new edge with conormal $(p'', q'') = (p + p', q + q')$. Thus the three fractions p/q are related by the identity

$$\frac{p''}{q''} = \frac{p + p'}{q + q'}.$$

Hence they are adjacent terms in the *Farey sequence* \mathcal{F}_K , $K := q + q'$. (Recall from Hardy–Wright [8, Ch.III] that \mathcal{F}_K is the finite sequence obtained by arranging the fractions p/q , where $0 \leq p \leq q$ are relatively prime and $q \leq K$, in order of increasing magnitude.) It is well known that \mathcal{F}_K can be constructed by starting with the fractions $\frac{0}{1}$ and $\frac{1}{1}$ and then repeatedly inserting the fractions $\frac{p+p'}{q+q'}$ with $q + q' \leq K$ between any two neighbors $\frac{p}{q}, \frac{p'}{q'}$. Since this precisely corresponds to the blow up procedure, it follows that one always can find a sequence of blow ups that starts from $\epsilon_0 = (1, 1)$ and $\epsilon_1 = (0, 1)$ and ends up with an edge ϵ_N with conormal (m, n) . Moreover, to find a minimal sequence one should include $\frac{p+p'}{q+q'}$ in the sequence only if $\frac{m}{n}$ lies between the points $\frac{p}{q}$ and $\frac{p'}{q'}$. We denote the corresponding connected chain of adjacent edges by $\mathcal{E}(m, n)$. This chain, when ordered by increasing $\frac{m}{n}$, starts with $\epsilon_1 = (0, 1)$, includes an edge ϵ_N with conormal (m, n) and ends with $\epsilon_0 = (1, 1)$. (Note that these edges are numbered according to the order in which the blow ups are performed, not by adjacency.)

For any pair (m, n) with $m < n$ we can construct a convex chain of edges $\mathcal{E}(m, n)$ of this form, such that all edges except for the last one ϵ_N are very short and so that ϵ_N is almost all of the slanted edge of $\Delta(m, n)$. To do this, first place ϵ_0 so that it meets the x axis at $x_0 := (n - \delta_0, 0)$ for small $\delta_0 > 0$ and make the first cut ϵ_1 so that it meets the y axis at $y_1 := (0, m - \delta_0 - \delta_1)$ for small $\delta_1 > 0$. Then perform all subsequent blow ups so that all the edges ϵ_i have positive length and so that ϵ_N coincides with part of the line $mx + ny = mn$: see Figure 3.2.

Note that if the conormal (p, q) of ϵ_i has $\frac{p}{q} < \frac{m}{n}$ then the line of the corresponding cut must meet the y axis at some point $(0, y_i)$ with $y_1 < y_i < m$, while if $\frac{p}{q} > \frac{m}{n}$ it meets the x -axis at $(x_i, 0)$ where $x_0 < x_i < n$. It follows that the edges for $i < N$ must be very short, while ϵ_N is almost all the slanted edge of $\Delta(m, n)$.

This chain of edges $\mathcal{E}(m, n)$ lies in $\Delta(m, n)$ and bounds a smooth subpolytope R_N of $\Delta(m, n)$. The above sequence of blow ups gives a way to construct the polytope R_N from $\Delta(n - \delta_0, n - \delta_0)$ by performing a sequence of blow ups with conormals $\nu(\epsilon_1), \dots, \nu(\epsilon_N)$. We shall denote by $R_i, i \geq 1$ the region obtained after the i th blow up and by γ_i the new edge of R_i formed by the i th blow up. Thus γ_i is an extension of ϵ_i . Notice that $R_i = R_{i-1} \setminus \Delta_i$, where the triangle Δ_i is equivalent under the action

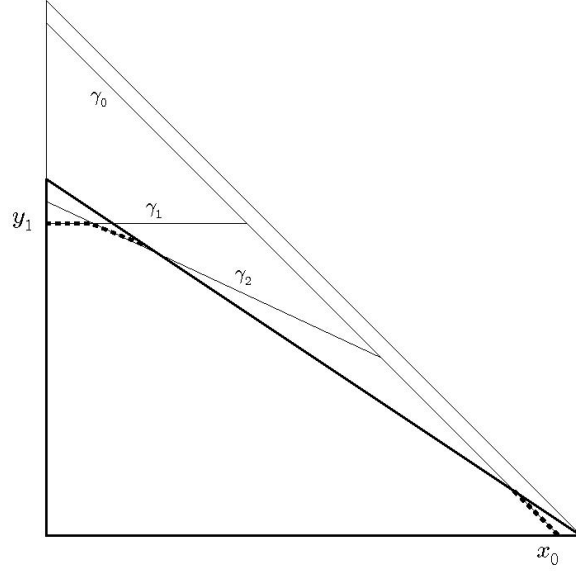


FIGURE 3.2. An inner approximation to $\Delta(2, 3)$. Here $x_0 = n - \delta_0$ and $y_1 = m - \delta_0 - \delta_1$ are the points where the edges ϵ_0, ϵ_1 meet the axes. The cuts γ_i are also labelled.

of $\text{SL}(2, \mathbb{Z})$ to $\mu_i \Delta(1, 1)$ and where μ_i is the affine length of the cut edge γ_i . It follows from this construction that the homology classes $h(\epsilon_i)$ of the edges ϵ_i are given by

$$(3.1) \quad h(\epsilon_i) = E_i - \sum_{j \in S_i} E_j, \text{ where } S_i = \{j > i : \text{the edge } \gamma_j \text{ intersects } \gamma_i\}.$$

