THE SYMPLECTOMORPHISM GROUP OF A BLOW UP

DUSA MCDUFF

Abstract. We study the relation between the symplectomorphism group $\mathrm{Symp}\, M$ of a closed connected symplectic manifold M and the symplectomorphism and diffeomorphism groups $\mathrm{Symp}\, \widetilde{M}$ and $\mathrm{Diff}\, \widetilde{M}$ of its one point blow up $\widetilde{M}$. There are three main arguments. The first shows that for any oriented M the natural map from $\pi_1(M)$ to $\pi_0(\mathrm{Diff} \widetilde{M})$ is often injective. The second argument applies when M is simply connected and detects nontrivial elements in the homotopy group $\pi_1(\mathrm{Diff} \widetilde{M})$ that persist into the space of self homotopy equivalences of $\widetilde{M}$. Since it uses purely homological arguments, it applies to c-symplectic manifolds (M, a), that is, to manifolds of dimension $2n$ that support a class $a \in H^2(M;\mathbb{R})$ such that $a^n \neq 0$. The third argument uses the symplectic structure on M and detects nontrivial elements in the (higher) homology of $B\mathrm{Symp}\, \widetilde{M}$ using characteristic classes defined by parametric Gromov–Witten invariants. Some results about many point blow ups are also obtained. For example we show that if M is the 4-torus with k-fold blow up $\widetilde{M}_k$ (where $k > 0$) then $\pi_1(\mathrm{Diff} \widetilde{M}_k)$ is not generated by the groups $\pi_1\big(\mathrm{Symp}\,(\widetilde{M}_k, \widetilde{\omega})\big)$ as $\widetilde{\omega}$ ranges over the set of all symplectic forms on $\widetilde{M}_k$.

1. Main results

In this note we investigate the relation between the symplectomorphism group of a closed connected symplectic manifold (M, ω) of dimension $2n \geq 4$ and of its one point blow up $(\widetilde{M}, \widetilde{\omega}_\epsilon)$. (Here $\epsilon > 0$ measures the size of the blow up.) When $\dim M = 4$ and ϵ is sufficiently small, it is known from work of Lalonde–Pinsonnault [14] and Kędra [8] that in several cases there is a homotopy equivalence between $\mathrm{Symp}\, \widetilde{M} := \mathrm{Symp}(\widetilde{M}, \widetilde{\omega}_\epsilon)$ and $\mathrm{Symp}(M, p) := \mathrm{Symp}(M, p, \omega)$, the group of symplectomorphisms of M that fix the point p. Our aim here is to understand both the role played by the symplectic form and what happens in higher dimensions. Throughout we investigate stable phenomena that persist in the diffeomorphism group Diff and under various kinds of deformation; see Remark 1.11.

We will present three different arguments. The first studies a natural map $\widetilde{f}_* : \pi_1(M) \to \pi_0(\mathrm{Diff} \widetilde{M})$ defined for any oriented manifold M, and shows that $\widetilde{f}_*$ is often injective. The second works best for simply connected M and detects nontrivial elements in the homotopy group $\pi_1(\mathrm{Diff} \widetilde{M})$ that persist into $\pi_1(\widetilde{M}^{\widetilde{M}})$, where X^X denotes the space of continuous selfmaps of X. Since it uses purely homological ideas, this argument applies to c-symplectic manifolds (M, a), that is, to manifolds of dimension $2n$ that support a class $a \in H^2(M;\mathbb{R})$ such that $a^n \neq 0$. In §6 we add some symplectic ideas to investigate the difference between $\pi_1(\mathrm{Diff} \widetilde{M}_k)$ and $\pi_1(\mathrm{Symp}\, \widetilde{M}_k)$ for a k-fold blow up of a 4-manifold. The third argument, presented in §4, needs the symplectic structure on M and detects nontrivial elements in

Date: October 4, 2006, with minor revisions May 29, 2007.

2000 *Mathematics Subject Classification.* 53D35, 57R17, 57S05.

Key words and phrases. symplectomorphism group, diffeomorphism group, symplectic blow up, parametric Gromov–Witten invariants, symplectic characteristic classes.

partially supported by NSF grants DMS 0305939 and 0604769.

the (higher) homology of $B\mathrm{Symp}\,\widetilde{M}$. It uses a characteristic class on $B\mathrm{Symp}\,\widetilde{M}$ defined by a suitable parametric Gromov–Witten invariant.

Before explaining our results in more detail, we need some preparation. Recall that the classifying space $B\mathrm{Symp}(M,p)$ is the total space of the universal M-bundle over $B\mathrm{Symp}\,M$

$$M \to B\mathrm{Symp}(M,p) \stackrel{\pi}{\to} B\mathrm{Symp}\,M. \tag{1.1}$$

(Later we sometimes denote this bundle by $M_{\mathrm{Symp}} \to B\mathrm{Symp}\,M$.) The pullback of this bundle over $\pi : B\mathrm{Symp}(M,p) \to B\mathrm{Symp}\,M$ is the universal M-bundle with section. In other words, a map $\phi : X \to B\mathrm{Symp}(M,p)$ determines and is determined up to homotopy by a triple (P_ϕ, π_ϕ, s), where $\pi_\phi : P_\phi \to X$ is a symplectic M-bundle and $s : X \to P_\phi$ is a section. Blowing up along this section gives a map[1]

$$\beta : B\mathrm{Symp}(M,p) \to B\mathrm{Diff}\,\widetilde{M}.$$

The composite map

$$\widetilde{f} : M \to B\mathrm{Diff}\,\widetilde{M} \tag{1.2}$$

represents the $\widetilde{M}$-bundle over M that is obtained from the trivial bundle $M \times M \to M$ by blowing up along the diagonal.

More generally, let M be an oriented even dimensional smooth manifold, choose a Hermitian metric on the tangent space T_pM at p that is compatible with the orientation (and with the symplectic form if one is given), and denote by $\mathrm{Diff}^U(M,p)$ the subgroup of $\mathrm{Diff}(M,p)$ consisting of diffeomorphisms whose derivative at p preserves the Hermitian metric and so is unitary. Then, maps $X \to B\mathrm{Diff}^U(M,p)$ classify smooth M-bundles over X provided with sections σ that have Hermitian normal bundles. Hence again one can blow up along the section to get a map

$$\beta^U : B\mathrm{Diff}^U(M,p) \to B\mathrm{Diff}\,\widetilde{M}.$$

Because a symplectic vector bundle has a complex structure that is unique up to homotopy, for symplectic M there is a commutative diagram

$$\begin{array}{ccc} B\mathrm{Symp}^U(M,p) & \stackrel{\simeq}{\to} & B\mathrm{Symp}(M,p) \\ \iota \downarrow & & \beta \downarrow \\ B\mathrm{Diff}^U(M,p) & \stackrel{\beta^U}{\to} & B\mathrm{Diff}\,\widetilde{M} \end{array}$$

in which the inclusion on the top line is a homotopy equivalence.

Suppose now that M is an oriented manifold of dimension $2n$, and consider the composite

$$\widetilde{f}_* : \pi_1 M \to \pi_1\big(B\mathrm{Diff}(M,p)\big) \stackrel{\beta^U}{\to} \pi_1(B\mathrm{Diff}\,\widetilde{M}).$$

(The first map here is induced by (1.1), while the second is well defined because an oriented $\mathbb{R}^{2n}$ bundle over S^1 has a unique homotopy class of complex structures.) Because there is a fibration

$$\mathrm{Diff}\,M \stackrel{\mathrm{ev}}{\to} M \to B\mathrm{Diff}(M,p), \quad \mathrm{ev}(\phi) := \phi(p),$$

the kernel of $\widetilde{f}_*$ contains the evaluation subgroup $\mathrm{ev}_*\big(\pi_1(\mathrm{Diff}\,M)\big)$.

[1]We take the target space to be $B\mathrm{Diff}$ rather than $B\mathrm{Symp}$ since then we can use the pointwise blow up of algebraic geometry. Because $B\mathrm{Symp}(M,p)$ is noncompact it is not clear that there is a symplectic blow up of uniform size $\epsilon > 0$ over the whole of $B\mathrm{Symp}(M,p)$.

Theorem 1.1. *For any closed oriented $2n$-manifold M the kernel of*

$$\widetilde{f}_*: \pi_1 M \to \pi_1(B\mathrm{Diff}\,\widetilde{M})$$

acts trivially on $\pi_i M$ for $1 \le i < \dim M$. In particular, it is contained in the center of $\pi_1 M$ and vanishes if M itself is a blow up. It also vanishes if the Euler characteristic $\chi(M)$ is nonzero.

Remark 1.2. We will see in §2 that the kernel of $\widetilde{f}_*$ is contained in the image of the evaluation map $\widetilde{\mathrm{ev}}_*: \pi_1(M^{\widetilde{M}}) \to \pi_1 M$ given by $\widetilde{\mathrm{ev}}(\phi) = \phi(\widetilde{p})$, where $\widetilde{p} \in \widetilde{M}$ is fixed. (Here $M^{\widetilde{M}}$ denotes the space of maps $\widetilde{M} \to M$, with base point at the blow down $\phi_0 : \widetilde{M} \to M$.) This evaluation map is very similar to the usual one $\mathrm{ev}_*: \pi_1(M^M) \to \pi_1 M$ considered by Gottlieb [6], though it might in principle have a larger image. The properties mentioned above are some of those that Gottlieb established for $\mathrm{im}\,\mathrm{ev}_*$.

We now consider π_2. Denote by

$$I_c: \quad \pi_2(B\mathrm{Diff}^U(M,p)) \to \pi_2(BU(n)) \cong \mathbb{Z}. \tag{1.3}$$

the homomorphism induced by taking the derivative at p. In the case when $[\sigma] \in \pi_2(B\mathrm{Diff}^U(M,p))$ is in the image of $B\mathrm{Symp}(M,p)$ and so corresponds to a section s_σ of a symplectic bundle $P \to S^2$ with its canonical Hermitian structure, then $I_c([\sigma])$ is the value on s_σ of the first Chern class c_1^V of the vertical tangent bundle.

