

# SYMPLECTIC EMBEDDINGS AND CONTINUED FRACTIONS: A SURVEY

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ABSTRACT. As has been known since the time of Gromov's Nonsqueezing Theorem, symplectic embedding questions lie at the heart of symplectic geometry. After surveying some of the most important ways of measuring the size of a symplectic set, these notes discuss some recent developments concerning the question of when a 4-dimensional ellipsoid can be symplectically embedded in a ball. This problem turns out to have unexpected relations to the properties of continued fractions and exceptional curves in blow ups of the complex projective plane. It is also related to questions of lattice packing of planar triangles.

## 1. OVERVIEW.

The standard symplectic structure on  $\mathbb{R}^{2n}$  is:

$$\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

By **Darboux's theorem** every symplectic form is locally diffeomorphic to this one, so it is crucial to understand its properties.

Let  $B := B^{2n}(a) \subset \mathbb{R}^{2n}$  be the standard (closed) ball of radius  $\sqrt{a}$  (so  $a$  is proportional to a 2-dimensional area), and let  $\phi : B \xrightarrow{s} \mathbb{R}^{2n}$  be a **symplectic embedding** (i.e. a smooth embedding onto  $\phi(B)$  that preserves  $\omega_0$ ). Since  $\omega_0^n = n! dx_1 \wedge \cdots \wedge dx_{2n}$  is a volume form, every symplectic embedding preserves volume.

**Gromov's question:** *What can one say about the images  $\phi(B)$  of a symplectic ball? How are the symplectic and volume preserving cases different?*

The following result can be easily proven using Moser's homotopy method.

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**Volume preserving embeddings:** *if  $V \subset \mathbb{R}^{2n}$  is diffeomorphic to a ball and  $\text{vol} V = \text{vol} B$  then there is a volume preserving diffeomorphism  $\psi : B \xrightarrow{\cong} V$ .*

The analogous statement is not true for symplectic diffeomorphisms — called *symplectomorphisms* for short — since the boundary of  $V$  has symplectic invariants given by the characteristic foliation.<sup>1</sup> Nevertheless one might ask if one can **fully fill**  $V$  by a symplectic ball. This means that for every  $\varepsilon > 0$  there is a ball  $B$  and a symplectic embedding  $\phi : B \xrightarrow{s} V$  such that  $\text{vol}(V \setminus \phi(B)) < \varepsilon$ .

Let  $Z(A)$  be the cylinder

$$B^2(A) \times \mathbb{R}^{2n-2} = \{(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n} : x_1^2 + x_2^2 \leq A\}$$

We provide  $Z(A)$  with the symplectic structure  $\omega_0$ . Thus its first two coordinates lie in symplectically paired directions. In particular,  $\omega_0$  has non zero integral over each disc  $B^2(A) \times \{pt\}$ .

**Gromov’s Nonsqueezing Theorem:** *There is a symplectic embedding*

$$\phi : B^{2n}(a) \xrightarrow{s} Z(A)$$

*if and only if  $a \leq A$ .*

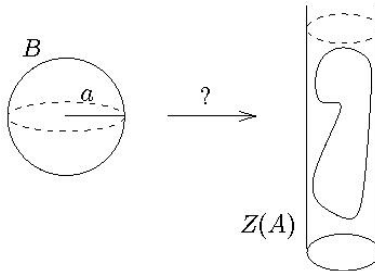


FIGURE 1.1. Does the ball embed symplectically in the cylinder?

This nonsqueezing property is fundamental. To a first approximation, it is true that a diffeomorphism with this nonsqueezing property for all balls and all symplectic cylinders must preserve the symplectic structure. (More precise versions of this statement were proven by Eliashberg [8] and Ekeland–Hofer [7].)

It follows also that symplectomorphisms are very different from volume preserving embeddings. For example, the largest ball that embeds symplectically in the polydisc  $B^2(1) \times B^2(1)$  is  $B^{2n}(1)$  which has volume just half that of the polydisc.

<sup>1</sup>The characteristic foliation on a smooth hypersurface  $Q$  is spanned by the unique null direction of  $\omega|_Q$ , i.e. by the vectors  $v \in TQ$  such that  $\omega(v, w) = 0$  for all  $w \in TQ$ .

**1.1. Symplectic capacities.** In their paper [7], Ekeland and Hofer formalized the idea of a *symplectic capacity*, which is a measurement  $c(U, \omega)$  of the size of a symplectic manifold  $(U, \omega)$  (possibly open or with boundary) with the following properties:

- (i)  $c$  is a function with values in  $[0, \infty]$  defined on some class of  $2n$ -dimensional symplectic manifolds<sup>2</sup>  $(M, \omega)$  that contains the closures of all open subsets of Euclidean space;
- (ii) (monotonicity) if there is a symplectic embedding  $(U, \omega) \rightarrow U', \omega'$  then  $c(U, \omega) \leq c(U', \omega')$ ;
- (iii) (scaling)  $c(U, \lambda\omega) = \lambda c(U, \omega)$  for all  $\lambda > 0$ ;
- (iv) (normalization)  $c(B^{2n}(a)) = \pi a$  and  $c(Z^{2n}(a)) < \infty$ .

The monotonicity property implies that  $c$  is a symplectic invariant, that is, it takes the same value on symplectomorphic sets, while the scaling property implies that it scales like a 2-dimensional invariant. One can satisfy the first three properties by considering an appropriate power of the volume; for example one could consider the function  $c(U, \omega) := (\int_U \omega^n)^{1/n}$ . However, this function  $c$  does not satisfy the second half of the normalization axiom. Indeed, the requirement that a cylinder has finite capacity is what makes this an interesting definition.

Here are some examples that illustrate some of the variety of possible definitions.

- (i) **The Gromov width.**  $c_G(U, \omega) = \inf\{\pi a \mid B^{2n}(a) \xrightarrow{s} (U, \omega)\}$ .