As well as this minimal blow up sequence, we shall need an integral homology class $V_{m,n} := k_0 L - \sum_{i=1}^N k_i E_i$ with the property that

$$(3.2) \quad V_{m,n} \cdot h(\epsilon_N) = 1, \quad V_{m,n} \cdot h(\epsilon_i) = 0, \quad 0 \leq i < N.$$

(This vector $V_{m,n}$ will give the weights of the corresponding ball embedding problem.) For example, if $(m, n) = (7, 12)$ then the conormals (p_i, q_i) to the sequence of edges starting at $\epsilon_1 = (0, 1)$ and going to $\epsilon_0 = (1, 1)$ have slopes

$$(3.3) \quad \frac{p}{q} = \frac{0}{1}, \frac{1}{2}, \frac{4}{7}, \frac{7}{12}, \frac{3}{5}, \frac{2}{3}, \frac{1}{1}.$$

These edges are numbered

$$\epsilon_1, \epsilon_2, \epsilon_5, \epsilon_6, \epsilon_4, \epsilon_3, \epsilon_0$$

according to the order of the blow ups. They have classes

$$\begin{aligned} h(\epsilon_6) &= E_6, & h(\epsilon_5) &= E_5 - E_6, & h(\epsilon_4) &= E_4 - E_5 - E_6, \\ h(\epsilon_3) &= E_3 - E_4, & h(\epsilon_2) &= E_2 - E_3 - E_4 - E_5, & h(\epsilon_1) &= E_1 - E_2, \\ h(\epsilon_0) &= L - E_1 - E_2 - E_3 \end{aligned}$$

as one can check by performing the relevant blow ups. Hence,

$$(3.4) \quad V_{7,12} = 12L - 5E_{12} - 2E_{34} - E_{56},$$

where $E_{j\dots k} := \sum_{i=j}^k E_i$. Note that $V_{7,12}^2 = 7 \cdot 12 = 84$. This is a general fact, which is known in other contexts; cf. Remark 3.9. However we include a proof here for completeness.

Lemma 3.1. *For each relatively prime pair (m, n) with $0 \leq m < n$ there is a unique primitive integral homology class $V_{m,n} := nL - \sum_{i=1}^N k_i E_i$ satisfying (3.2). Moreover, $V_{m,n}^2 = mn$.*

Proof. Let us call the coefficient k_i of E_i in $V_{m,n}$ the *label* of the edge ϵ_i . Equation (3.1) implies that $V_{m,n}$ will satisfy (3.2) if the labels k_N, k_{N-1}, \dots, k_0 are assigned as follows.

Set $k_N = 1$. Given $k_j, j > i$, define k_i to be the sum of the labels of the edges $\epsilon_j, j \in S_i$.

Since the classes $h(\epsilon_i), i = 0, \dots, N$, generate $H_2(X_N)$ there is obviously a unique $V_{m,n}$ satisfying (3.2). Therefore, it remains to check that $k_0 = n$ and that $V_{m,n}^2 = mn$.

Using induction, we shall show that for all (m, n) such that $m + n \leq K$ then $V_{m,n}$ as defined above has the required properties. Moreover if (m', n') is another pair with $m' + n' \leq K$ and if $|mn' - m'n| = 1$ then

$$(3.5) \quad 2V_{m,n} \cdot V_{m',n'} = 1 + mn' + m'n.$$

The base case is $K = 2$. Then $V_{0,1} = L - E_1$ and $V_{1,1} = L$. The required properties are easily verified.

Suppose the result is known for $m + n < K$ and consider a pair (m'', n'') with $m'' + n'' = K$. Let $(m, n), (m', n')$ be the neighbors of (m'', n'') in the Farey sequence $\mathcal{F}_{n''}$, named so that $n < n'$. Then $m + n < K, m' + n' < K$ and also $|mn' - m'n| = 1$. (Note that this implies $|m''n - mn''| = 1, |m''n' - m'n''| = 1$.) To complete the inductive step, we will show that

$$(3.6) \quad V'' := V_{m,n} + V_{m',n'} - E_{N''}$$

satisfies all the conditions required of $V_{m'',n''}$, where the edge with conormal (m'', n'') is called $\epsilon_{N''}$.

To make sense of this formula, note that because $n < n', \frac{m}{n}$ occurs as part of the Farey sequence $\mathcal{F}_{n'}$. Hence, because $|mn' - m'n| = 1, \frac{m}{n}$ and $\frac{m'}{n'}$ are adjacent in this Farey sequence. It follows that all the edges in $\mathcal{E} := \mathcal{E}(m, n)$ occur in $\mathcal{E}' := \mathcal{E}(m', n')$: indeed \mathcal{E}' may be obtained from \mathcal{E} by repeatedly blowing up at the vertex of the edge ϵ_N “closest” to $\frac{m'}{n'}$. (For example, if $\frac{m}{n} < \frac{m'}{n'}$ then one blows up at the vertex of ϵ_N closest to the x -axis.) In particular the edge ϵ_N of \mathcal{E} , when considered as part of \mathcal{E}' , is adjacent to $\epsilon_{N'}$. Therefore we may consider the classes E_i that occur in $V := V_{m,n}$ to be a subset of those occurring in $V' := V_{m',n'}$ so that the above formula for V'' makes sense. Moreover, if $h'(\epsilon_i)$ denotes the class in $X_{n'}$ represented by ϵ_i then, for all $i \leq N$,

$h(\epsilon_i)$ is the class obtained from $h'(\epsilon_i)$ by setting $E_j = 0, j > N$. Similarly, V is the class obtained from V' by setting all $E_j, j > N$, to 0.