If (M,a) is c-symplectic and simply connected, there is another useful homomorphism

$$I_a : \pi_2(B\mathrm{Diff}^U(M,p)) \to \mathbb{R}$$

that we now define. Consider the universal M-bundle

$$M \to B\big(\mathrm{Diff}(M,p) \cap \mathrm{Diff}_0 M\big) \xrightarrow{\pi} B\mathrm{Diff}_0 M, \tag{1.4}$$

where Diff_0 denotes the identity component of Diff. Denote by $\mathcal{C} := \mathcal{C}_M$ the (open) cone consisting of all classes $a \in H^2(M;\mathbb{R})$ such that $a^n \neq 0$.

Lemma 1.3 ([10])**.** *If $H^1(M;\mathbb{R}) = 0$ the restriction map*

$$H^2\Big(B\big(\mathrm{Diff}(M,p) \cap \mathrm{Diff}_0 M\big);\mathbb{R}\Big) \to H^2(M;\mathbb{R})$$

is surjective. Moreover, each $a \in \mathcal{C}$ has a unique extension to a class $\widehat{a}$ such that the fiberwise integral $\pi_!(\widehat{a}) := \int_M \widehat{a}^{n+1} \in H^2(B\mathrm{Diff}_0 M)$ vanishes.

Sketch of proof. Consider the Leray–Serre spectral sequence for the fibration (1.4). Since $E_2^{2,1} = 0$ and $\mathcal{C}$ is open, the first statement will hold provided that the differential $d := d^3_{0,2}$ vanishes on all $a \in \mathcal{C}$. But $0 = d(a^{n+1}) = (n+1)a^n da$ which implies that $da = 0$ since multiplication by a^n induces an isomorphism $E_2^{3,0} \to E_2^{3,2n}$. If $\widehat{a}'$ is any extension of a, it is easy to check that

$$\widehat{a} := \widehat{a}' - \frac{1}{\lambda(n+1)}\pi^*\big(\pi_!(\widehat{a}')^{n+1}\big),$$

satisfies the conditions if $\lambda := \int_M a^n$. (Here $\pi_! : H^{2n+2} \to H^2$ denotes the cohomology push forward, i.e. integration over the fiber.) □

The pullback of $\widehat{a}$ to the total space of an M-bundle $P \to Z$ is often called the **coupling class**; we will also call it the **normalized extension** of a. On a product bundle, for example, it is just the obvious pullback $pr_M^*(a)$ of a. Since each $[\sigma] \in \pi_2(B\mathrm{Diff}^U(M,p))$

has a unique lift to an element (also called $[\sigma]$) of $\pi_2(B(\mathrm{Diff}^U(M,p)\cap \mathrm{Diff}_0))$, we may define I_a by setting:

$$I_a([\sigma]) := \int_{S^2} \sigma^*(\widehat{a}). \tag{1.5}$$

If we think of $[\sigma]$ as an M-bundle $P \to S^2$ with section s_σ, then $I_a([\sigma])$ is given by evaluating the coupling class $\widehat{a}$ on the section s_σ.

We say that $[\sigma] \in \pi_2(B\mathrm{Diff}^U(M,p))$ is **homologically visible** if it has nonzero image under the homomorphism $I_c \oplus I_a$ for some $a \in \mathcal{C}$.

Theorem 1.4. *Let M be a simply connected c-symplectic manifold. Then every homologically visible class $[\sigma] \in \pi_2(B\mathrm{Diff}^U(M,p))$ has nonzero image under the composite map $\pi_2(B\mathrm{Diff}^U(M,p)) \to \pi_2\big(B\mathrm{Diff}\widetilde{M}\big) \to \pi_2\big(B(\widetilde{M}^{\widetilde{M}})\big)$.*

Corollary 1.5. *Let (M,a) be a c-symplectic manifold. Then there is a homomorphism $\widetilde{f}_* : \mathbb{Z}\oplus\pi_2(M) \to \pi_2\big(B\mathrm{Diff}\widetilde{M}\big)$ whose kernel is contained in the kernel of the rational Hurewicz homomorphism $\pi_2(M)\to H_2(M;\mathbb{Q})$.*

Proof. First assume that $\pi_1(M)=0$. The boundary map $\pi_{*+1}(B\mathrm{Diff}_0)\to\pi_*(M)$ in the fibration (1.4) desuspends to the map $\mathrm{ev}_* : \pi_*(\mathrm{Diff}_0)\to\pi_*(M)$ induced by evaluation

$$\mathrm{ev} : \mathrm{Diff}_0 \to M, \quad \phi \mapsto \phi(p).$$

Since ev induces the trivial map on $\pi_2\otimes\mathbb{Q}$, the kernel of the induced map $f_* : \pi_2(M) \to \pi_2(B\mathrm{Diff}(M,p))$ consists of torsion elements. This map f_* can be described explicitly as follows. Given $\sigma : S^2 \to M$ the element $f_*[\sigma]$ corresponds to the trivial bundle $M\times S^2 \to S^2$ with section $gr_\sigma : z \mapsto (\sigma(z),z)$. Since the normalized extension of the class $a\in\mathcal{C}_M$ to $M\times S^2$ is just $pr_M^*(a)$, we find that $I_a(f_*[\sigma]) = a([\sigma])$. Therefore, because $\mathcal{C}$ is open, the elements in $\mathrm{im} f_*$ that are not detected by some homomorphism I_a have the form $f_*[\sigma]$ where $[\sigma]$ maps to zero in $H_2(M;\mathbb{Q})$. Since $\pi_2(SO(2n)/U(n))\cong\mathbb{Z}$, each element in $\pi_2(B\mathrm{Diff}(M,p))$ has $\mathbb{Z}$ different lifts to $\pi_2(B\mathrm{Diff}^U(M,p))$, distinguished by the values of I_c. Now apply Theorem 1.4. The general case is proved in §3. □

Remark 1.6. When $\pi_1(M)=0$, the above argument shows that the elements persist in $\pi_2\big(B(\widetilde{M}^{\widetilde{M}})\big)$. However this may not hold in general.

When (M,ω) is symplectic, the map $\widetilde{f} : M \to B\mathrm{Diff}\widetilde{M}$ defined in (1.2) factors through $B\mathrm{Symp}(\widetilde{M},\widetilde{\omega}_\epsilon)$ for suitably small ϵ. One can construct this lift explicitly as follows. Consider the product $M\times M$ with symplectic form $\Omega := \omega\times\omega$. Then the diagonal is a symplectic submanifold, and so one can blow up normal to it by some amount $\epsilon_0>0$ to get a symplectic fibration

$$(\widetilde{M},\widetilde{\omega}_{\epsilon_0}) \to (\widetilde{P}_{\epsilon_0},\widetilde{\Omega}_{\epsilon_0}) \to M. \tag{1.6}$$

By shrinking the size of the blow up, one gets a corresponding fibration $(\widetilde{P}_\epsilon,\widetilde{\Omega}_\epsilon)\to M$ for every $\epsilon\le\epsilon_0$. Note that the initial fibration $(\widetilde{P}_{\epsilon_0},\widetilde{\Omega}_{\epsilon_0})\to M$ might depend on the choice of blow up. However any two choices become equivalent for small enough $\epsilon>0$. Define

$$\widetilde{f}_\epsilon : M \to B\mathrm{Symp}(\widetilde{M},\widetilde{\omega}_\epsilon) \tag{1.7}$$

to be the classifying map of this fibration. Then for each $\sigma : S^2\to M$ the class $(\widetilde{f}_\epsilon)_*([\sigma])$ is represented by the pullback of $\widetilde{P}_\epsilon \to M$ over σ. (Kędra gives another description of this map in [8].)

Theorem 1.4 implies that the subgroup $\widetilde{f}_*(\mathbb{Z}\oplus 0)$ is disjoint from the image of $(\widetilde{f}_\epsilon)_*(\pi_2(M))$ in $\pi_2(B\mathrm{Diff}\widetilde{M})$. It seems very likely that for any form $\widetilde{\omega}_\epsilon$ there are no elements in $\pi_2\big(B\mathrm{Symp}(\widetilde{M},\widetilde{\omega}_\epsilon)\big)$ that map into $\widetilde{f}_*(\mathbb{Z}\oplus 0)$. This is true when $M = \mathbb{C}P^2$ since in this case $\pi_2\big(B\mathrm{Symp}(\widetilde{M},\widetilde{\omega}_\epsilon)\big)$ is known to be $\mathbb{Z}$. It is not clear how to prove this in general, though §6 makes some progress in the 4-dimensional case: see Proposition 6.11 and Corollary 6.12.

The next result is based on detailed knowledge of the symplectic geometry of the one point blow up $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2 =: X$ of $\mathbb{C}P^2$. In particular, it uses the fact (proved in [11]) that every symplectic form on X is diffeomorphic to some $\widetilde{\omega}_\epsilon$.

Proposition 1.7. *Let $X = \mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. Then $\pi_2(B\mathrm{Diff}X)$ is not generated by the images of $\pi_2(B\mathrm{Symp}(X,\omega))$, as ω varies over the space of all symplectic forms on X.*

Remark 1.8. (i) During the proof of this proposition we show that the obvious inclusion $S^2 := \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$ gives rise not to the generator of $\pi_2(B\mathrm{Symp}X)$ (as one might initially expect) but to three times this generator.
(ii) There are many blow up manifolds that do not admit circle actions. For example, in dimension 4 Baldridge [2] has shown that if X has $b^+ > 1$ and admits a circle action with a fixed point then its Seiberg–Witten invariants must vanish. Since manifolds that admit fixed point free circle actions must have zero Euler characteristic, this implies that no simply connected Kähler surface with $b^+ > 1$ admits a circle action. On the other hand, Corollary 1.5 implies that if X is a blow up and $\pi_1 X = 0$ the groups $\pi_1(\mathrm{Symp})$ and $\pi_1(\mathrm{Diff})$ do not vanish. Thus, as pointed out in Kędra [9], there are plenty of examples of nonzero elements of $\pi_1(\mathrm{Symp})$ or $\pi_1(\mathrm{Diff})$ that cannot be represented by a circle action.[2] One of the motivations of the present work was to understand the limits of Kędra's argument.