The nonsqueezing theorem implies that this does satisfy the axioms. Moreover, it satisfies the following strong normalization condition:

$$c(B^{2n}(a)) = c(Z^{2n}(a)).$$

- (ii) **The Hofer–Zehnder capacity** [17]. Every smooth function  $H : M \rightarrow \mathbb{R}$  on a closed symplectic manifold generates a 1-parameter subgroup  $\phi_t^H, t \in \mathbb{R}$ , of the group of symplectomorphisms of  $(M, \omega)$  by the following procedure; define the vector field  $X_H$ , the *symplectic gradient* of  $H$ , by requiring that

$$\omega(X_H, \cdot) = dH(\cdot),$$

and then define  $\phi_t^H$  to be the flow generated by  $X_H$  with  $\phi_0^H = id$ . The identity

$$\mathcal{L}_{X_H}(\omega) = d(\iota(X_H)\omega) + \iota(X_H)d\omega = d(dH) = 0,$$

(where  $\mathcal{L}$  denotes the Lie derivative) shows that  $(\phi_t^H)^*\omega = \omega$  for all  $t$ . In other words, the flow preserve the symplectic form. A point  $x$  for which there are times  $0 < t < T$  such that  $\phi_T^H(x) = x$  but  $\phi_t^H(x) \neq x$  is said to be

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<sup>2</sup>A smooth manifold  $M$  is said to be symplectic if it is provided with a symplectic form  $\omega$ . Here  $\omega$  is any closed 2-form such that  $\omega^n$  never vanishes. A fundamental result due to Darboux is that every symplectic form is locally diffeomorphic to Euclidean space  $\mathbb{R}^{2n}$  with its standard structure  $\omega_0$ . Hence the importance of understanding the symplectic properties of subsets of  $\mathbb{R}^{2n}$ .

a nontrivial periodic orbit of  $\phi_t^H$  of period  $T$ . Further we define  $\mathcal{H}$  to be the set of functions  $H : M \rightarrow \mathbb{R}$  with the following properties:

- $H(x) \geq 0$  for all  $x \in M$ ,
- there is an open subset of  $M$  on which  $H = 0$ ;
- $H$  is constant outside a compact subset of the interior of  $M$ ;
- every nontrivial periodic orbit of  $H$  has period  $T \geq 1$ .

Then we define the Hofer–Zehnder capacity  $c_{HZ}$  as follows:

$$c_{HZ}(U, \omega) = \sup_{H \in \mathcal{H}} \left( \sup_{x \in M} H(x) \right)$$

This function  $c_{HZ}$  obviously satisfies the first three conditions for capacities. Moreover as  $\lambda$  increases the flow of  $\lambda H$  moves faster so that the periods of the periodic orbits decrease. Therefore in order to prove that a set such as  $B^{2n}$  has finite capacity one needs a mechanism to prove that periodic orbits for  $\phi_t^H$  must exist under suitable circumstances (for example, if  $H = \lambda K$  where  $K$  satisfies the first three conditions to be in  $\mathcal{H}$  and  $\lambda$  is sufficiently large.) In the original papers this mechanism involved subtle arguments in variational analysis; however one can also prove such results using  $J$ -holomorphic methods.

(iii) **The displacement energy.** If  $H_t : M \rightarrow \mathbb{R}, t \in [0, 1]$ , is a smooth family of functions then one can define a path of vector fields  $X_{H_t}$  by requiring that

$$\omega(X_{H_t}, \cdot) = d(H_t)(\cdot) \text{ for each } t \in [0, 1],$$

and then integrate this family of vector fields to get a flow  $\phi_t^H, t \in [0, 1]$ . As before, each diffeomorphism  $\phi_t^H$  preserves  $\omega$ , and the family is called a Hamiltonian path, or Hamiltonian isotopy. Define the length  $\mathcal{L}(\{\phi_t^H\})$  of such a path by setting

$$\mathcal{L}(\{\phi_t^H\}) := \int_0^1 \left( \sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x) \right) dt.$$

(Thus this is a measure of the total variation of  $H_t$ .)

If  $U \subset (M, \omega)$  then a Hamiltonian isotopy  $\phi_t^H$  of  $M$  is said to disjoin  $U$  in  $M$  if  $\phi_1^H(U) \cap U = \emptyset$ . The *displacement energy* of a subset  $(U, \omega)$  of a symplectic manifold  $(M, \omega)$  is defined to be:

$$d_M(U, \omega) = \inf \{ \mathcal{L}(\{\phi_t^H\}) : \{\phi_t^H\} \text{ disjoins } U \text{ in } M \},$$

where we take the infimum of an empty set of real numbers to be  $\infty$ . It is a deep fact that the displacement energy of a ball is always positive, no matter what  $M$  is. Using this, it is easy to see that this energy defines a capacity on the set of all subsets of a given symplectic manifold  $(M, \omega)$ . Interestingly enough, as explained in [19] there are relations between the displacement energy of a ball and the nonsqueezing theorem.

Although the displacement energy of a set  $(U, \omega)$  in principle depends on the choice of ambient manifold  $(M, \omega)$ , I am not aware of any examples of

sets  $U$  that have different displacement energies when considered as subsets of two different manifolds  $(M_1, \omega_1)$  and  $(M_2, \omega_2)$ , except in the trivial case when  $U$  is displaceable in  $M_1$  but not in  $M_2$ . As a refinement of this question one could consider two different symplectic embeddings  $\iota_1$  and  $\iota_2$  of one set  $(U, \omega)$  into another  $(M, \omega)$ . Are there any examples where  $d_M(\iota_1(U, \omega)) \neq d_M(\iota_2(U, \omega))$ ? This will not happen if the two embeddings are Hamiltonian isotopic (i.e. there is a Hamiltonian isotopy  $\phi_t^H$  of  $M$  such that  $\iota_2 = \phi_1^H \circ \iota_1$ ). But there are cases in which nonisotopic embeddings are known to exist; in other words, the space of embeddings  $(U, \omega) \xrightarrow{s} (M, \omega)$  need not be connected. Such examples (of embeddings of one polydisc  $B^2(a_1) \times B^2(a_2)$  into another) were constructed in [9]. This leads into the whole question of what is the topology of such embedding spaces. Rather little is known about such questions.

The above three capacities are by now classical invariants. There are many other more recently introduced functions that share some of the properties of capacities. For example, Biran and Cornea [3, §6.6] are interested in exploring the “size” of Lagrangian submanifolds  $L$  of  $(M, \omega)$ . A Lagrangian submanifold is a submanifold of dimension  $n$  on which the symplectic form vanishes identically. A basic example is the subspace  $V_{\mathbb{R}}$  of  $(\mathbb{R}^{2n}, \omega_0)$  spanned by the coordinates  $x_1, x_3, \dots, x_{2n_1}$ . Lagrangian submanifolds are fundamental objects of study in symplectic geometry.