Now observe that, because (m, n) and (m', n') are the neighbors of (m'', n'') in $\mathcal{F}_{n''}$, $\mathcal{E}'' := \mathcal{E}(m'', n'')$ is obtained from \mathcal{E}' by adding one extra edge $\epsilon_{N''}$ between ϵ_N and $\epsilon_{N'}$. Hence, $N'' = N' + 1$. Further, if we denote the class of ϵ_i in \mathcal{E}'' by $h''(\epsilon_i)$,

$$h''(\epsilon_{N'}) = h'(\epsilon_{N'}) - E_{N''}, \quad h''(\epsilon_N) = h'(\epsilon_N) - E_{N''}, \quad h''(\epsilon_i) = h'(\epsilon_i), \quad i \neq N, N'.$$

It follows easily that if V'' is defined by equation (3.6) then the relations (3.2) hold. Hence $k''_0 = k_0 + k'_0 = n + n' = n''$. Moreover by equation (3.5)

$$\begin{aligned} (V'')^2 &= (V + V' - E_{N''})^2 \\ &= mn + m'n' + 2VV' - 1 \\ &= (m + m')(n + n') = m''n''. \end{aligned}$$

It remains to check that VV'' and $V'V''$ satisfy the analog of equation (3.5). This is left to the reader. \square

Example 3.2. To illustrate this, consider the case $(m'', n'') = (10, 17)$ with Farey neighbors $(m, n) = (3, 5), (m', n') = (7, 12)$. The edges in $\mathcal{E}(3, 5)$ have slopes

$$\frac{p}{q} = \frac{0}{1}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{1}{1},$$

and their classes are

$$E_1 - E_2, \quad E_2 - E_3 - E_4, \quad E_4, \quad E_3 - E_4, \quad L - E_1 - E_2 - E_3.$$

Thus $V_{3,5} = 5L - 2E_{12} - E_{34}$. Therefore, by (3.4), formula (3.6) gives

$$(3.7) \quad V_{10,17} = 17L - 7E_{12} - 3E_{34} - E_{567}.$$

In the above discussion each edge ϵ_i was assumed to have positive length, although it was very short for $i < N$. Now imagine performing these cuts so that these edges have zero length. In other words, at each stage construct R_i^0 by cutting out a vertex of R_{i-1}^0 together with the whole of the shorter adjacent edge. Thus each cut γ_i^0 has a vertex at $(0, m)$ or at $(n, 0)$ and the end result is $R_N^0 := \Delta(m, n)$. Thus these cuts decompose the triangle $T(m, n) := \Delta(n, n) \setminus \overset{\circ}{\Delta}(m, n)$ into a union of triangles, each equivalent to a multiple $\mu_i^0 \Delta(1, 1)$ of the standard triangle: see Figure 3.3. We will think of this decomposition of $T(m, n)$ as corresponding to a singular (nonsmooth) blow up of $\Delta(n, n)$.

We now show that the multiplicities μ_i^0 are precisely the weights k_i . Since the area of the cut triangles is $\sum_{i=1}^N (\mu_i^0)^2$ this shows that

$$n^2 - mn = \sum_{i=1}^N k_i^2,$$

which gives a geometric explanation for the quadratic relation $V_{m,n}^2 = mn$.

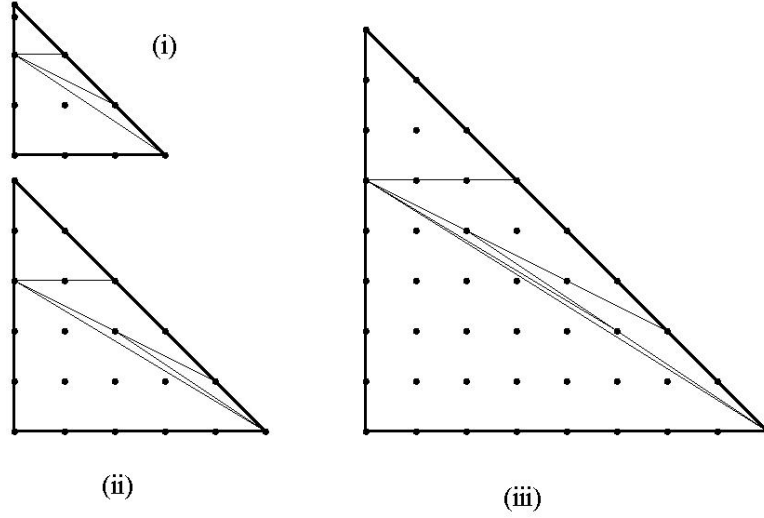


FIGURE 3.3. Blowing up $\Delta(n, n)$ to $\Delta(m, n)$ for (m, n) equal to (i) $(2, 3)$, (ii) $(3, 5)$ and (iii) $(5, 8)$. The points of the integral lattice are marked. Note that $V_{2,3} = 3L - E_{123}$, $V_{3,5} = 5L - 2E_{12} - E_{34}$ and $V_{5,8} = 8L - 3E_{12} - 2E_3 - E_{45}$. The sizes of the triangles in each decomposition are given by the coefficients of the E_i in each $V_{m,n}$.