Our final argument detects parts of the image of $H_*(M)$ in the homology of $B\mathrm{Symp}\,\widetilde{M}$. If $\pi_1 M = 0$, the classifying map $\widetilde{f}_\epsilon : M \to B\mathrm{Symp}\,\widetilde{M}$ of the fibration (1.6) factors through the simply connected space $B\mathrm{Symp}_0\widetilde{M} = B\mathrm{Ham}\,\widetilde{M}$, where Ham denotes the Hamiltonian subgroup $\mathrm{Ham} \subseteq \mathrm{Symp}_0$. We denote by $\mathcal{C}^*$ the subring of $H^*(M;\mathbb{Q})$ generated[3] by the classes in $H^2(M)$, and define

$$\mathcal{C}_* := \{\gamma \in H_*(M;\mathbb{Q}) : \kappa(\gamma) = 0 \text{ for all } \kappa \in \mathcal{C}^*\}.$$

Theorem 1.9. *Suppose that $\pi_1(M) = 0$. Then, the kernel of*

$$(\widetilde{f}_\epsilon)_* : H_*(M;\mathbb{Q}) \to H_*\big(B\mathrm{Ham}(\widetilde{M},\widetilde{\omega}_\epsilon);\mathbb{Q}\big)$$

is contained in $\mathcal{C}_*$.

When $\pi_1(M) \neq 0$, Theorem 1.1 implies that the bundle (1.6) is usually not trivial over its 1-skeleton. Thus its classifying map need not lift to to $B\mathrm{Ham}$ (or even to $B\mathrm{Symp}_0$). This is important because we detect the image of $(\widetilde{f}_\epsilon)_*$ by using characteristic classes, and typically these live on $B\mathrm{Ham}$ rather than $B\mathrm{Symp}_0$ or $B\mathrm{Symp}$; cf. §4. One way one might try to deal with this is to note that because the fiberwise symplectic forms on (1.6) extend

[2] By using the geometric decomposition of a Hamiltonian S^1 manifold provided by the moment map, it is easy to show when $\dim M = 4$ that certain elements η in $\pi_1(\mathrm{Ham})$ cannot be represented by an S^1 action. For example, if $(M,\omega) = (S^2\times S^2, \lambda pr_1^*\sigma + pr_2^*\sigma)$, where $\lambda > 1$, one can take $\eta := \eta_1 + \eta_2$, where $\eta_1 \in \pi_1\big(SO(3)\times\{id\}\big) \subset \pi_1\big(SO(3)\times SO(3)\big)$ rotates the first sphere and η_2 is one of the circle actions of infinite order. Since η_1 has finite order, the rational Samelson product $[\eta,\eta]_{\mathbb{Q}}$ vanishes. Buse in [5, Prop 2.1] gives an example of an element η in $\pi_1(\mathrm{Ham}(\mathbb{T}^2\times S^2))$ for which $[\eta,\eta]_{\mathbb{Q}}$ does not vanish.

[3] Since the cone $\mathcal{C} \subset H^2(M)$ is open, this is the same as the subring generated by $\mathcal{C}$.

to the closed form $\widetilde{\Omega}_\epsilon$, one can lift the classifying map to the space $B\mathrm{Ham}^s$ defined in [22]. One can then try to use characteristic classes coming from the coupling class on M_{Ham^s}. The problem here is that when $H^1(M;\mathbb{R}) \neq 0$ the normalization condition in Lemma 1.3 does not uniquely determine the extension $\widehat{a}$ of $a = [\omega]$ to the total space of an M-bundle $P \to Z$ unless $H^1(Z;\mathbb{R}) = 0$.

We shall adopt an easier approach, restricting consideration to classes $[\sigma] \in H_d(M;\mathbb{Q})$ that are represented by smooth maps $\sigma : Z \to M$ where Z is a closed and *simply connected* oriented d-manifold. (For short we will call these smooth simply connected cycles. If $\dim M > 4$ these generate the image of the homology of the universal cover of M; see Lemma 4.4.) The corresponding $\widetilde{M}$-bundles $\widetilde{P} \to Z$ are then classified by maps $Z \to B\mathrm{Symp}_0\, \widetilde{M}$. These lift further to $B\mathrm{Ham}\, \widetilde{M}$ because of the existence of the closed extension $\widetilde{\Omega}_\epsilon$ of the fiberwise symplectic forms. (This is another place where it is important that $\pi_1(Z) = 0$; see [12, Erratum].)

Even with these conditions the question of which classes can be detected in $B\mathrm{Ham}\, \widetilde{M}$ is quite subtle; see Example 4.6. Recall that (M, ω) is said to have the **hard Lefschetz property** if for all $k \le n$, the map $\cup[\omega]^k : H^{n-k}(M;\mathbb{R}) \to H^{n+k}(M;\mathbb{R})$ is an isomorphism.

Proposition 1.10. *Let $\sigma : Z \to M$ be a smooth and simply connected cycle representing a nonzero homology class $[\sigma] \in H_d(M;\mathbb{Q})$. Let $\widetilde{f}_\epsilon : Z \to B\mathrm{Ham}(\widetilde{M}, \widetilde{\omega}_\epsilon)$ classify the bundle $\widetilde{P}_\epsilon \to Z$ obtained by blowing up $M \times Z \to Z$ along gr_σ. Then $(\widetilde{f}_\epsilon)_*[\sigma] \neq 0$ if one of the following conditions holds.*

(i) *There is $\kappa \in \mathcal{C}^*$ such that $\kappa([\sigma]) \neq 0$.*

(ii) *$Z = S^{2k}$ and (M, ω) has the hard Lefschetz property.*

Remark 1.11. (i) When we work in the symplectic category, we cannot control the size of the blow ups and so must assume that they are arbitrarily small. We can say something about size only when we make several blow ups; for example, if we blow up twice, the second blow up may have to be smaller than the first. (For more precise statements see Propositions 5.3 and 6.4 and Corollary 6.7.)

(ii) Our methods detect very stable symplectic phenomena that persist even in the diffeomorphism group. This should be contrasted with the work of Seidel [27] who in the 4-dimensional case detects nonzero elements in $\pi_0(\mathrm{Symp}\,(X, \omega_X))$ that disappear in $\pi_0(\mathrm{Diff} X)$. In his Example 1.13, (X, ω_X) is the monotone blow up of $\mathbb{C}P^2$ at 5 points in which each point is blown up to exactly one third of the size of the line L in $\mathbb{C}P^2$. If one blows up each point slightly less (but still keeps all the blow ups the same size) then at least some of these elements in $\pi_0(\mathrm{Symp}(X, \omega_X))$ become zero: for example, the class $L - E_1 - E_2 - E_3$ is represented by a Lagrangian sphere in the monotone case, but by a symplectic sphere when the size is reduced. Therefore the square of the Dehn twist in this sphere becomes trivial. If the sizes of the blow ups are all different, then the above argument applies to a set of generators for $\pi_0(\mathrm{Symp}^H(X, \omega_X))$ so that this whole group vanishes. (Here Symp^H denotes the subgroup acting trivially on $H_*(X)$.)

It would be interesting to investigate the map

$$\pi_0(\mathrm{Symp}(X, \omega_X)) \to \pi_0(\mathrm{Symp}(X \times S^2, \omega_X + \omega_{S^2}))$$

induced by $\phi \mapsto \phi \times id$ to see if it has nontrivial image. In contrast, the elements of $\pi_*(B\mathrm{Diff}\widetilde{M})$ considered in this paper do remain nontrivial when $\widetilde{M}$ is stabilized to a product $\widetilde{M} \times Y$.

(iii) There is still rather little work that compares the homotopy of Symp and Diff, or that discusses the dependence of $\pi_k(\mathrm{Symp})$ on the symplectic form for manifolds of dimension > 4. Here are two notable exceptions. Ruan [25] gives examples of 6-manifolds such that $\pi_0(\mathrm{Symp}_d)$ does not surject onto $\pi_0(\mathrm{Diff})$, where Symp_d is the group of diffeomorphisms that preserve that symplectic form up to deformation, while Seidel [26] studies the changes in $\pi_k\big(\mathrm{Symp}(\mathbb{C}P^n \times \mathbb{C}P^m, \omega)\big)$ as $[\omega]$ varies.

Acknowledgements. I thank Jarek Kędra for useful comments on an earlier draft of this paper, Daniel Gottlieb for help in understanding the evaluation subgroup and the referee for pointing out various small inconsistencies.

2. The fundamental group

In this section we first discuss some homotopy theoretic approaches to the above questions, and then prove Theorem 1.1. Let M be a closed oriented manifold and consider the diagram

$$\begin{array}{ccccccc} \pi_k(\mathrm{Diff}\,\widetilde{M}) & \longrightarrow & \pi_k(\mathcal{E}/U) & \longrightarrow & \pi_k\big(B\mathrm{Diff}^U(\widetilde{M},\Sigma)\big) & \longrightarrow & \pi_k(B\mathrm{Diff}\,\widetilde{M}) \\ & & & & \simeq\uparrow & & \\ \pi_k(\mathrm{Diff}\,M) & \xrightarrow{\mathrm{ev}} & \pi_k(JM) & \longrightarrow & \pi_k\big(B\mathrm{Diff}^U(M,p)\big) & \longrightarrow & \pi_k(B\mathrm{Diff}\,M). \end{array}$$

Here JM denotes the space of pairs (x, J_x), where $x \in M$ and $J_x : T_xM \to T_xM$ is an almost complex structure compatible with the orientation of M. Thus there is a Hurewicz fibration $SO(2n)/U(n) \to JM \to M$. Since $\mathrm{Diff}\,M$ acts transitively on JM the bottom row in the above diagram is exact. The top row is also exact. Here we identify the exceptional divisor Σ with $\mathbb{C}P^{n-1}$ and denote by $\mathcal{E}$ the space of all smooth embeddings of Σ into $\widetilde{M}$ that extend to diffeomorphisms of $\widetilde{M}$. Thus an element of $\mathcal{E}$ factors (nonuniquely) as

$$\mathbb{C}P^{n-1} \equiv \Sigma \xrightarrow{f} \Sigma \hookrightarrow \widetilde{M} \xrightarrow{\tilde{g}} \widetilde{M}.$$

If $\mathcal{E}/U$ is the quotient of $\mathcal{E}$ by the action of $U(n)$ on the domain, $\mathrm{Diff}(\widetilde{M})$ acts transitively on $\mathcal{E}/U$ with kernel $\mathrm{Diff}^U(\widetilde{M},\Sigma)$ equal to the stabilizer of the inclusion map $\Sigma \hookrightarrow \widetilde{M}$. By standard arguments the groups $\mathrm{Diff}^U(M,p)$ and $\mathrm{Diff}^U(\widetilde{M},\Sigma)$ are homotopy equivalent to the subgroups where the diffeomorphisms are unitary in a neighborhood of p and Σ. Thus these two groups are homotopy equivalent.