Let us write  $B_{\mathbb{R}}^{2n}$  for the intersection  $B^{2n}(a) \cap V_{\mathbb{R}}$ . Then the *width*  $w(L)$  of a Lagrangian submanifold  $L \subset (M, \omega)$  is defined to be the supremum of  $\pi a$  for which there is a symplectic embedding

$$(B^{2n}(a), B_{\mathbb{R}}^{2n}(a)) \xrightarrow{s} (M, L).$$

Although some calculations can be made, it is not even known whether every compact Lagrangian submanifold of  $\mathbb{R}^{2n}$  must have finite width.

Because so little is known about these general embedding questions, we shall now return to considering balls and ellipsoids. As we shall see, their embedding properties are very closely related, at least in dimension 4. We shall first consider the problem of embedding several equal balls, and then discuss ellipsoids in §2.

**1.2. Symplectic ball packing. Question:** *How much of the volume of  $B(1)$  can be filled by  $k$  equal symplectic balls?*

[Gromov [11]]: *there are obstructions when  $k = 2$ :*

$$\text{if } B(a) \sqcup B(a) \xrightarrow{s} B(1) \text{ then } a \leq 1/2;$$

in other words, the volume of the image of  $B(a) \sqcup B(a)$  is  $\leq \frac{1}{2^{n-1}} \text{vol}(B(1))$ .

[McDuff and Polterovich [22]]: *for any  $d \geq 1$  the  $2n$ -dimensional ball  $B(1)$  can be fully filled by  $k = d^n$  equal balls.*

One can explicitly see these embeddings in the following way.

**Describing embeddings by toric models:**

Consider the (moment) map  $\Phi : \mathbb{C}^2 \rightarrow \mathbb{R}^2, \Phi(z_1, z_2) = (|z_1|^2, |z_2|^2)$ . Then  $\Phi(B(1))$  is the (standard) triangle  $\{0 \leq x_1, x_2; x_1 + x_2 \leq 1\}$ . Thus the inverse image of the closed triangle can obviously be fully filled by a ball. Somewhat surprisingly, Traynor [27] showed that it is possible to fully fill the inverse image of the *interior* of any standard triangle. Hence, we can obtain full fillings of  $B^4(1)$  by  $d^2$  equal balls by cutting a standard triangle into  $d^2$  standard pieces as in Figure 1.2.



FIGURE 1.2. Embedding  $d^2$  standard triangles into a triangle of size 1.

However, when  $k \neq d^2$  the story is different:

[McDuff–Polterovich [22]]: *In dimension 4 there are obstructions to full fillings by  $k$  balls when  $k < 9, k \neq 1, 4$ ;*

[Biran [1]]  *$B^4(1)$  can be fully filled by  $k$  equal balls for all  $k \geq 9$ .*

Thus, with enough balls, the obstructions disappear. In later work [2], Biran showed that for all (closed) rational<sup>3</sup> symplectic 4-manifolds  $(M, \omega)$  there is an integer  $N = N(M, \omega)$  such that  $(M, \omega)$  can be fully filled by  $k$  balls for all  $k \geq N$ . This is rather surprising. Note that  $N$  depends on  $\omega$  and can diverge to infinity even when  $M$  is fixed. For example, in the case when  $M = S^2 \times S^2$  and  $\omega_k = \frac{1}{k}pr_1^*\sigma \times pr_2^*\sigma$  for some area form  $\sigma$  on  $S^2$  of total area  $\pi$ , the nonsqueezing theorem shows that one cannot embed a ball larger than  $B^4(\frac{1}{k})$ . Since the volume of  $(M, \omega_k)$  is  $\frac{\pi^2}{k}$  while that of  $B^4(\frac{1}{k})$  is  $\frac{\pi^2}{2k^2}$ , we find that  $N(M, \omega_k) \geq 2k$ . Thus, it is not at all clear whether one can remove the rationality condition on  $[\omega]$ . It is also not clear whether there is an analogous result in higher dimensions.

## 2. EMBEDDING ELLIPSOIDS INTO BALLS

From now on we shall work in 4 (real) dimensions, and shall write  $E \overset{s}{\hookrightarrow} B$  to mean that there is a symplectic embedding of  $E$  into  $B$ . Define the ellipsoid  $E(a, 1)$  by setting:

$$E(a, 1) = \{(x_1, \dots, x_4) \in \mathbb{R}^4 : \frac{x_1^2 + x_2^2}{a} + x_3^2 + x_4^2 \leq 1\}.$$

<sup>3</sup> $(M, \omega)$  is called *rational* if  $[a] \in H^2(M; \mathbb{Q})$ .

Consider the embedding capacity function  $c$  for  $a \geq 1$ :

$$c(a) := \inf\{\mu : E(a, 1) \xrightarrow{s} B(\mu)\}.$$

Note that  $c(a) \geq \sqrt{a}$  because  $\text{vol } E(a, 1) = \text{vol } B(\sqrt{a})$ .

In [5], Cieliebak, Hofer, Latschev and Schlenk describe a wide variety of symplectic embedding problems in quantitative terms, formulating many interesting questions, but providing relatively few answers. (See also Schlenk [26].) In particular, they define the function  $c(a)$  described above, but could say rather little about its properties. The first most significant open question concerned the value of  $c(4)$ . This was in fact first calculated by Opshtein [24], though he did not make this explicit in his paper.

As we shall see, the function  $c(a)$  turns out to be surprisingly interesting. It is the first of the capacity functions of [5] to be calculated.<sup>4</sup> Note also that although the nonsqueezing theorem stated above is valid in all dimensions, we only understand the embedding of ellipsoids in 4-dimensions. As is shown by the work of Guth [12] and Hind–Kerman [15], very interesting new phenomena appear in higher dimensions.