Remark 3.3. It turns out that this geometric blow up procedure is very closely related to the construction of an appropriate continued fraction. This is easiest to see in the context of outer approximations: see Remark 3.9.

Lemma 3.4. Write $V_{m,n} = nL - \sum k_i E_i$. Then $\Delta(m, n)$ is the (nonsmooth) blow up of $\Delta(n, n)$ where the cuts have conormals $\nu(\epsilon_1), \dots, \nu(\epsilon_N)$ and weights k_1, k_2, \dots, k_N .

Proof. The coefficient μ_i^0 is just the affine length of the cut γ_i , i.e. it is the length of the corresponding edge of R_i^0 . This edge represents the class $\mu_i^0 E_i$ in $H_2(X_N)$. Thus $\mu_N^0 = 1$, the affine length of the slanted edge γ_N of $\Delta(m, n)$. One can now argue that $\mu_i^0 = k_i$ for $i = n_1, n_2, \dots, 1$ in turn. The point is that the i th cut leaves an “edge” ϵ_i in class $h(\epsilon_i)$ of length 0. But ϵ_i is the result of cutting γ_i by cuts of length $\mu_j^0, j \in S_i$. Therefore the result follows from formula (3.1) and the definition of the k_i given in the proof of Lemma 3.1. \square

Corollary 3.5. The open subset of $\mathbb{C}P^2(n)$ with moment polytope $\mathring{\Delta}(n, n) \setminus \Delta(m, n)$ contains N open balls of sizes k_1, \dots, k_N .

Proof. This holds as in Lemma 2.6. \square

With this notation in hand, we can now discuss inner approximations with more precision. As explained above, an inner approximation to $\Delta(m, n)$ is obtained by moving the edge of $\Delta(n, n)$ with conormal $(1, 1)$ a little closer to the origin (so that it meets

the x -axis at the point $(n - \delta_0, 0)$ for some $\delta_0 > 0$), and then slightly adjusting the size of all the subsequent blow ups from k_i to $k_i + \delta_i$ so as not to cut out quite all of an edge at each blow up. (See Figure 3.2. Note that some of the δ_i may be negative; cf. the construction of $\widehat{\mathcal{C}}_4(\lambda, \delta)$ in §2.) For suitable δ_i this will create a smooth polygonal arc $\mathcal{E}(m, n; \delta)$ whose edges $\epsilon_i, 0 \leq i \leq N$, have conormals $\nu(\epsilon_i)$ as described above.

Define

$$(3.8) \quad a_{m,n;\delta} := (n - \delta_0)\ell - \sum_{i=1}^N (k_i + \delta_i)e_i \in H^2(X_N),$$

where e_i is Poincaré dual to E_i , i.e. $e_j(E_i) = -\delta_{ij}$. Since $(n - \delta_0)\ell$ is the cohomology class of the symplectic form on $\mathbb{C}P^2(n - \delta_0)$, $a_{m,n;\delta}$ is the class of the symplectic form on X_N obtained from $\mathbb{C}P^2(n - \delta_0)$ by the blow up procedure explained above with the i th blow up of size $k_i + \delta_i$. Hence the affine length $\alpha(\epsilon_i)$ of the edge ϵ_i is

$$\alpha(\epsilon_i) = a_{m,n;\delta}(E_i) > 0.$$

Because $a_{m,n;0}$ is the Poincaré dual of $V_{m,n}$, all the edges of $\mathcal{E}(m, n; \delta)$ are very short except for ϵ_N which has affine length nearly 1.

Definition 3.6. We say that $\delta := (\delta_0, \dots, \delta_N)$ is **admissible** if:

- (i) $\delta_0, \delta_1 > 0$;
- (ii) the edges $\epsilon_0, \dots, \epsilon_N$ of $\mathcal{E}(m, n; \delta)$ have positive lengths $\alpha(\epsilon_i)$;
- (iii) $\mathcal{E}(m, n; \delta) \subset \Delta(m, n) \setminus \left(r \overset{\circ}{\Delta}(m, n) \right)$ where $r := 1 - \delta_0 - \delta_1$.

Note that condition (ii) implies that $\mathcal{E}(m, n; \delta)$ is a chain of edges with the same intersection properties as $\mathcal{E}(m, n)$. Hence the slopes decrease as one moves along $\mathcal{E}(m, n; \delta)$ from ϵ_1 to ϵ_0 , so that it is a convex polygonal arc. Therefore because $\mathcal{E}(m, n; \delta)$ has endpoints $(0, m - \delta_0 - \delta_1)$ and $(n - \delta_0, 0)$ where $\delta_0, \delta_0 + \delta_1 > 0$, it lies outside $\frac{m - \delta_0 - \delta_1}{m} \overset{\circ}{\Delta}(m, n)$. Therefore, to prove (iii) one must simply check that it lies inside $\Delta(m, n)$.

Lemma 3.7. If δ is admissible, so is $t\delta$ for all $0 < t \leq 1$.

Proof. Condition (i) in the definition obviously holds. To check (ii), let us denote the edges of $\mathcal{E}(m, n; \delta)$ by ϵ_i^δ . Then, if $0 \leq i < N$,

$$\alpha(\epsilon_i^\delta) = a_{m,n,\delta}(h(\epsilon_i)) = (a_{m,n,\delta} - a_{m,n,0})(h(\epsilon_i))$$

is a homogeneous linear function of δ and hence is positive for $t\delta$ if it is positive for δ . Further $\alpha(\epsilon_N^\delta) = 1 +$ a homogenous linear function of δ , and so again condition (ii) is satisfied by $t\delta$.