We are interested in understanding the zig-zag composite

$$\pi_k(JM) \longrightarrow \pi_k\big(B\mathrm{Diff}^U(M,p)\big) \xrightarrow{\simeq} \pi_k\big(B\mathrm{Diff}^U(\widetilde{M},\Sigma)\big) \longrightarrow \pi_k\big(B\mathrm{Diff}\,\widetilde{M}\big).$$

The first map is relatively understandable, since the previous discussion shows that its kernel is closely related to the image of the evaluation map

$$\mathrm{ev} : \mathrm{Diff}\,M \to M, \quad g \mapsto g(p).$$

Therefore, the problem is to understand the kernel of the last map, i.e. the image of $\pi_k(\mathcal{E}/U) \to \pi_k\big(B\mathrm{Diff}^U(\widetilde{M},\Sigma)\big)$. In the 4-dimensional symplectic situation considered by Kędra, the space analogous to $\mathcal{E}/U$ consists of symplectically embedded 2-spheres and is often contractible.[4] However, this need not hold in the smooth case or in higher dimensions.

[4] Suppose for example, that $[\omega]$ is integral and ϵ has the form $1/N$ for some integer N. Then the exceptional divisor Σ is a curve of minimal energy in $(\widetilde{M}, \widetilde{\omega}_\epsilon)$ and so has a unique embedded J-holomorphic representative for each ω-tame J. This implies that there is a homotopy equivalence from the contractible space $\mathcal{J}(\omega)$ of ω-tame J to $\mathcal{E}/U$: see Lalonde–Pinsonnault [14, §2].

We now restrict to the case $k = 1$. We will prove Theorem 1.1 by showing that the kernel of $\pi_1 M \to \pi_1(B\mathrm{Diff}\widetilde{M})$ is the image of $\pi_1(\mathcal{B})$, where $\mathcal{B} \subset M^{\widetilde{M}}$ is the space of smooth blow down maps.

To this end, fix a blow down map $\phi_0 : (\widetilde{M}, \Sigma) \to (M, p)$ and denote by $\mathcal{B}$ the set of smooth maps conjugate to ϕ_0. More precisely

$$\mathcal{B} := \{h \circ \phi_0 \circ \widetilde{g} \in \mathrm{Map}^\infty(\widetilde{M}, M) : h \in \mathrm{Homeo}_0 M, \widetilde{g} \in \mathrm{Diff}_0 \widetilde{M}\},$$

where Homeo$M$ is the group of homeomorphisms[5] of M and the subscript 0 denotes that we consider the identity component. Thus $\phi : \widetilde{M} \to M$ is an element of $\mathcal{B}$ iff

- there is $q \in M$ such that $\phi : \widetilde{M} \setminus \{\phi^{-1}(q)\} \to M \setminus q$ is a diffeomorphism, and
- $\phi^{-1}(q) \in \overline{\mathcal{E}}_0$, where $\overline{\mathcal{E}}_0$ is the identity component of $\overline{\mathcal{E}} := (\mathrm{Diff}\widetilde{M})|_\Sigma / (\mathrm{Diff}\Sigma)$, i.e. the space of smoothly embedded submanifolds isotopic to Σ.

(For then there is $\widetilde{g} \in \mathrm{Diff}_0 \widetilde{M}$ such that $\widetilde{g}(\phi^{-1}(q)) = \Sigma$ so that ϕ can be written as $h \circ \phi_0 \circ \widetilde{g}$ for suitable h.)

Although the elements $\phi \in \mathcal{B}$ do not have unique factorizations into pairs $(h, \widetilde{g})$, each ϕ induces a diffeomorphism $\widetilde{M} \setminus \widetilde{g}^{-1}(\Sigma) \to M \setminus \{h(p)\}$. In particular, there are well defined maps

$$\mathrm{ev}_B : \mathcal{B} \to M, \quad h \circ \phi_0 \circ \widetilde{g} \mapsto h(p),$$

and

$$F : \mathcal{B} \to \overline{\mathcal{E}}_0, \quad h \circ \phi_0 \circ \widetilde{g} \mapsto \widetilde{g}^{-1}(\Sigma).$$

Observe also that $h \circ \phi_0 \circ \widetilde{g} = \phi_0$ iff $h = \widetilde{g}^{-1}$ on $M \setminus \{p\} \equiv \widetilde{M} \setminus \Sigma$.

Theorem 1.1 follows immediately from the next two results. As in Remark 1.2, $\widetilde{\mathrm{ev}}_* : \pi_1(M^{\widetilde{M}}) \to \pi_1 M$ denotes the evaluation map given by $\widetilde{\mathrm{ev}}(\phi) = \phi(\widetilde{p})$, where $\widetilde{p} \in \widetilde{M}$ is fixed. Note that this does *not* restrict to ev_B on $\mathcal{B}$ since the base point $\widetilde{p}$ need not lie in $\widetilde{g}^{-1}(\Sigma_0)$. Nevertheless, the next result shows that the two maps have comparable effects on $\pi_1(\mathcal{B})$.

Proposition 2.1. (i) *For all $k \geq 1$ there is an exact sequence*

$$\pi_k \mathcal{B} \xrightarrow{(\mathrm{ev}_B)_*} \pi_k M \xrightarrow{\widetilde{f}_*} \pi_k(B\mathrm{Diff}\widetilde{M}).$$

(ii) $(\mathrm{ev}_B)_*\big(\pi_1(\mathcal{B})\big) \subseteq \widetilde{\mathrm{ev}}_*\big(\pi_1(M^{\widetilde{M}})\big)$.

The following lemma explains some of the elementary properties of the generalized evaluation subgroup $\mathrm{im}\, \widetilde{\mathrm{ev}}_*$. The present formulation of (i) is due to Kędra. The results here can be be considerably extended; see Gottlieb [7].

Lemma 2.2. (i) *Each $\alpha \in \widetilde{\mathrm{ev}}_*(\pi_1(M^{\widetilde{M}}))$ acts trivially on the elements in the image of $\pi_*(\widetilde{M})$ in $\pi_*(M)$. In particular, α acts trivially on $\pi_i M$ for $1 \leq i < 2n - 1$, where $2n := \dim M$.*
(ii) $\widetilde{\mathrm{ev}}_*(\pi_1(M^{\widetilde{M}}))$ *is trivial if M is a blow up.*
(iii) $\widetilde{\mathrm{ev}}_*(\pi_1(\mathcal{B}))$ *is trivial if $\chi(M) \neq 0$.*

We begin the proofs with the following lemma.

[5]It is convenient to relax the smoothness conditions on h since then for any diffeomorphism $\widetilde{g} : (\widetilde{M}, \Sigma) \to (\widetilde{M}, \Sigma)$ there is h satisfying $h \circ \phi_0 = \phi_0 \circ \widetilde{g}$. However, because $h \circ \phi_0 \circ \widetilde{g}$ and $\widetilde{g}$ are both smooth, h is smooth except at the basepoint p.

Lemma 2.3. *Fix a point $\widetilde{p} \in \Sigma$. Any element $\alpha \in \pi_1(\overline{\mathcal{E}})$ can be represented by a path $g_t : \Sigma \to \widetilde{M}$ such that $g_t(\widetilde{p}) = \widetilde{p}$ for all $t \in S^1$.*

Proof. Lift α to a path $g_t : \Sigma \to \widetilde{M}, t \in I$, in $\mathcal{E}$. Extend it to a path $\widetilde{g}_t$, $t \in I$, in $\mathrm{Diff}\widetilde{M}$ such that $\widetilde{g}_0 = id$ and consider the mapping torus $\widetilde{P} \to S^1$ of $\widetilde{g}_1$:

$$\widetilde{P} := \widetilde{M} \times I/\big((\widetilde{x}, 1) \sim (\widetilde{g}_1(\widetilde{x}), 0)\big).$$

Since $\widetilde{g}_1$ lies in the identity component of $\mathrm{Diff}\widetilde{M}$, this is a smoothly trivial bundle and hence its homology is the product $H_*(\widetilde{M}) \otimes H_*(S^1)$.

Let $\Sigma_t := g_t(\Sigma)$ and denote by $\widetilde{\Sigma} \subset \widetilde{P}$ the union of the submanifolds $\Sigma_t \times \{t\}, t \in S^1$. Then $Y_t := \Sigma_t \cap \Sigma$ is nonempty for all t. Moreover, by jiggling the $\widetilde{g}_t$ we may arrange that $\widetilde{Y} := \cup_t Y_t \times \{t\} \subset \widetilde{P}$ is a manifold of dimension $2n-3$ and that Y_0 is connected and contains $\widetilde{p}$. We claim that there is a smooth map $\gamma : S^1 \to \widetilde{Y}$ whose projection $\pi \circ \gamma$ onto S^1 has degree 1. To see this, note first that $\pi^*(dt)$ is nonzero on $\widetilde{Y}$ because it is Poincaré dual (in $\widetilde{Y}$) to $[\widetilde{Y} \cap \pi^{-1}(t)] = [\Sigma] \cap [\Sigma] \neq 0$. Hence $\pi^*(dt)$ does not vanish on some cycle $\gamma : S^1 \to \widetilde{Y}$ and we can arrange that its projection to S^1 has degree 1 because Y_0 is path connected. Let $\beta := \pi \circ \gamma : S^1 \to S^1$. Then

$$\gamma(t) \in Y_{\beta(t)}, \quad t \in S^1.$$

Replacing $\widetilde{g}_t$ by the homotopic loop $\widetilde{g}_{\beta(t)}$, we can assume that $\gamma(t) \in Y_t \subset \Sigma$ for all $t \in S^1$. But γ contracts in Σ to the point $\widetilde{p}$. Therefore we can homotop the loop $g_t(\Sigma)$ in $\overline{\mathcal{E}}$ so that $\widetilde{p} \in g_t(\Sigma)$ for all t, and then choose g_t to fix $\widetilde{p}$. □

Proof of Proposition 2.1. Let $\widetilde{P} \to S^k$ be the blow up of $M \times S^k$ along s_σ, where $\sigma : S^k \to M$. Suppose that $\pi : \widetilde{P} \to S^k$ is smoothly trivial. Then there is a smooth family of diffeomorphisms $\widetilde{g}_t : \widetilde{M} \to \widetilde{M}_t := \pi^{-1}(t)$ for $t \in S^k$ such that $\widetilde{g}_0 = id$. Define $\phi_t \in \mathcal{B}$ by

$$\phi_t := pr_M \circ \phi_P \circ \widetilde{g}_t, \quad t \in S^k,$$

where $\phi_P : \widetilde{P} \to M \times S^k$ is the blow down map and pr_M is the projection to M. By construction $\mathrm{ev}_B(\phi_t) = \sigma(t) \in M$, for all $t \in S^k$. Therefore $\ker \widetilde{f}_* \subset \mathrm{im}(\mathrm{ev}_B)_*$.