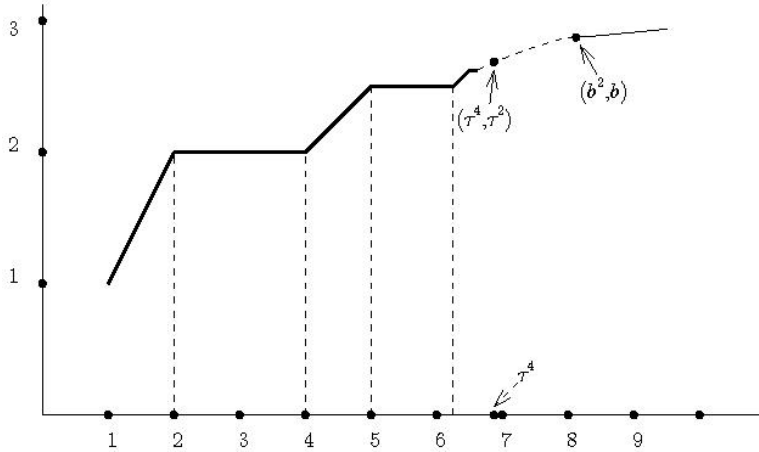


FIGURE 2.1. The approximate graph of  $c(a)$ . Here  $b := \frac{17}{6}$ .

**Theorem 2.1** (McDuff–Schlenk). *Let  $\tau = \frac{1+\sqrt{5}}{2}$ . The graph of  $c(a)$  divides into three parts:*

- if  $1 \leq a < \tau^4$  the graph is piecewise linear – an infinite Fibonacci staircase converging to  $(\tau^4, \tau^2)$ ;
- $\tau^4 \leq a < 8\frac{1}{36}$  is a transitional region;  $c(a) = \sqrt{a}$  except on a finite number of short intervals;
- if  $a \geq 8\frac{1}{36} = (\frac{17}{6})^2$  then  $c(a) = \sqrt{a}$ .

<sup>4</sup>In fact, the proofs of the results stated below are not yet quite complete, so that at this stage they should still be considered conjectural.

**Description of the Fibonacci stairs:** Let

$$g_0 = g_1 = 1, \quad g_2 = 2, \quad g_3 = 5, \quad g_4 = 13, \quad g_5 = 34, \quad g_6 = 89, \dots,$$

be the odd terms in the sequence of Fibonacci numbers; set

$$a_n := (g_{n+1}/g_n)^2, \quad b_n := g_{n+2}/g_n$$

so that  $a_n < b_n < a_{n+1}$ , and  $a_n \rightarrow \tau^4$ . Then the claim is that

$$c(x) = x/\sqrt{a_n} \text{ on } [a_n, b_n], \quad \text{and} \quad c(x) = \sqrt{a_{n+1}} \text{ on } [b_n, a_{n+1}].$$

Since  $\frac{b_n}{\sqrt{a_n}} = \sqrt{a_{n+1}}$  this gives a continuous graph on the interval  $1 \leq a < \tau^4$ . For example,

$$c(2) = c(b_0) = \sqrt{a_2} = 4 = c(4), \quad c(5) = c(b_1) = \frac{5}{2} = c(6\frac{1}{4}).$$

Note that the function  $c(a)/a$  is continuous and nonincreasing. (This holds because, for all  $\lambda \geq 1$  and  $\varepsilon > 0$ , we have  $E(\lambda a, 1) \subset E(\lambda a, \lambda) = \lambda E(a, 1) \subset \lambda B(c(a) + \varepsilon)$ .) It follows that the function  $c$  is determined on the interval  $[1, \tau^4]$  by its values at the points  $a_n, b_n$ .

As we describe in more detail below, the first part of Theorem 2.1 is mostly proven. The proof of the second statement is also almost complete, while the third statement is fully proven. The proofs are based on the two Propositions 2.2 and 2.4 stated below. The first relates the ellipsoidal embedding problem to a ball packing problem, while the second gives a recipe for solving ball packing problems.

**Proposition 2.2** (McDuff [21]). *For each rational  $a \geq 1$  there is a finite weight expansion  $\mathbf{w}(a) = (w_1, \dots, w_M)$  such that  $E(a, 1)$  embeds symplectically in the interior of  $B^4(\mu)$  if and only if the disjoint union of balls  $\bigsqcup_{i \leq M} B(w_i)$  embeds symplectically in the interior of  $B^4(\mu)$ .*

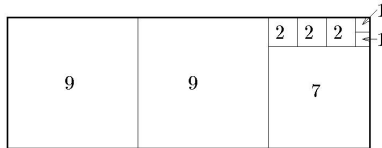


FIGURE 2.2.  $\mathbf{w}(25/9) = (1, 1, 7/9, 2/9, 2/9, 2/9, 1/9, 1/9)$  which we abbreviate as  $(1^{\times 2}, 7/9, (2/9)^{\times 3}, (1/9)^{\times 2})$ . The multiplicities 2, 1, 3, 2 of the weights are the terms (or partial quotients) of the continued fraction representation  $[2; 1, 3, 2]$  of  $25/9$ .

Weight expansions are perhaps best explained pictorially as in Figure 2.2, but here is a formal definition.

**Definition 2.3.** Let  $a = p/q \in \mathbb{Q}$  written in lowest terms. The **weight expansion**  $\mathbf{w} := (w_i) := (w_1, \dots, w_k)$  of  $a \geq 1$  is defined recursively as follows:

- $w_1 = 1$ , and  $w_n \geq w_{n+1} > 0$  for all  $n$ ;
- if  $w_i > w_{i+1} = \dots = w_n$  (where we set  $w_0 := a$ ) then
 
$$w_{n+1} = \begin{cases} w_n & \text{if } w_{i+1} + \dots + w_{n+1} = (n-i+1)w_{i+1} \leq w_i \\ w_i - (n-i)w_{i+1} & \text{otherwise;} \end{cases}$$
- the sequence stops at  $w_n$  if the above formula gives  $w_{n+1} = 0$ .

The number  $k$  of nonzero entries in  $\mathbf{w}(a)$  is called the **length**  $\ell(a)$  of  $a$ .

The second main proposition solves the ball packing problem. Given  $k$  write  $\mathbf{m} := (m_1, \dots, m_k) \in \mathbb{N}^k$ ,  $m_i \geq m_{i+1}$ . Let

$$\mathcal{E} := \mathcal{E}_k := \left\{ (d; \mathbf{m}) : d^2 + 1 = \sum_i m_i^2, \quad 3d - 1 = \sum_i m_i, \quad \text{and } (*) \right\}.$$

We shall refer to the first two conditions above as Diophantine conditions. The third condition  $(*)$  is more algebraic, requiring that tuple  $(d; \mathbf{m})$  can be reduced to  $(0; 1)$  by repeated Cremona moves. Such a move takes  $(d; \mathbf{m})$  to  $(d'; \mathbf{m}')$ , where  $(d'; \mathbf{m}')$  is obtained by first transforming  $(d; \mathbf{m})$  to  $(2d - m_1 - m_2 - m_3; d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, m_5, \dots)$ , and then reordering the new  $m_i$  (discarding zeros) so that they do not increase.