To check (iii), observe that the positions of the edges depend linearly on δ , i.e. for each i there are constants c_{i0} and homogenous linear functions $c_{i1}(\delta)$ such that the edge ϵ_i^δ lies in the line

$$\{\mathbf{x} \in \mathbb{R}^2 : \nu(\epsilon_i) \cdot \mathbf{x} = c_{i0} + c_{i1}(\delta)\}.$$

Moreover, by Lemma 3.4, the line $\nu(\epsilon_i) \cdot \mathbf{x} = c_{i0}$ goes through one of the points $(0, m)$ or $(n, 0)$. Hence (iii) holds iff $c_{i1}(\delta) \leq 0$ for all i . The result follows. \square

There is an analogous discussion for outer approximations. These approximate $\Delta(m, n)$ by a polytope with a concave chain $\widehat{\mathcal{E}}(m, n; \delta)$ of edges \widehat{e}_i lying just outside the slanted edge of $\Delta(m, n)$. We did the case $(1, k)$ in §2: $\widehat{\mathcal{E}}(1, 4)$ (which in the notation of §2 corresponds to the chain of spheres $\widehat{\mathcal{C}}_4$) is illustrated in Figure 2.3. Note that this chain of edges goes between the edges with conormals $(-1, 0)$ and $(0, -1)$ but does not include them, so that the classes $h(\widehat{e}_i)$ of the edges in $\widehat{\mathcal{E}}$ are linear combinations of the exceptional divisors \widehat{E}_i , with no mention of L . Again we assume that $\widehat{\mathcal{E}}(m, n)$ is minimal, i.e. the last edge $\widehat{e}_{\widehat{N}}$ with conormal $(-m, -n)$ is the only edge whose class $h(\widehat{e}_i)$ has self intersection -1 . We shall denote the analog of $V_{m,n}$ by $\widehat{V}_{m,n}$. Thus $\widehat{V}_{m,n} = \sum_{i=1}^{\widehat{N}} \widehat{k}_i \widehat{E}_i$ is such that

$$(3.9) \quad \widehat{V}_{m,n} \cdot h(\widehat{e}_{\widehat{N}}) = -1, \quad \widehat{V}_{m,n} \cdot h(\widehat{e}_j) = 0, \quad j < \widehat{N}, \quad \widehat{V}_{m,n}^2 = -mn.$$

For example $\widehat{V}_{1,k} = \sum_{j=1}^k \widehat{E}_j$. We leave the proof of the following statement to the reader.

Lemma 3.8. *Let (m, n) be relatively prime positive integers with $m < n$. Then:*

- (i) *For small admissible $\widehat{\delta}$, $\Delta(m, n)$ has an outer approximation $\widehat{\mathcal{E}}(m, n; \widehat{\delta})$ that lies in $(1 + \widehat{\delta}_0) \Delta(m, n) \setminus \overset{\circ}{\Delta}(m, n)$, where $\widehat{\delta}_0 > 0$.*
- (ii) *If $\widehat{\delta}$ is admissible, so is $t\widehat{\delta}$ for all $0 < t \leq 1$.*
- (iii) *There is a vector $\widehat{V}_{m,n}$ satisfying (3.9).*

Remark 3.9. Just as in the case of inner approximations, the conormals occurring in an outer approximation to $\Delta(m, n)$ give rise to a decomposition of a triangle, which this time is $\Delta(m, n)$ itself. Moreover, as we shall prove in [19], the sequence of labels $\widehat{k}_1, \widehat{k}_2, \dots, \widehat{k}_{\widehat{N}}$ that occur as coefficients in the vector $\widehat{V}_{m,n}$ can be obtained by the following version of the Euclidean algorithm.

First write down a_1 copies of $p_1 := m$ where $a_1 m \leq n < (a_1 + 1)m$, then write down a_2 copies of $p_2 := n - a_1 p_1$ where $a_2 p_2 \leq m = p_1 < (a_2 + 1)p_2$, and so on. At the i th step one writes down a_i copies of $p_i := p_{i-2} - a_{i-1} p_{i-1}$ where $a_i p_i \leq p_{i-1} < (a_i + 1)p_i$. The process stops as soon as some $p_i = 0$.

As pointed out Dylan Thurston³ the combinatorics of the resulting decomposition of $\Delta(m, n)$ are precisely the same as the combinatorics of one of the standard ways of getting the continued fraction expansion: see Figure 3.4. Hence the multiplicities a_i of the labels \widehat{k}_j give the continued fraction expansion of $\frac{m}{n}$. For example, $\frac{5}{3}$ has labels $\widehat{k}_1, \dots, \widehat{k}_4 = 3, 2, 1, 1$ with multiplicities $a_1, a_2, a_3 = 1, 1, 2$, and

$$\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}}.$$

³Private communication.

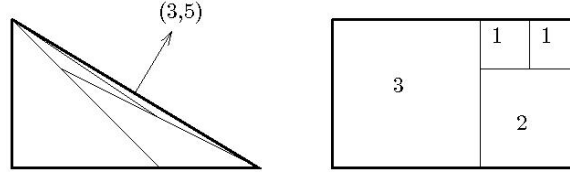


FIGURE 3.4. $\frac{5}{3}$ has labels 3, 2, 1, 1 with multiplicities 1, 1, 2. Note that in the diagram on the right one starts by expanding horizontally because the rectangle is wider than it is high; at the second step one rotates by 90° and then continues. This rotation is equivalent to taking the reciprocal of the aspect ratio of the rectangle. Hence this expansion mirrors the continued fraction.