To prove the converse, suppose that $[\sigma] = (\mathrm{ev}_B)_*([\phi_t])$. Then

$$\Phi : \widetilde{M} \times S^k \to M \times S^k, \quad (x,t) \mapsto (\phi_t(x), t),$$

is a smooth map that is a diffeomorphism over $M \times S^k \setminus s_\sigma$ and is a blow down over each point in the section s_σ. Hence $P \cong \widetilde{M} \times S^k$ by the uniqueness of the blow up construction. This proves (i).

Now suppose that $k = 1$. Let $\Sigma_t' \subset \widetilde{M}_t, t \in S^1$, be the copy of the exceptional divisor that is blown down by ϕ_P and define $\Sigma_t := \widetilde{f}_t^{-1}(\Sigma_t')$. This is a loop in $\overline{\mathcal{E}}$. Choose a family of decompositions $\phi_t := h_t \circ \phi_0 \circ \widetilde{g}_t, t \in I$, for the loop $\phi_t \in \mathcal{B}$, starting with $h_0 = \widetilde{g}_0 = id$. Then $\widetilde{g}_t^{-1}(\Sigma) = \Sigma_t$. Lemma 2.3 implies that we may assume that $\widetilde{g}_t(\widetilde{p}) = \widetilde{p}$ for all t. Therefore $\mathrm{ev}_B(\phi_t) := h_t(p) = \widetilde{g}_t(\widetilde{p}) =: \widetilde{\mathrm{ev}}(\phi_t)$. This proves (ii). □

Proof of Lemma 2.2. $\alpha \in \pi_1 M$ acts trivially on $\pi_i(M, p)$ if every based map $f : (S^i, x_0) \to (M, p)$ extends to a map $F : S^i \times S^1 \to M$ such that $F|_{pt \times S^1} = \alpha$. But if $\alpha = \widetilde{\mathrm{ev}}_*([\phi_t])$ we may define $F : S^i \times S^1 \to M$ by setting

$$F(x,t) = \phi_t(\widetilde{f}(x)),$$

where $\widetilde{f}:(S^i,x_0)\to(\widetilde{M},\widetilde{p})$ lifts f, i.e. satisfies $\phi_0\circ\widetilde{f}=f$. This proves the first statement.

Next observe that when $i<2n$ every element in π_iM lifts to $\widetilde{M}$. In fact, since $\widetilde{M}$ is diffeomorphic to the connected sum of M with $\mathbb{C}P^n$ (oppositely oriented), $\widetilde{M}$ has a cell decomposition with one vertex at $\widetilde{p}$, one $2n$-cell e^{2n}, and $(2n-1)$-skeleton of the form $X\vee\Sigma$, where X is the $(2n-1)$-skeleton of M. This proves (i).

(ii) holds because when M is the blow up $\widetilde{X}$ of X every nonzero element of π_1M acts nontrivially on $\pi_2(M)$. This follows immediately from the fact that M is the connected sum of X with $(\mathbb{C}P^n)^{opp}$, so that the universal cover of M is the connected sum of the universal cover of X with $|\pi_1(X)|$ copies of $(\mathbb{C}P^n)^{opp}$ on which the group of deck transformations acts effectively.

To prove (iii), consider

$$Z:=gr\,\phi_0=\big(\Sigma\times p\big)\cup Z_0\subset\widetilde{M}\times M,$$

where $Z_0=\{(\widetilde{q},\phi_0(\widetilde{q})):\widetilde{q}\in\widetilde{M}\setminus\Sigma\}$. Let $h:M\to M$ be a diffeomorphism C^1 close to the identity, with nondegenerate fixed points and such that $h(p)\neq p$. Then Z intersects the graph of $\phi_0':=h\circ\phi_0$ transversally in $\chi(M)$ points. (Since Z is a stratified space with top stratum of half the dimension of the ambient manifold and one other stratum of codimension 2, it makes sense to talk of transversality here. Note that all the intersection points must lie in Z_0.) Thus $[Z]\cdot[Z]=\chi(M)\in H_0(\widetilde{M}\times M)$.

Now suppose that $\phi_t, t\in S^1$, is a loop in the space of smooth maps $\mathrm{Map}^\infty(\widetilde{M},M)$ based at ϕ_0. We may perturb it to a loop $\phi':=\{\phi_t'\}$, based at $\phi_0':=h\circ\phi_0$ (where h is as above) and such that its graph in $\widetilde{M}\times M\times S^1$ is transverse to $Z\times S^1$. We will show that $\widetilde{\mathrm{ev}}_*(\phi)=\widetilde{\mathrm{ev}}_*(\phi')$ vanishes by finding a loop $\widetilde{p}_t\in\widetilde{M}$ of *coincidence points*, i.e. points where

$$p_t:=\phi_t'(\widetilde{p}_t)=\phi_0(\widetilde{p}_t),\qquad t\in S^1. \tag{2.1}$$

(In fact, we will achieve this after a further deformation of ϕ'.) Then the loop $t\mapsto\phi_t'(\widetilde{p}_t)$ is homotopic in (M,p_0) to the sum of the loops $t\mapsto\phi_0(\widetilde{p}_t)$ and $t\mapsto\phi_t'(\widetilde{p}_0)$. Since equation (2.1) implies that the first two loops are equal, the third loop $t\mapsto\phi_t'(\widetilde{p}_0)$ is contractible.

To find the coincidence points we use smooth intersection theory (which is why we restrict to the space $\mathcal{B}$ of smooth maps.) It is convenient to take $t\in[0,1]=:I$ instead of $t\in S^1$. Note that for generic $t\in I$, the signed number of points in the intersection of $gr\,\phi_t'$ with $Z\times t$ is $\chi(M)\neq 0$. Hence, the intersection of $gr\,\phi'$ with $Z_0\times I$ contains at least one connected component that projects to $(I,\partial I)$ by a map of degree 1. Reparametrize ϕ' so that this projection is the identity. Then, because all intersection points lie in $Z_0\times S^1$, there is a path $\widetilde{p}_t, t\in I$, in $M\setminus\{p\}$ such that $\phi_t'(\widetilde{p}_t)=\phi_0(\widetilde{p}_t)$ for all t. We now extend the path $\widetilde{p}_t, t\in I$, to a loop $\widetilde{p}_t, t\in[0,2]$, by choosing $\widetilde{p}_t, t\in[1,2]$ to be any path in $\widetilde{M}$ from $\widetilde{p}_1$ to $\widetilde{p}_0$. Since $\phi_0'=\phi_1'=h\circ\phi_0$, $\widetilde{p}_0$ and $\widetilde{p}_1$ are fixed by h. We may assume there is an isotopy h_t from $h_0=id$ to $h_1=h$ that also fixes $\widetilde{p}_0,\widetilde{p}_1$. Then extend ϕ' by the contractible loop $\phi_t':=h_{\beta(t)}\circ\phi_0, t\in[1,2]$, where $\beta(1)=\beta(2)=1$ and $\beta(t)=0$ for $t\in[1+\epsilon,2-\epsilon]$. After reparametrizing $\widetilde{p}_t, t\in[1,2]$, Eq. (2.1) is satisfied for $t\in[0,2]$. □

Remark 2.4. It is not clear how to extend the arguments in this section to the case $k>1$. Even if one could prove some analog of Proposition 2.1 (ii) for $k>1$, the map $\widetilde{\mathrm{ev}}_*:\pi_k(M^{\widetilde{M}})\to\pi_kM$ is not easy to work with. In particular, since $M^{\widetilde{M}}$ is not an H-space, we cannot immediately claim that the image of this map is torsion when k is even (as is the case for the usual evaluation map $M^M \to M$.) All the basic questions can of course be phrased in terms of the space $\mathcal{E}/U$, but this seems no more accessible. In the next section we develop some homological tools that work when M is c-symplectic and $k = 2$. One of their advantages is that they adapt easily to the case of many blowups; see §5.

3. The c-symplectic case

This section proves Theorem 1.4, Proposition 1.7 and Corollary 1.5. Unless explicit mention is made to the contrary, in this section (M, a) is a connected and simply connected closed c-symplectic manifold.

Denote by $\widetilde{Q} \to B\mathrm{Diff}^U(M,p)$ the pullback of the universal $\widetilde{M}$-bundle over $B\mathrm{Diff}\,\widetilde{M}$ by the blow up map $B\mathrm{Diff}^U(M,p) \to B\mathrm{Diff}\,\widetilde{M}$. It is obtained by blowing up the universal M-bundle over $B\mathrm{Diff}^U(M,p)$ along its canonical section. For each map $\sigma : S^2 \to B\mathrm{Diff}^U(M,p)$, define $(\lambda, \ell) := \big(I_a([\sigma]), I_c([\sigma])\big)$. Denote by

$$\widetilde{P}_{\lambda,\ell} \to S^2$$

the pullback of $\widetilde{Q} \to B\mathrm{Diff}^U(M,p)$ by σ, and by $P_{\lambda,\ell} \to S^2$ the corresponding M-bundle with section s_σ.