It was shown in Li–Li [20] (using Seiberg–Witten theory) that  $\mathcal{E}_k$  is the set of homology classes  $dL - \sum m_i E_i$  represented by symplectic exceptional divisors in the  $k$ -fold blow up of  $\mathbb{C}P^2$ .<sup>5</sup> It follows that the intersection number  $(d; \mathbf{m}) \cdot (d'; \mathbf{m}') := dd' - \sum m_i m'_i$  of any two elements in  $\mathcal{E}$  is nonnegative.

**Proposition 2.4** (McDuff–Polterovich [22], Biran [1]).

$$\bigsqcup_{i \leq k} B(w_i) \xrightarrow{s} B(\mu) \iff a < \mu^2 \text{ and } \mu d \geq \sum m_i w_i \quad \forall (d; \mathbf{m}) \in \mathcal{E}_k.$$

**Corollary 2.5.**  $c(a) = \sup \left\{ \sqrt{a}, \sup_{(d, \mathbf{m}) \in \mathcal{E}_k} \frac{\sum m_i w_i(a)}{d} \right\}$ .

**Example:**  $\mathcal{E}_4$  has the single element  $(1; 1, 1)$  (corresponding to  $L - E_1 - E_2$ ) and  $\mathbf{w}(4) = (1, 1, 1, 1) = (1^{\times 4})$ . Therefore  $c(4) = 2$ . However  $\mathcal{E}_5$  also contains  $(2; 1, \dots, 1) = (2; 1^{\times 5})$  corresponding to  $2L - \sum_{i=1}^5 E_i$ . Thus  $c(5) = \sup \{ \sqrt{5}, 2, 5/2 \} = 5/2$ .

<sup>5</sup>Here we denote by  $L$  the class of a line  $[\mathbb{C}P^1]$  and by  $E_i$  the class of the  $i$ th exceptional divisor. For more about blow ups see §2.2. Observe also that this is a place where symplectic geometry shows how flexible it is in comparison with algebraic geometry. In algebraic (or complex) geometry, one would want to describe the homology classes that can be represented by holomorphically embedded exceptional divisors for a “generic” complex structure on the blow up. This question is far from being understood.

The existence of the Fibonacci staircase in Theorem 2.1 is based on the somewhat surprising discovery that there are elements of  $\mathcal{E}$  related to the weight expansions of the odd Fibonacci numbers.

**Proposition 2.6** (McDuff–Schlenk [23]). *Denote by  $g_n$  the  $n$ th odd Fibonacci number, and consider  $b_n = g_{n+2}/g_n$ . Then*

$$E(b_n) := (g_{n+1}; g_n \mathbf{w}(b_n)) \in \mathcal{E}_M.$$

**Example:**  $n = 2$  gives:  $(5; 2\mathbf{w}(13/2)) = (5; 2^{\times 6}, 1^{\times 2})$ , while

$$5L - 2(E_1 + \cdots + E_6) - E_7 - E_8 \in \mathcal{E}_8.$$

Similarly,  $n = 4$  gives

$$(34; 13\mathbf{w}(89/13)) = (34; 13^{\times 6}, 11, 2^{\times 5}, 1^{\times 2}),$$

and one can check that

$$34L - 13(E_1 + \cdots + E_6) - 11E_7 - 2(E_8 + \cdots + E_{12}) - E_{13} - E_{14} \in \mathcal{E}.$$

Using Proposition 2.6, we can find  $c(b_n)$  for all  $n$ :

$$\begin{aligned} d\mu \geq \sum m_i w_i &\implies g_{n+1}\mu \geq g_n \sum w_i^2 = g_n a \\ &\implies \mu \geq (g_n g_{n+2}) / (g_{n+1} g_n) = \sqrt{a_{n+1}}. \end{aligned}$$

This shows  $c(b_n) \geq \sqrt{a_{n+1}}$ . We get equality because  $dd' \geq \sum m_i m'_i$  for all distinct pairs  $(d, \mathbf{m}), (d', \mathbf{m}') \in \mathcal{E}_M$  (by positivity of intersections).

**Claim I:** *If  $a := a_n = (g_{n+1}/g_n)^2$  then*

$$E(a_n) := (g_n g_{n+1}, g_n^2(\mathbf{w}(a), 1)) \in \mathcal{E}_k.$$

It is not hard to show that the element  $E(a_n)$  satisfies the Diophantine conditions needed to be in  $\mathcal{E}_k$ . However, at the time of writing the third condition (\*), though proven for even  $n$ , has not yet been fully proven for odd  $n$ . It turns out that, although the element  $E(b_n)$  of Proposition 2.6 reduces to  $E(b_{n-2})$  under five Cremona moves, the corresponding behavior of  $E(a_n)$  is much more complicated.

If Claim I were true (which seems very likely), then using positivity of intersections one could easily show that  $c(a_n) = \sqrt{a_n}$ . The graph of  $c(a)$  for  $a \in [1, \tau^4]$  would then be as described in Theorem 2.1.

We also claim that  $c(a) = \frac{1+a}{3}$  on the interval  $[\tau^4, 7]$ , with obstruction given by the class  $(d; \mathbf{m}) = (3, 2, 1^{\times 6}) \in \mathcal{E}_7$ . So far this has been proved for  $a > 6\frac{11}{12}$  and for the (even) convergents to  $\tau$ . The proof is quite hard because there are “fake” elements of  $\mathcal{E}$ . More precisely, by looking at ratios of even Fibonacci numbers one can construct tuples  $(d; \mathbf{m})$  that satisfy the

Diophantine conditions to be in  $\mathcal{E}$  but that fail condition (\*); in fact they have negative intersection with  $(3, 2, 1^{\times 6})$ , that is

$$3d < 2m_0 + m_1 + \cdots + m_7.$$

There are also infinitely many other elements of  $\mathcal{E}$ , again constructed from the even Fibonacci numbers, for which  $(\sum m_i w_i(a))/d = \frac{1+a}{3}$  on some interval  $[\tau^4, \tau^4 + \varepsilon]$ . Thus one cannot understand  $c(a)$  on this interval simply by making estimates using the Diophantine conditions. On the other hand, this approach is sufficient to show that there are no obstructions on the interval  $a \geq 8\frac{1}{36}$ . Full details of the proof will be given in [23].