Note also that the multiplicative relation $\widehat{V}_{m,n}^2 = \sum \widehat{k}_i^2 = mn$ is obvious from this point of view. Since, as we show in Theorem 3.11 below, the labels \widehat{k}_i determine the weights of the corresponding ball embedding problem, this multiplicative relation corresponds to the geometric fact that the total volume of the balls corresponding to an ellipsoid E must be the same as the volume of E .

Remark 3.10. (i) We saw above that an inner approximation to $\Delta(m, n)$ is constructed from a decomposition of the triangle $T(m, n) = \Delta(n, n) \setminus \overset{\circ}{\Delta}(m, n)$ with outward conormals $(1, 1), (-1, 0), (-m, -n)$, while an outer approximation to $\Delta(m, n)$ is constructed from a decomposition of $\Delta(m, n)$ itself. The matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

takes the conormals of $T(m, n)$ to those of $\Delta(n, n - m)$. It is easy to check that this transformation takes the first decomposition into the second. Thus inner and outer approximations are essentially the same thing, though they are related in a slightly different way to the ambient triangle $\Delta(n, n)$.

(ii) From the point of view of singularity theory, our construction of the inner (or outer) approximation to $\Delta(m, n)$ can be considered as a kind of joint resolution of the two singular points of the corresponding toric variety. Usually, one would resolve them separately, in which case, it is the Hirzebruch-Jung continued fractions (with $-$ rather than $+$ signs) that are relevant: see Fulton [4, §2.6]. In the standard resolution of a single singularity one performs the blow ups near just one of the vertices getting half of our conormals. For example Fulton's method of resolving the vertex with outward conormals $(0, 1)$ and $(n, -m)$ (where $0 < m < n$) begins by a cut with conormal $(1, 0)$. Hence if we rotate his picture anticlockwise by 90° we get the half of the inner approximation to $\Delta(m, n)$ near the vertex $(0, m)$; cf. Figure 3.1. Note that the orbifold structure of this vertex has stabilizer of order n , with generator ζ acting on \mathbb{C}^2 via $(z_1, z_2) \mapsto (\zeta^{-m} z_1, \zeta z_2)$ where $\zeta = e^{2\pi i/n}$. The other half of this inner approximation corresponds to the vertex $(n, 0)$ which has stabilizer of order m acting

via $(z_1, z_2) \mapsto (\eta z_1, \eta^{-n} z_2)$ where $\eta = e^{2\pi i/m}$. Hence this corresponds to Fulton's resolution of the vertex with outward conormals $(0, 1)$ and $(m, km - n)$ where we choose k so that $0 < n - km < m$; i.e. we simply interchange the roles of m and n . Note that the interpretation of the coefficients of the continued fraction expansion is rather different in the two cases.

3.2. Proof of Theorem 1.5. We shall prove the following more precise form of Theorem 1.5. Recall that V and \widehat{V} are defined in equations (3.2) and (3.9) respectively.

Theorem 3.11. *Suppose that each pair (m, n) and (m', n') is mutually prime, and let*

$$\widehat{V}_{m,n} = \sum_{1 \leq i \leq \widehat{N}} \widehat{k}_i \widehat{E}_i, \quad V_{m',n'} = n' L - \sum_{1 \leq i \leq N} k_i E_i.$$

Set

$$k = N + \widehat{N}, \quad \underline{w}_\lambda = \left(\frac{k_1}{n'}, \dots, \frac{k_N}{n'}, \frac{\lambda \widehat{k}_1}{n'}, \dots, \frac{\lambda \widehat{k}_{\widehat{N}}}{n'} \right).$$

Then the question of whether $\lambda E(m, n)$ embeds into the open ellipsoid $\mathring{E}(m', n')$ is equivalent to the symplectic packing problem for k balls with weights \underline{w}_λ .

Proof. Suppose first that $E(\lambda m, \lambda n)$ embeds into the open ellipsoid $\mathring{E}(m', n')$. Since $\mathring{E}(m', n') \subset \mathring{B}(n')$ we may consider $\mathring{E}(m', n')$ as a subset of $\mathbb{C}P^2(n')$. By Corollary 3.5 the complement of $\mathring{E}(m', n')$ in $\mathbb{C}P^2(n')$ contains the N open balls $\mathring{B}(k_i)$ (cf. Lemma 2.6.) Moreover these balls can be embedded disjointly from the line in $\mathbb{C}P^2(n')$ represented by the edge of $\Delta(n', n')$ with conormal $(1, 1)$. Similarly, $\lambda E(m, n)$ contains the \widehat{N} open balls $\lambda \mathring{B}(\widehat{k}_j)$. Hence, rescaling by $1/n'$, we see that the given symplectic packing problem has a solution with open balls.

However, the problem was formulated in terms of embedding closed balls. To deal with this, observe that when $\lambda E(m, n) \xrightarrow{s} \mathring{E}(m', n')$ there is $\kappa > 0$ such that $(1 + \kappa)\lambda E(m, n)$ embeds in $\frac{1}{1+\kappa} \mathring{E}(m', n')$. This means that the sizes of all the open balls can be slightly increased so that they contain closed balls of the correct size.