The following result clearly implies Theorem 1.4.

Proposition 3.1. *Let (M, a) be a simply connected c-symplectic manifold. Then the bundle $\widetilde{P}_{\lambda,\ell}$ is smoothly trivial only if $(\lambda, \ell) = (0, 0)$.*

Proof. Consider the M-bundle $P := P_{\lambda,\ell} \xrightarrow{\pi} S^2$, and denote its coupling class by $\widehat{a} \in H^2(P)$. This is the unique class that restricts to a on the fiber M and is such that the fiberwise integral $\pi_!(a^{n+1}) := \int_M \widehat{a}^{n+1} \in H^2(S^2)$ vanishes. Choose an area form β on S^2 so that the product class $\widehat{a} + \pi^*([\beta])$ evaluates positively over the section $s_\sigma \subset P$. Then the normalization condition on $\widehat{a}$ implies that

$$\mathrm{vol}\big(P, \widehat{a} + \pi^*([\beta])\big) = \mathrm{vol}(M, a) \int_{S^2} \beta \;=:\; V\mu_0,$$

where $\mu_0 := \int_{S^2} \beta$ and $V := \mathrm{vol}(M, a) = \frac{1}{n!}\int_M a^n$. Further

$$\int_{s_\sigma} \widehat{a} + \pi^*\beta = \mu_0 + I_a([\sigma]) = \mu_0 + \lambda. \tag{3.1}$$

Now choose a representative Ω of the class $\widehat{a}+\pi^*([\beta])$ that restricts in some neighborhood U of s_σ to a symplectic form that induces the given almost structure on its normal bundle.[6] Next choose ϵ_0 so that a neighborhood of s_σ of capacity ϵ_0 (and hence radius $r_0 := \sqrt{\epsilon_0/\pi}$) embeds symplectically in U. Then, for any $\epsilon < \epsilon_0$ we can perform the ϵ-blow up along s_σ symplectically in U; more details are below. Denote the resulting bundle by $(\widetilde{P}_{\lambda,\ell}, \widetilde{\Omega}_\epsilon)$. We show below that

$$\mathrm{vol}(\widetilde{P}_{\lambda,\ell}, \widetilde{\Omega}_\epsilon) = \mu_0 V - v_\epsilon \left(\mu_0 + \lambda - \frac{\ell}{n+1}\epsilon\right), \tag{3.2}$$

where $v_\epsilon := \frac{\epsilon^n}{n!}$ is the volume of a ball of capacity ϵ.

[6] It is not necessary to use symplectic geometry at all in this proof. We do so because it is well understood how symplectic forms behave under blow up.

Note that the underlying smooth bundle $\widetilde{P} \to S^2$ does not depend on the choice of ϵ. Therefore, if $\widetilde{P}_{\lambda,\ell} \to S^2$ were a smoothly trivial fibration, then, for all ϵ, the volume of $\widetilde{P}_{\lambda,\ell}$ would be the product of $V - v_\epsilon := \mathrm{vol}(\widetilde{M}, \widetilde{\omega}_\epsilon)$ with the "size" μ of the base. Since μ could be measured by integrating $\widetilde{\Omega}_\epsilon$ over a section of $\widetilde{P}$ which is the same for all ϵ, $\mu = \mu_1 + k\epsilon$ would be a linear function of ϵ. Therefore, the two polynomial functions $\mathrm{vol}(\widetilde{P}_{\lambda,\ell}, \widetilde{\Omega}_\epsilon)$ and $(V - v_\epsilon)(\mu_1 + k\epsilon)$ would have to be equal. This is possible only if $\lambda = 0$ and also $k = \ell = 0$. The result follows.

It remains to derive the formula for $\mathrm{vol}(\widetilde{P}_{\lambda,\ell}, \widetilde{\Omega}_\epsilon)$. First assume that $\ell = 0$. Then the section s_σ has trivial (complex) normal bundle in $M \times S^2$. Hence it has a neighborhood $U_\epsilon \subset U$ that is symplectomorphic to a product $B^{2n}(\epsilon) \times s_\sigma$. Thus, by equation (3.1), the volume of U_ϵ with respect to the form Ω is $\mathrm{vol}(U_\epsilon) = v_\epsilon(\mu_0 + \lambda)$. We may obtain the blow up by cutting out U_ϵ from (P, Ω) and identifying the boundary via the Hopf map; see Lerman [16]. Then

$$\mathrm{vol}(\widetilde{P}_{\lambda,0}, \widetilde{\Omega}_\epsilon) = \mu_0 V - v_\epsilon(\mu_0 + \lambda),$$

as claimed.

Now consider the case when $\ell \neq 0$. Then the normal bundle to s_σ in $M \times S^2$ is isomorphic to the product $\mathbb{C}^{n-1} \oplus L_\ell$, where $L_\ell \to S^2$ is the holomorphic line bundle with $c_1 = \ell$. Therefore, we can choose Ω so that it restricts in some neighborhood of s_σ to the product of a ball in $\mathbb{C}^{n-1}$ with a δ-neighborhood $\mathcal{N}_\delta(L_\ell)$ of the zero section of L_ℓ. Identifying $\mathcal{N}_\delta(L_\ell)$ with part of a 4-dimensional symplectic toric manifold, we can see that its volume is $h\delta - \ell\delta^2/2$, where $h =$ area of zero section and $\delta = \pi r^2$ is the capacity of the disc normal to s_σ. Therefore, since $h = \mu_0 + \lambda$ here,

$$\begin{aligned}\mathrm{vol}(U_\epsilon) &= \int_0^{\sqrt{\frac{\epsilon}{\pi}}} \mathrm{vol}(S^{2n-3}(r)) \cdot \mathrm{vol}(\mathcal{N}_{\epsilon - \pi r^2}(L_\ell))\, dr \\ &= (\mu_0 + \lambda)\epsilon^n/n! - \ell\epsilon^{n+1}/(n+1)! \\ &= v_\epsilon\big(\mu_0 + \lambda - \ell\epsilon/(n+1)\big).\end{aligned}$$

Everything in the previous calculation remains valid except that we have to add $v_\epsilon\ell\epsilon/(n+1)$ to the volume of $\widetilde{P}_{\lambda,\ell}$. This completes the proof. □

Proof of Corollary 1.5. The above proof of Theorem 1.4 uses the fact that $\pi_1(M) = 0$ to assert the existence of a coupling class $\widehat{a}$ on the initial M-bundle $P \to S^2$, from which we construct the form $\widetilde{\Omega}_\epsilon$ on the blow up $\widetilde{P}_{\lambda,\ell} \to S^2$. In the situation of this corollary we are starting with a trivial bundle $M \times S^2 \to S^2$ which always has a coupling class, namely $pr_M^*(a)$. Hence the required form $\widetilde{\Omega}_\epsilon$ exists on the blow up $\widetilde{P}_{\lambda,\ell} \to S^2$. The previous argument now applies to show that this bundle is nontrivial whenever (λ, ℓ) is nonzero. The corollary now follows as in the case when $\pi_1(M) = 0$. □

Proof of Proposition 1.7. Let $X := \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, the one point blow up of $\mathbb{C}P^2$. Let $a \in H^2(\mathbb{C}P^2)$ be the class of the Kähler form τ_0 on $\mathbb{C}P^2$ normalized to integrate to 1 over the line. By [11], any symplectic form τ on X is diffeomorphic to a form obtained by blow up from τ_0.

Suppose first that $\tau = \tau_\epsilon$ is the ϵ-blow up of τ_0. By Abreu–McDuff [1], $\pi_1(\mathrm{Symp}(X, \tau)) = \mathbb{Z}$ is generated by the circle action that rotates the fibers of the ruling $X := \mathcal{P}(L_1 \oplus \mathbb{C}) \to \mathbb{C}P^1$ once. If $X \to P \to S^2$ is the fibration corresponding to this generator, then one can blow down the exceptional divisors in the fiber to obtain a $\mathbb{C}P^2$-bundle over S^2 with section s. Note that $I_c(s) = \pm 1$, and we can choose the sign of the generator so that $I_c(s) = 1$.

The blown down bundle is nontrivial, but has order 3 since $\pi_1(\mathrm{Symp}(\mathbb{C}P^2)) = \pi_1(PU(3)) = \mathbb{Z}/3\mathbb{Z}$; see [23, Ch 9]. Thus three times the generator of $\pi_1(\mathrm{Symp}(X,\tau)) = \pi_2(B\mathrm{Symp}(X,\tau))$ is the bundle given by blowing up the product $\mathbb{C}P^2 \times S^2$ along the graph of the map $\sigma : S^2 \to \mathbb{C}P^2$ with image equal to the positively oriented line. Thus it corresponds to the bundle $\widetilde{P}_{\lambda,\ell}$ where $(\lambda,\ell) = (1,3)$. Therefore, the image of $\pi_2(B\mathrm{Symp}(X,\tau))$ in $\pi_2(B\mathrm{Diff}X)$ corresponds to the set of fibrations $\widetilde{P}_{\ell/3,\ell}$, $\ell \in \mathbb{Z}$.

Now consider the form $\tau' := \phi^*(\tau_\epsilon)$, where $\phi : X \to X$ is complex conjugation. Then (X,τ') is obtained by blowing up the complex conjugate $(\mathbb{C}P^2, \bar{\jmath}, \overline{\tau}_0)$ of the standard $\mathbb{C}P^2$. But the first Chern class of the line in $(\mathbb{C}P^2, \bar{\jmath}, \overline{\tau}_0)$ is still three times the value of the normalized Kähler form on the line. Hence the image of $\pi_2(B\mathrm{Symp}(X,\tau))$ in $\pi_2(B\mathrm{Diff}X)$ again corresponds[7] to the bundles $\widetilde{P}_{\ell/3,\ell}$, $\ell \in \mathbb{Z}$.

When appropriately scaled, every symplectic form on X is diffeomorphic either to τ or τ' by a diffeomorphism that acts trivially on cohomology and hence fixes both the symplectic class and the first Chern class. Since the volume computation is a calculation in the cohomology of X, it is also unchanged by such a diffeomorphism. It follows that if τ'' is an arbitrary symplectic form on X, normalized so that its integral over the line has absolute value 1, the image of $\pi_2(B\mathrm{Symp}(X,\tau''))$ in $\pi_2(B\mathrm{Diff}X)$ corresponds to a set of bundles over S^2 with volume forms equal to those of $\widetilde{P}_{\ell/3,\ell}$, $\ell \in \mathbb{Z}$.