**The special properties of  $a = \tau^4$ .**

$a = \tau^4$  is the positive root of  $3\sqrt{a} = a + 1$ . The sharpest constraints come from  $(d, \mathbf{m})$  where  $\mathbf{m} \approx \lambda \mathbf{w}(a)$ . Since  $\sum w_i^2 = a$  and  $\sum m_i^2 \approx d^2$ , we need  $\lambda \approx d/\sqrt{a}$ . But also

$$\sum w_i = 1 + a - 1/q \approx 1 + a, \quad \sum \lambda w_i \approx \sum_i m_i \approx 3d$$

implies  $3d \approx \lambda(1 + a)$  or  $3\sqrt{a} \approx 1 + a$ .

**Why there are no constraints for  $a > 9$ .**

We saw above that  $c(a) = \sup_{(d, \mathbf{m}) \in \mathcal{E}} \frac{\sum m_i w_i(a)}{d}$ . Hence  $c(a) > \sqrt{a}$  only if there is  $(d, \mathbf{m}) \in \mathcal{E}$  such that  $\sum m_i w_i(a)/d > \sqrt{a}$ . But  $\sqrt{a} \geq 3$ , and

$$\sum m_i w_i \leq \sum m_i = 3d - 1 \quad \text{since all } w_i \leq 1.$$

So  $\sum m_i w_i/d \leq 3$  always. This was Biran's argument in [1]. More complicated versions of this argument also show there are no constraints for  $a \geq 8\frac{1}{36}$ .

**2.1. Relation to lattice packing and embedded contact homology.**

The embedded contact homology theory of Hutchings and Taubes [18] is not yet fully developed. But, according to Hutchings, the index calculations in this theory should provide a series of obstructions to symplectically embedding  $E(a, 1)$  into  $B(\mu)$  of the following nature.

Let  $a \geq 1$  be irrational. For each pair of integers  $A, B \geq 0$ , consider the triangle

$$T_{A,B}^a := \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x + ay \leq A + aB\}.$$

We define

$$k_{A,B}(a) := \frac{A+Ba}{d},$$

where  $d$  is the positive smallest integer such that

$$\#(T_{A,B}^a \cap \mathbb{Z}^2) \leq \frac{1}{2}(d+1)(d+2).$$

Further, set

$$c_{ECH}(a) := \sup_{A,B} \{k_{A,B}(a)\}.$$

Since  $c_{ECH}(a)$  is nondecreasing, we may extend it to rational  $a$  by defining

$$c_{ECH}(a) = \sup_{z < a, z \text{ irrat}} c_{ECH}(z).$$

**Claim II:**  $c_{ECH}(a) \leq c(a)$  for all  $a$ .

As mentioned above, Claim II is not yet fully proven, but it is expected to hold. Further, it seems probable that these two functions actually coincide. As evidence for this, we have the following results whose proofs were explained to me by Hutchings.

**Lemma 2.7.** *Suppose that  $a$  is rational, define  $T = T_{A,B}^a$  as above and suppose that*

$$\#(T \cap \mathbb{Z}^2) \leq \frac{1}{2}(d+1)(d+2) + k - 1 = \frac{1}{2}(d^2 + 3d) + k,$$

where  $k \geq 1$  is the number of integral vertices on the slant edge of  $T$ . Assume that  $(A, B)$  (resp.  $(A', B')$ ) is the integral point on the slant edge with smallest (resp. largest)  $x$ -coordinate. Then there is  $\varepsilon > 0$  such that

$$c_{ECH}(z) \geq \frac{A+zB}{d} \text{ if } z \in (a - \varepsilon, a), \quad c_{ECH}(z) \geq \frac{A'+zB'}{d} \text{ if } z \in (a, a + \varepsilon).$$

*Proof.* To prove the statement for  $z < a$  it suffices by continuity to consider irrational  $z$  of the form  $z = a - \varepsilon$ . Then, for small enough  $\varepsilon > 0$ , the triangle  $T_{A,B}^z$  contains  $k - 1$  fewer integral points than  $T$ . Therefore  $k_{A,B}(z) \geq \frac{A+zB}{d}$ , which proves the first statement. Similarly, the second statement holds because if  $z = a + \varepsilon$  is irrational and  $\varepsilon > 0$  is sufficiently small, the triangle  $T_{A',B'}^z$  contains  $k - 1$  fewer integral points than  $T$ .  $\square$

**Lemma 2.8.** *For all  $n \geq 1$ ,  $c_{ECH}(b_n) \geq \sqrt{a_{n+1}} = c(b_n)$ .*

*Proof.* Consider the triangle  $T_n \subset \mathbb{R}^2$  with vertices  $(0, 0)$ ,  $(g_{n+2}, 0)$  and  $(0, g_n)$ , where  $g_n$  is the  $n$ th odd Fibonacci number. Because  $g_n, g_{n+2}$  are mutually prime and satisfy the identities

$$g_n + g_{n+2} = 3g_{n+1}, \quad g_n g_{n+2} = g_{n+1}^2 + 1,$$

we find that

$$\begin{aligned} \#(T_n \cap \mathbb{Z}^2) &= \frac{1}{2}(g_n + 1)(g_{n+2} + 1) + 1 \\ &= \frac{1}{2}(g_{n+1}^2 + 3g_{n+1}) + 2. \end{aligned}$$

Therefore, because  $b_n = \frac{g_{n+2}}{g_n}$ , we can apply Lemma 2.7 with  $k = 2$ . Hence for some  $\varepsilon > 0$  we have

$$c_{ECH}(z) \geq \frac{zg_n}{g_{n+1}} \text{ when } z \in (b_n - \varepsilon, b_n],$$

and

$$c_{ECH}(z) \geq \frac{g_{n+2}}{g_{n+1}} \text{ when } z \in [b_n, b_n + \varepsilon).$$

In particular,  $c_{ECH}(b_n) = \frac{g_{n+2}}{g_{n+1}} = \sqrt{a_{n+1}}$ .  $\square$

The arguments that prove Claim II should also prove that  $c_{ECH}$  has the same *scaling property* as  $c$ , namely the function  $\frac{c_{ECH}(a)}{a}$  should be nonincreasing. If so, one can conclude that  $c_{ECH}(a) = c(a)$  for  $a \in [1, \tau^4]$ . To see this, observe first that Lemma 2.8 would imply that the two functions agree at  $a = b_n$ , and second that  $c(a)$  is the smallest nondecreasing function with the scaling property that takes the given values at  $b_n$ .