Conversely, suppose that the ball packing problem has a solution, i.e. that there is a symplectic form on the k -fold blow up X_k of $\mathbb{C}P^2$ in class

$$(3.10) \quad a_\lambda := \ell - \sum_{i=1}^k w_i e_i.$$

Since the space of symplectic forms is open, we may suppose without loss of generality that λ is rational. Moreover, it suffices to prove that

$$(3.11) \quad \lambda_0 E(m, n) \xrightarrow{s} \mathring{E}(m', n') \quad \text{for all } \lambda_0 < \lambda.$$

Before proceeding further, it is convenient to introduce some notation. We will denote the divisors of X_k by E_1, \dots, E_k as usual; hence the E_{N+i} , $1 \leq i \leq \widehat{N}$, correspond to the exceptional divisors \widehat{E}_i associated to $E(m, n)$. Further δ will denote a tuple of small constants, whose length (either $N+1$, \widehat{N} or $k+1 = N+1+\widehat{N}$) will depend on the

context. When it is necessary to be more specific we shall denote the $(k+1)$ -tuple δ by $(\delta', \widehat{\delta})$. Further $\delta = (\delta', \widehat{\delta})$ is admissible if its first $N+1$ components δ' are admissible for $E(m', n')$ while its last \widehat{N} components $\widehat{\delta}$ are admissible for $E(m, n)$.

Given an inner approximation $\mathcal{E}(m', n'; \delta)$, we shall denote by U_δ the T^2 invariant open subset of $\mathbb{C}P^2(n')$ whose moment image is the component of $\Delta(n', n') \setminus \mathcal{E}(m', n'; \delta)$ that lies in $\Delta(m', n')$. Thus $U_\delta \subset \mathring{E}(m', n')$ is a smooth approximation to $E(m', n')$. We shall denote the chain of spheres corresponding to an inner approximation $\mathcal{E}(m', n'; \delta)$ by \mathcal{C}_δ and that corresponding to an outer approximation $\widehat{\mathcal{E}}(m, n; \delta)$ by $\widehat{\mathcal{C}}_\delta$.

Finally if U is any subset of a symplectic manifold (M, ω) and $r > 0$ we shall denote by rU the set U provided with the form $r\omega|_U$. Note that if Ω is a symplectic form on X_k such that the disjoint union $\mathcal{C}_\delta \sqcup \lambda \widehat{\mathcal{C}}_\delta$ embeds in (X_k, Ω) , then the class $[\Omega]$ is determined by δ and is close to $n(\ell - \sum w_i e_i)$.

Claim 1: *If there is a symplectic form Ω on X_k such that $\mathcal{C}_\delta \sqcup \lambda_0 \widehat{\mathcal{C}}_\delta$ embeds in (X_k, Ω) for some admissible δ , then $\lambda_0 E(m, n) \xrightarrow{s} \mathring{E}(m', n')$.*

Proof. Let $\mathcal{N}(\mathcal{C}_\delta)$ be a T^2 -invariant neighborhood of \mathcal{C}_δ whose moment image is a neighborhood of $\mathcal{E}(m', n'; \delta)$. As in Lemma 2.2 we may suppose that $\mathcal{N}(\mathcal{C}_\delta)$ embeds in (X_k, Ω) . Then a neighborhood of infinity in $(W, \Omega) := (X_k \setminus \mathcal{C}_\delta, \Omega)$ may be identified with a neighborhood of infinity in U_δ , where U_δ is as above. (In fact, $(X_k \setminus \mathcal{C}_\delta, \Omega)$ can be obtained from U_δ by further blowing up near the inverse image of $(0, 0)$.) Moreover (W, Ω) contains a copy of $\lambda_0 \widehat{\mathcal{C}}_\delta$ and hence, as in Lemma 2.3, blows down to an open set (Z, ω) containing $\lambda_0 E(m, n)$. But $H_2(Z) = 0$ by construction, and (Z, ω) is symplectomorphic to U_δ at infinity. Hence, by the uniqueness of symplectic forms on starshaped subsets of \mathbb{R}^4 that are standard near the boundary (see [18, Thm. 9.4.2]), (Z, ω) is symplectomorphic to U_δ . Therefore

$$\lambda_0 E(m, n) \xrightarrow{s} (Z, \omega) \cong U_\delta \subset E(m', n'),$$

which proves the claim. \square

Claim 2: *If the ball packing problem has a solution with weights \underline{w}_λ , then for all $\lambda_0 < \lambda$, there is a symplectic form Ω on X_k such that $\mathcal{C}_\delta \sqcup \lambda_0 \widehat{\mathcal{C}}_\delta$ embeds in (X_k, Ω) for some admissible δ .*

Proof. If $r < \frac{m'}{n}$ then $rE(m, n)$ embeds linearly in $\mathring{E}(m', n')$ and so, for small admissible δ there is a symplectic form $\Omega_{r, \delta}$ on X_k such that $\mathcal{C}_\delta \sqcup r \widehat{\mathcal{C}}_\delta$ embeds in $(X_k, \Omega_{r, \delta})$. Note that $[\Omega_{r, \delta}] = a' + r \widehat{a} + c(\delta)$, where

$$a' := n' \ell - \sum_{1 \leq i \leq N} w_i e_i, \quad \widehat{a} := -\frac{1}{\lambda} \sum_{N < i \leq k} w_i e_i,$$

and $c(\delta) = c'(\delta') + \widehat{c}(\widehat{\delta}) \in H^2(X_k)$ is a homogeneous linear function of δ .