Let $\mathcal{A}$ be the subgroup of $\pi_2(\mathrm{Diff}X)$ generated by the images of $\pi_2(B\mathrm{Symp}(X,\tau''))$ as τ'' varies over all symplectic forms on X. Addition in $\pi_2(B\mathrm{Diff}X)$ corresponds to taking the fiber sum of the corresponding bundles. Hence, because the volume of $(\widetilde{P}_{\ell/3,\ell}, \widetilde{\Omega}_\epsilon)$ depends linearly on ℓ, any bundle $\widetilde{P} \to S^2$ given by $\alpha \in \mathcal{A}$ has volume equal to that of some bundle $(\widetilde{P}_{\ell/3,\ell}, \widetilde{\Omega}_\epsilon)$. Hence the bundles $\widetilde{P}_{0,\ell}, \ell \in \mathbb{Z}$, do not correspond to elements of $\mathcal{A}$. This completes the proof. □

Remark 3.2. (i) It would be interesting to find a way to establish the nontriviality of the elements in $\pi_k(\mathrm{Diff}\widetilde{M})$ for $k > 2$ constructed by blowing up the product $M \times S^k$ along $pt \times S^k$ with respect to a complex structure given by a nonzero element in $\pi_k(SO(2n)/U(n))$. For example if $n \geq 4$ one might use a nonzero element from $\pi_6(SO(2n)/U(n))$.

(ii) One might also attempt to use a similar construction with the manifold $X = S^2 \times S^2$, considering this as the blow up of the orbifold Y given by collapsing the antidiagonal in $S^2 \times S^2$ to a point y_*. Since a neighborhood of y_* is diffeomorphic to $\mathbb{R}^4/(\pm 1)$, we may construct a bundle $X \to P \to S^2$ from $Y \times S^2$ by blowing up along the constant section $y_* \times S^2$ with respect to a nontrivial family of almost complex structures $J_z, z \in S^2$, on $T_{y_*}Y$. For example, we could identify S^2 with the unit sphere in $\mathbb{R}^3 \equiv (\mathbb{R}e_1)^\perp \subset \mathbb{R}^4$ (where $e_1 = (1,0,0,0)$) and define J_z by setting it equal to the unique element of $SO(4)/U(2)$ such that $J_z(e_1) = z$. The resulting loop in $\mathrm{Diff}(S^2 \times S^2)$ is certainly not homotopic to a symplectic loop. However, it seems to represent the sum $[\Lambda] - [\sigma\Lambda\sigma]$, where $\sigma : (z,w) \mapsto (w,z)$ is the obvious involution of X and Λ is the symplectic loop that rotates X once about its diagonal and antidiagonal, i.e. if we identify X with the Hirzebruch surface

[7]Observe that the parameter λ in the bundle $\widetilde{P}_{\lambda,\ell}$ is determined by the fixed choice of c-symplectic class a on X. Hence if $\overline{\sigma} : S^2 \to (\mathbb{C}P^2, \overline{\tau}_0)$ is an $\overline{\tau}_0$-symplectic embedding onto the line then $\int \overline{\sigma}^*(\widehat{a}) = -1$ and the corresponding X-bundle is $\widetilde{P}_{-1,-3}$.

$\mathbf{P}(L_2 \oplus \mathbb{C})$, where $L_2 \to S^2$ is the holomorphic line bundle with Chern number 2, then $\Lambda_t[z_1 : z_2] = [e^{2\pi it}z_1 : z_2]$. Therefore it is in the subgroup $\mathcal{A} \subset \pi_2(\mathrm{Diff}\, X)$ considered above.

4. The symplectic case

This section proves Theorem 1.9. We begin with a discussion of characteristic classes defined by parametric Gromov–Witten invariants. This approach was first suggested by Le–Ono [15] and was then used by Buse [4].

In the easiest case, we have a class $A \in H_2(M;\mathbb{Z})$ such that the formal dimension of the space $\mathcal{M}_{g,0}(J,A)$ of unparametrized J-holomorphic A-curves of genus g is $-k < 0$. Let $\langle A\rangle$ denote the orbit of this class under the action of $\pi_0(\mathrm{Symp}\, M)$. Then there is a corresponding class $c(\langle A\rangle, g) \in H^k(B\mathrm{Symp}\, M;\mathbb{Q})$ defined as follows. Because rational bordism equals rational homology, it suffices to evaluate $c(\langle A\rangle, g)$ on cycles of the form $\sigma : Z \to B\mathrm{Symp}\, M$ where Z is a k-dimensional closed oriented smooth manifold. (For short, we call such cycles smooth.) The pullback of the universal M-bundle $M_{\mathrm{Symp}} \to B\mathrm{Symp}\, M$ is a smooth bundle $\pi_\sigma : P_\sigma \to Z$. Then

$$\Big\langle c(\langle A\rangle, g),\ \sigma_*[Z]\Big\rangle := \mathrm{PGW}^{P_\sigma, Z}_{g,0},$$

the "number" of unparametrized holomorphic $\langle A\rangle$-curves of genus g in the fibers of P_σ. If Symp_H denotes the subgroup of Symp that acts trivially on $H_2(M;\mathbb{Q})$ there is a similar class $c(A,g) \in H^k(B\mathrm{Symp}_H\, M)$ that only counts curves in class A.[8]

Invariants of the above type were used in [15, 4]. More generally, one might want to count curves that satisfy certain constraints that are pulled back from the cohomology of the universal M-bundle $M_{\mathrm{Symp}} \to B\mathrm{Symp}$. In the problem at hand we cannot use arbitrary cohomology classes on M_{Symp} since we need there to be a relation between the characteristic classes for M and those for $\widetilde{M}$. We therefore consider two different kinds of constraints.

First, we use constraints given by classes in $H^*(M)$ that have canonical extensions to $H^*(M_{\mathrm{Symp}})$. The most obvious classes with this property are the Chern classes of the tangent bundle of M, which extend to the Chern classes of the tangent bundle of the fibers of $M_{\mathrm{Symp}} \to B\mathrm{Symp}$. However, these do not behave well under blowup and so are not very useful here.

Another possibility is to use the classes of $\mathcal{C}^*$, the subring of $H^*(M)$ generated by $\mathcal{C}$. If $H^1(M;\mathbb{R}) = 0$ these classes extend to the universal bundle $M_{\mathrm{Symp}_0} \to B\mathrm{Symp}_0 M$ by Lemma 1.3. For general M, they extend to the universal Hamiltonian bundle $M_{\mathrm{Ham}} \to B\mathrm{Ham}\, M$ by [13, 12]. The proof of Lemma 1.3 shows that in both cases one can pick a unique normalized extension to M_{Ham}. For each $\kappa \in \mathcal{C}^*$, choose a representing polynomial $q_\kappa := q_\kappa(a_1,\dots,a_m)$ in the elements $a_i \in \mathcal{C}$. Then κ extends to $\widehat{q}_\kappa := q_\kappa(\widehat{a}_1,\dots,\widehat{a}_\ell)$. (This extension *depends* on the choice of polynomial q_κ.) We then get classes of the form

$$c(A, g; \widehat{q}_{\kappa_1},\dots,\widehat{q}_{\kappa_\ell}) \in H^{k+d-2\ell}(B\mathrm{Ham}\, M;\mathbb{Q}),\quad d := \sum_i \deg \kappa_i, \tag{4.1}$$

[8] If Z is symplectic and $\pi_1(M) = \pi_1(Z) = 0$ then Lemma 1.3 shows that the fiberwise symplectic forms ω_z extend to a symplectic form Ω on P_σ. In this case, $H_2(M;\mathbb{Q})$ injects into $H_2(P_\sigma;\mathbb{Q})$ and when $g = 0$ the parametric GW invariants are just the usual GW invariants for classes $A \in H_2(M;\mathbb{Q}) \subset H_2(P_\sigma;\mathbb{Q})$; cf. [23, Rem.6.7.8].

where $-k = 2n + 2c_1(A) - 6 \geq 0$. The value of such a class on $\sigma_*[Z]$ is (intuitively) the number of ℓ-pointed genus g A-curves in the fibers of $P_\sigma \to Z$ meeting cycles representing the Poincaré duals (with respect to P_σ) of the classes $\widehat{q}_{\kappa_1}, \ldots, \widehat{q}_{\kappa_\ell}$.

Second, we consider the case when $(M, [\omega])$ has the c-splitting property, i.e. the rational cohomology of the total space M_{Ham} of the universal M-bundle over $B\mathrm{Ham}\, M$ is additively isomorphic to $H^*(M) \otimes H^*(B\mathrm{Ham}\, M)$; cf. [12]. For example, by Blanchard [3] the hard Lefschetz property mentioned in Theorem 1.9 implies the c-splitting property. In this case, all elements of $H^*(M;\mathbb{R})$ extend to M_{Ham}. Therefore there are classes

$$c(A, g; \widehat{b}_1, \ldots, \widehat{b}_\ell) \in H^{k+d-2\ell}(B\mathrm{Ham}\, M; \mathbb{Q}), \quad d := \sum_i \deg \widehat{b}_i, \tag{4.2}$$

that depend on the chosen extensions $\widehat{b}_i$ of the classes $b_i \in H^*(M;\mathbb{R})$.

The proof that these classes are well defined when $g = 0$ and (M, ω) satisfies a suitable semi-positivity hypothesis is given in Buse [4] and may also be extracted from [23]. The general case is established by the usual methods of the virtual moduli cycle; cf. Li–Tian [18] for example.