Numerical evidence suggests that  $c_{ECH}(a) = c(a)$  for all  $a$ . It seems very likely that, once  $c(a)$  is fully calculated, one could verify this by finding suitable triangles as in Lemma 2.8.

To my knowledge, rather little is known about lattice counting functions such as  $c_{ECH}$ . Here is one very simple result.

**Lemma 2.9.**  $c_{ECH}(a) \geq \sqrt{a}$  for all  $a \geq 1$ .

*Proof.* This holds because when  $s$  is large the number of integral points in  $T_{s,0}^a$  is estimated by its area  $\frac{1}{2}s^2/a$  with error  $O(s)$ . Therefore  $k_{s,0} \rightarrow \sqrt{a}$  as  $s \rightarrow \infty$ . Since  $A + Ba = s$  in this case, the result follows.  $\square$

In [13], Hardy and Littlewood<sup>6</sup> consider rather different asymptotic questions about the number of lattice points in  $T_{A,B}^a$ , looking at asymptotical behavior for fixed  $a$  rather than comparing the triangles for a given  $a$  with those for  $a = 1$  as we do. Interestingly enough the golden ratio  $\tau$  also plays a prominent role in their work.

**Remark 2.10.** Given positive numbers  $a, b$  with  $a \leq b$ , denote by  $N(a, b)$  the list (with repetitions) of all nonzero numbers of the form  $ma + nb$ , where  $m, n$  are nonnegative integers, put in nondecreasing order. Thus  $N(1, 1) = (1, 1, 2, 2, 2, 3, \dots)$ . Write

$$N(a, b) \preceq N(c, d)$$

if for all  $k \geq 1$  the  $k$  entry of  $N(a, b)$  is no greater than the  $k$ th entry of  $N(c, d)$ . Then it is easy to check that

$$c_{ECH}(a) = \inf \{ \mu : N(a, 1) \preceq N(\mu, \mu) \}.$$

Thus, if  $c_{ECH}(a) = c(a)$ , it would follow that

$$E(a, 1) \xrightarrow{s} B(\mu, \mu) \iff N(a, 1) \preceq N(\mu, \mu).$$

As pointed out by Hofer,<sup>7</sup> it is an easy step from here to wonder whether

$$E(a, 1) \xrightarrow{s} E(c, d) \iff N(a, 1) \preceq N(c, d).$$

Note that the embedding results of Guth [11] and Hind–Kerman [15] imply that the analogous result in dimensions  $> 4$  does *not* hold. However, embedded contact homology also does not exist in dimensions  $> 4$ . The question of what the obstructions actually are in higher dimensions is very interesting.

<sup>6</sup>I am indebted to Peter Sarnak for this reference.

<sup>7</sup>at a problem session at the Introductory Workshop to the Symplectic Topology program, August 21, 2009 at MSRI, Berkeley.

**2.2. Ideas behind the proofs.** Finally, we describe the basic ideas behind the proofs of the two key Propositions 2.2 and 2.4.

**I: Embedding balls and blowing up:**

To blow up a point in complex geometry you remove the point and replace it by the set of all (complex) lines through that point. In two complex dimensions, this set is a complex line  $E \cong \mathbb{C}P^1$  called the **exceptional divisor**. Its normal bundle  $\pi : L \rightarrow E$  has Chern class  $-1$ . In this context the basic fact is that  $L \setminus E$  is biholomorphic to  $\mathbb{C}^2 \setminus \{0\}$ .

On the other hand, in symplectic geometry the picture of blowing up is somewhat different. Darboux's theorem shows that a neighborhood of a point  $p$  in  $(M^4, \omega)$  can be identified with a neighborhood of  $\{0\}$  in  $\mathbb{R}^4 \equiv \mathbb{C}^2$ . Therefore one can blow up as before (with respect to a compatible complex structure) to get a manifold  $\widetilde{M}$  with blow down map  $bl : \widetilde{M} \rightarrow M$ . But one must put a symplectic form on  $\widetilde{M}$ , and the pullback form  $bl^*(\omega)$  vanishes on  $E$ . So the symplectic structure  $\widetilde{\omega}_a$  on  $\widetilde{M}$  must have the form  $bl^*(\omega) + a\pi^*(\sigma_E)$  near  $E$ , where  $\int_E \sigma_E = 1$  and  $\pi : \text{nbhd}(E) \rightarrow E$ . In this context the basic fact is the following:

$$((\text{nbhd } E \setminus E), \widetilde{\omega}_a) \text{ is symplectomorphic to } \mathbb{C}^2 \setminus B(a).$$

This means that to blow up symplectically with weight  $a$  one removes an embedded copy of the open ball  $\text{int } B(a)$  and then collapses  $\partial B(a) = S^3$  to  $S^2 \equiv E$  by the Hopf map (along the leaves of the characteristic foliation).

Proposition 2.4 is proved by using this description of symplectic blowing up to convert the ball packing problem into a question about the existence of symplectic forms on the  $k$ -fold blow up of  $\mathbb{C}P^2$ . The latter problem is solved using the theory of  $J$ -holomorphic curves as in [22, 1].

**II: Cutting  $E(a, 1)$  into balls via toric models**

We now describe the basic ideas behind the proof of Proposition 2.2.

If  $a = p/q$  the image of  $E(a, 1)$  under the moment map  $\Phi : \mathbb{C}^2 \rightarrow \mathbb{R}^2, (z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)$  is the triangle

$$\{0 \leq x_1, x_2; qx_1 + px_2 \leq 1\}.$$

As illustrated on the left of Figure 2.3, this corresponds to a singular variety, with singular points of orders  $p, q$ . (In contrast, the triangles in Figure 1.2 are standard, i.e. equivalent under an integral affine transformation to a triangle with vertices  $(0, 0), (a, 0), (0, a)$ , and so represent the smooth variety  $\mathbb{C}P^2$ .) We can cut this singular triangle into standard triangles of different sizes.