By assumption the class $a_\lambda := n' \ell - \sum w_i e_i = a' + \lambda \widehat{a}$ of equation (3.10) is rational and represented by a symplectic form. Therefore, as in the proof of Proposition 1.10, the homology class $PD(qa_\lambda)$ has nontrivial Gromov invariant for large q . Choose an $\Omega_{r, \delta}$

tame almost complex structure J on X_k such that both \mathcal{C}_δ and $r\widehat{\mathcal{C}}_\delta$ are J -holomorphic. If J is sufficiently generic, then as in the proof of Theorem 1.1 in §2 we may suppose that the class $PD(qa_\lambda)$ is represented by a connected J -holomorphic submanifold Q that intersects \mathcal{C}_δ and $r\widehat{\mathcal{C}}_\delta$ transversally. The inflation procedure gives a family of symplectic forms Ω_t on X_k that are nondegenerate on the two configurations of spheres and lie in class

$$\begin{aligned} [\Omega_t] &= [\Omega_{r,\delta}] + tq a_\lambda \\ &= (1 + tq) \left(a' + \left(\frac{r + \lambda tq}{1 + tq} \right) \widehat{a} + c'(\kappa') + \widehat{c}(\widehat{\kappa}) \right) \end{aligned}$$

where κ' and $\widehat{\kappa}$ are multiples of δ' and $\widehat{\delta}$. and so are admissible by Lemmas 3.7 and 3.8. Observe that as $t \rightarrow \infty$ the class $\frac{1}{1+qt}[\Omega_t]$ converges to a_λ . Moreover, for appropriate δ , \mathcal{C}_δ embeds in (X_k, Ω_t) for all t , while $\lambda_0 \widehat{\mathcal{C}}_\delta$ embeds in (X_k, Ω_t) if $\lambda_0 = \frac{r + \lambda tq}{1 + tq}$. By equation (3.11) this completes the proof. \square

Proof of Corollary 1.6. Since the targets of the embeddings are open ellipsoids, an easy continuity argument implies that it suffices to prove these statements when a, b, a', b' are integers. Part (i) is equivalent to saying that all deformation equivalent symplectic forms on $X_k \setminus \mathcal{N}(\mathcal{C}_\delta \cup \widehat{\mathcal{C}}_\delta)$ are isotopic. It can be proved in the same way as the uniqueness of symplectic forms on X_k . One just needs to inflate along curves Q that intersect \mathcal{C}_δ and $\widehat{\mathcal{C}}_\delta$ transversally, which is possible as in the proof of Claim 2 above. For more details, see [15].

(ii) follows from (i) just as the analogous statement for balls follows from the fact that the space of embeddings of one ball into another is connected. Let $\lambda_n, n \geq 1$, be an increasing sequence with limit λ . From a sequence of embeddings

$$\iota_n : \lambda_n E(m, n) \xrightarrow{s} \mathring{E}(m', n')$$

one first uses (i) with target ellipsoid $\mathring{E}(m', n')$ to construct a sequence ι'_n such that $\text{Im } \iota'_n \subset \text{Im } \iota'_{n+1}$. By using (i) again, this time with target ellipsoids $\text{Im } \iota'_{n+1}$, one makes a further adjustment so that ι'_{n+1} restricts to ι'_n on $\lambda_n E(m, n)$. The result follows. \square

Proof of Proposition 1.7 (i). We saw above that $V_{2,3} = 3L - F_1 - F_2 - F_3$ and $\widehat{V}_{1,4} = \widehat{E}_1 + \widehat{E}_2 + \widehat{E}_3 + \widehat{E}_4$. Hence the first statement follows from Theorem 3.11.

For simplicity, let us rename the blow up classes in X_7 as $E_i, i = 1, \dots, 7$, where $F_i := E_i$ for $1 \leq i \leq 3$ and $E_i := \widehat{E}_{i-3}$ for $4 \leq i \leq 7$. By Propositions 1.9 and 1.10, the second statement will follow if we show that the class

$$a_{\underline{w}} = \ell - \frac{1}{3}e_{123} - \frac{\lambda}{3}e_{4567}$$

takes positive values on all the elements in $\mathcal{E}_K(X_7)$, where $e_{j\dots k} := \sum_{i=j}^k e_i$. But $\mathcal{E}_K(X_7)$ is generated by classes of the form $E_i, L - E_i - E_j$ together with classes that are equivalent to the following (after permutation of indices):

$$(i) \quad 2L - E_{1\dots 5}, \quad (ii) \quad 3L - 2E_1 - E_{2\dots 7}.$$

Evaluating a_w on $3L - 2E_7 - E_{1\dots 6}$ we find that we need $3 > \frac{5}{3}\lambda + 1$ i.e. $\lambda < \frac{6}{5}$. Since the other curves in $\mathcal{E}_K(X_7)$ give weaker inequalities, the result follows. \square

Proof of Proposition 1.7 (ii). The first statement holds as before. To prove the second, recall that $\mathcal{E}_K(X_8)$ is generated by the classes in $\mathcal{E}_K(X_7)$ together with those of the following three forms:

- (iii) $4L - 2E_{123} - E_{4\dots 8}$;
- (iv) $5L - 2E_{1\dots 6} - E_{78}$;
- (v) $6L - 3E_1 - 2E_{2\dots 8}$.

(These structural results on $\mathcal{E}_K(X_7)$ and $\mathcal{E}_K(X_8)$ are classical and not hard to prove directly from the definition.) One gets the sharpest inequality on λ from elements of the form (v), which give $\lambda < \frac{12}{11}$. Hence the result. \square

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