Figure 2.3 also shows that the triangle decomposition on the left is structurally the same as that giving the weights for  $5/3$ . One needs to remove the top right corner of the rectangle and then collapse its top and right side. As we explain in more detail in [21], it also corresponds to a joint resolution

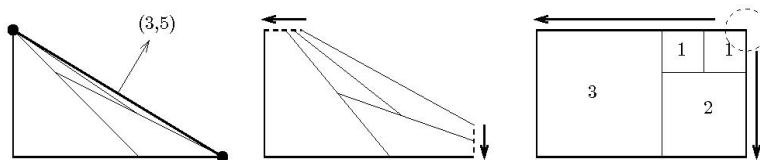


FIGURE 2.3. Cutting triangles into standard triangles as compared to cutting a rectangle into squares.

of the two singularities of the toric variety corresponding to the *complement* of the triangle in the positive quadrant.



FIGURE 2.4. The lines on the right hand figure are the cuts needed to resolve the singularities of the complement of the triangle in the left hand figure.

Figure 2.4 illustrates another key point. The cuts needed to resolve the two singularities in the shaded polytope (the result of removing the singular triangle) are illustrated on the right. Note that they are *parallel* to the cuts that decompose the singular triangle on the left of Figure 2.3 into standard triangles. (These cuts resolve the singularities because the matrices whose rows are adjacent normal vectors all have determinant  $\pm 1$ .)

**Remark 2.11.** The information contained in the weight expansion of  $a$  can be summarized in a diagram called the *Riemenschneider staircase*. This usually arises in the context of resolving singularities so that the interesting numbers are the coefficients of the Hirzebruch–Jung continued fraction expansions of the two related fractions  $p/q$  and  $p/(p - q)$  (where we assume  $p > 2q$ ); cf. Fulton [10]. However one can also construct the diagram from the multiplicities  $n_1, \dots, n_S$  of the entries in the weight expansion  $\mathbf{w}(a)$ . We illustrate this by the two examples in Figure 2.5. See Popescu-Pampu [25] for an extended discussion of these combinatorics; and Craw–Reid [6] for an example of the use of this diagram in current algebraic geometry. Also compare [21, Remark 3.12(ii)].

Observe first that if  $p, q$  are relatively prime with  $p > q$  then exactly one of the numbers  $p/q, p/(p - q)$  is greater than two. Thus, without loss of generality, we may assume that  $p/q > 2$  so that  $n_0 \geq 2$ . Given positive

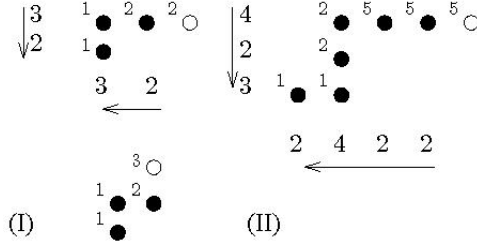


FIGURE 2.5. (I) illustrates  $a = p/q = 5/2 = [2; 2]$  while (II) illustrates  $a = p/q = 17/5 = [3; 2, 2]$ . The labels  $W_i(a)$  are the smaller numbers by the dots; the vertical and horizontal numbers give the components of the Hirzebruch–Jung continued fractions for  $p/q > 2$  and  $p/(p - q) < 2$ . The arrows give the direction in which one should read the numbers.

integers  $n_0, \dots, n_k$  define

$$\frac{p}{q} = [n_0; n_1, n_2, \dots, n_k] = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \dots + \frac{1}{n_k}}}.$$

It is not hard to see that the entries of the weight expansion  $\mathbf{w}(\frac{p}{q}) = (w_0, \dots, w_N)$  have multiplicities  $n_0, \dots, n_k$ . Moreover  $qw_i \in \mathbb{Z}$  for all  $i$ . In other words, the vector  $W(\frac{p}{q}) = (W_0, \dots, W_N) := q\mathbf{w}(\frac{p}{q})$  has the form

$$W(\frac{p}{q}) = \left( \underbrace{X_0, \dots, X_0}_{n_0}, \underbrace{X_1, \dots, X_1}_{n_1}, \dots, \underbrace{X_k, \dots, X_k}_{n_k} \right),$$

where  $X_i \in \mathbb{Z}$  and  $X_k = 1$ . We shall call these renormalized weights  $W_i = qw_i$  the labels of  $a = \frac{p}{q}$ .

The staircase is constructed from the upper right, proceeding to the left and down, by first placing  $n_0$  horizontal dots, then  $n_1$  vertical dots (starting in the row of the first dots), then  $n_2$  horizontal dots (starting in the row of the second dots), and so on. The first dot is white, and the others are black. One can recover the labels from the staircase by starting with 1 at the bottom left and moving back up the staircase according to the rule:

*at a horizontal move, the new label is the sum of the labels in the preceding column; at a vertical move, the new label is the sum of the labels in the preceding row.*

The coefficients of the Hirzebruch–Jung continued fraction for  $p/q$  are one more than the number of black dots in each row (read downwards). If one moves the white dot, putting it in a new row instead of a new column, and constructs the labels as described above, then one gets the labels for the associated fraction  $p/(p - q)$ . Correspondingly the coefficients of the

Hirzebruch–Jung continued fraction for  $p/(p - q)$  are one more than the number of black dots in each column (read from right to left). Thus Figure 2.5 (I) gives  $W(\frac{5}{2}) = (2, 2, 1, 1)$  and

$$\frac{5}{2} = 2 + \frac{1}{2} = 3 - \frac{1}{2}, \quad \frac{5}{3} = 1 + \frac{1}{1+\frac{1}{2}} = 2 - \frac{1}{3};$$

while Figure 2.5 (II) gives  $W(\frac{17}{5}) = (5, 5, 5, 2, 2, 1, 1)$  and

$$\frac{17}{5} = 3 + \frac{1}{2+\frac{1}{2}} = 4 - \frac{1}{2-\frac{1}{3}}, \quad \frac{17}{12} = 2 - \frac{1}{2-\frac{1}{4-\frac{1}{2}}}.$$

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