We now return to the specific case at hand. In our examples, $g = 0$ and so we will often omit it from the notation. Let us first suppose that $[\sigma]$ is detected by some element $\kappa \in \mathcal{C}^*$. Let $-e \in H^2(\widetilde{M})$ be Poincaré dual to the exceptional divisor, so that $\mathrm{PD}(e^{n-1}) = E$, the class of a line in the exceptional divisor. Denote by $\widetilde{\kappa} := \phi_M^*(\kappa)$ the pullback of $\kappa \in \mathcal{C}_M^*$ to $\mathcal{C}_{\widetilde{M}}^*$. Then the characteristic class of the form (4.1)

$$c(E; \widehat{e}^{n-1}, \widehat{e}^{n-1}\, \widehat{q_{\widetilde{\kappa}}}) \in H^{2k}(B\mathrm{Symp}_0 \widetilde{M})$$

is well defined. As we see below, when the bundle $\widetilde{P} \to Z$ is constructed as a blow up, the value of $c(E; \widehat{e}^{n-1}, \widehat{e}^{n-1}\, \widehat{q_{\widetilde{\kappa}}})$ on $\sigma_*[Z]$ is given by integrating $\widehat{q_{\widetilde{\kappa}}}$ over a suitable section of the bundle $\widetilde{\Sigma} \to Z$ of exceptional divisors.

The basic reason why the next proposition holds is that the extensions $\widehat{q}_\kappa$ are well behaved under blow up. More precisely, suppose that the $\widetilde{M}$-bundle $\widetilde{P} \to Z$ is the blow up of the trivial bundle $P = M \times Z \to Z$ along gr_σ, and let $\phi_P : \widetilde{P} \to P$ be the blow down. Then, if $H^1(M;\mathbb{Q}) = 0$, it is easy to see that ϕ_P pulls the coupling class $\widehat{a} \in H^2(P)$ of $a \in \mathcal{C}_M$ back to the coupling class of $\phi_M^*(a)$ in $\widetilde{P} \to Z$. (One just needs to check this when $\dim Z = 2$ and then the normalization condition for the coupling class $\widehat{c}$ is that $\widehat{c}^{n+1} = 0$.) Hence

$$\phi_P^*(\widehat{q}_\kappa|_P) = \widehat{q_{\widetilde{\kappa}}}|_{\widetilde{P}} \quad \text{where } \widetilde{\kappa} := \phi_M^*(\kappa). \tag{4.3}$$

Proposition 4.1. *Assume that $H^1(M;\mathbb{R}) = 0$. Let $\sigma : Z \to M$ be a smooth cycle representing the homology class $[\sigma] \in H_{2k}(M)$, and denote by $\widetilde{f}_\epsilon : Z \to B\mathrm{Symp}_0(\widetilde{M}, \widetilde{\omega}_\epsilon)$ the cycle given by the ϵ-blow up of $M \times Z \to Z$ along gr_σ. Then, for every polynomial representative q_κ of $\kappa \in \mathcal{C}^*$,*

$$\Big\langle c(E; \widehat{e}^{n-1}, \widehat{e}^{n-1}\, \widehat{q_{\widetilde{\kappa}}}),\ (\widetilde{f}_\epsilon)_*[Z] \Big\rangle = \kappa([\sigma]).$$

The proof uses the following lemma.

Lemma 4.2. *Let $E \in H_2(\widetilde{M};\mathbb{Z})$ be the class of a line in the exceptional divisor. Then $\mathrm{GW}_{0,2}^{\widetilde{M},E}(e^{n-1}, e^{n-1}) = 1$.*

Proof. Choose J_0 on $\widetilde{M}$ that equals the standard integrable structure near the exceptional divisor Σ. Let ℓ_1, ℓ_2 be two disjoint embedded 2-spheres in $\widetilde{M}$ representing E that meet Σ once transversally (with sign -1). Consider the moduli space $\mathcal{M}_{0,2}(E, J_0)$ of J_0-holomorphic E-spheres with 2-marked points in $\widetilde{M}$ (quotiented out by $\mathbb{C}$, the reparametrizations that fix $0, \infty \in S^2$.) Since the exceptional divisor Σ is J_0-holomorphic and $E \cdot \Sigma = -1$, the only J_0-holomorphic E-curves in $\widetilde{M}$ are the obvious curves in Σ. Thus $\mathcal{M}_{0,2}(E, J_0)$ is compact, except that the two marked points can come together. Hence it can be completed to a *manifold* by adding in the relevant stable maps, that have a ghost bubble with the two marked points on it. We call the compactified space $\overline{\mathcal{M}}_{0,2}(E, J_0)$.

All the elements in $\overline{\mathcal{M}}_{0,2}(E, J_0)$ are regular by construction. So $\overline{\mathcal{M}}_{0,2}(E, J_0)$ is a manifold of dimension $2n + 2c_1(E) - 2 = 4n - 4$. Consider the evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(E, J_0) \to \widetilde{M} \times \widetilde{M}.$$

It meets $\ell_1 \times \ell_2$ exactly once and transversally. This proves the lemma. □

Proof of Proposition 4.1. Consider the bundle $\pi_Z : (\widetilde{P}, \widetilde{\Omega}_\epsilon) \to Z$ given by the ϵ-blow up of $M \times Z \to Z$ along gr_σ. Choose a family J of fiberwise $\widetilde{\Omega}_\epsilon$-tame almost complex structures J_z, $z \in Z$, on $\widetilde{P}$, that are standard near each exceptional divisor Σ_z. Denote by $\widetilde{\Sigma}$ the union of the exceptional divisors. Then the moduli space $\mathcal{M}_{0,0}(E, J)$ of unparametrized J-holomorphic E-spheres is a compact manifold of dimension $4n - 4 + 2k$ consisting of regular curves. The evaluation map $\mathrm{ev} : \mathcal{M}_{0,2}(E, J) \to \widetilde{P} \times \widetilde{P}$ is a pseudocycle of dimension $4n + 2k$.[9] Further $\langle c(E; \widehat{e}^{n-1}, \widehat{e}^{n-1}\, \widehat{q}_{\widetilde{\kappa}}),\ (\widetilde{f}_\epsilon)_*[Z]\rangle$ is the intersection number of $\mathrm{ev} : \mathcal{M}_{0,2}(E, J) \to \widetilde{P} \times \widetilde{P}$ with the product $X_1 \times (X_2 \cap X_3)$, where X_1, X_2 represent the Poincaré dual of $\widehat{e}^{n-1}$ and X_3 represents that of $\widehat{q}_{\widetilde{\kappa}}$.

To calculate this, choose a smoothly embedded[10] submanifold $X \subset \widetilde{P}$ that represents the Poincaré dual of $\widehat{e}^{n-1}$, choose two copies X_1, X_2 of X that intersect $\widetilde{\Sigma}$ in transversally intersecting $2k$-dimensional submanifolds Y_1, Y_2, and consider the submanifold $\mathcal{M}^{cut} := \mathrm{ev}^{-1}(X_1 \times X_2)$ of $\mathcal{M}_{0,2}(E, J)$. Evaluation at the second marked point $\mathrm{ev}_2 : \mathcal{M}^{cut} \to \widetilde{P}$ is a $2k$-dimensional pseudocycle (with image in $\widetilde{\Sigma} \cap X_2$) and, by definition,

$$\langle c(E; \widehat{e}^{n-1}, \widehat{e}^{n-1}\, \widehat{q}_{\widetilde{\kappa}}),\ (\widetilde{f}_\epsilon)_*[Z]\rangle = \langle \mathrm{ev}_2^*(\widehat{q}_{\widetilde{\kappa}}),\ \mathcal{M}^{cut}\rangle = \langle \mathrm{ev}_2^* \phi_P^*(\widehat{q}_\kappa),\ \mathcal{M}^{cut}\rangle,$$

where the second equality holds by (4.3).

We now claim that $\phi_P \circ \mathrm{ev}_2 : \mathcal{M}^{cut} \to P = M \times Z$ represents $[gr_\sigma]$. To see this, consider the codimension 2 submanifold $Z_1 := \pi_Z(Y_1 \cap Y_2) \subset Z$. Because there is a unique J-holomorphic E-curve through each pair of distinct points in Σ_z,

$$\mathcal{M}^{cut} \cap \mathrm{ev}^{-1}\big(\pi_Z^{-1}(Z \setminus Z_1)\big) \cong Y_1 \times_{Z \setminus Z_1} Y_2.$$

But by construction the map $\phi_P : Y_i \to gr_\sigma$ has degree 1. Hence $\phi_P \circ \mathrm{ev}_2 : \mathcal{M}^{cut} \to gr_\sigma$ also has degree 1.

Therefore,

$$\langle \mathrm{ev}_2^* \phi_P^*(\widehat{q}_\kappa),\ \mathcal{M}^{cut}\rangle = \langle \widehat{q}_\kappa,\ [gr_\sigma]\rangle = \langle \kappa,\ [\sigma]\rangle,$$

[9] Note that we do not compactify here, i.e. the 2 marked points are assumed distinct, but we do divide out by the reparametrization group. Pseudocycles are discussed in [23, 29].

[10] The Poincaré dual of $\widehat{e}$ can be represented by an embedded submanifold Q: take the inverse image of a hyperplane by a map $P \to \mathbb{C}P^N$ representing $\widehat{e}$. Hence we can take X to be an iterated intersection. Since $[X] \cap [\widetilde{M}] = E$, roughly speaking X intersects a generic fiber of $\widetilde{P} \to Z$ in a line.

where the second equality holds because, for the trivial bundle $P = M \times Z \to Z$, we have $\widehat{q}_\kappa = pr_M^*(\kappa)$. □

Corollary 4.3. *Theorem 1.9 holds.*

Proof. This is an immediate consequence of Proposition 4.1. □

We next turn to the proof of Proposition 1.10, which concerns classes $[\sigma]$ that can be represented by smooth, simply connected cycles, i.e. smooth maps $\sigma : Z \to M$ where the manifold Z is simply connected. The next lemma is no doubt well known. We include a proof for completeness.

Lemma 4.4. *If $\dim M > 4$ then the classes represented by smooth, simply connected cycles generate the image in $H_*(M;\mathbb{Q})$ of the homology of the universal cover of M.*

Proof. It is obvious that these classes lie in this image since any such map σ factors through the universal cover. To prove the converse, it suffices to prove that if $\pi_1(M) = 0$ and $Z \subset M$ is a smooth oriented embedded[11] d-manifold with $d \geq 3$ then one can do surgery on Z to make it simply connected also. Since $\dim M > 4$ every embedded loop γ in Z is the boundary of an embedded 2-disc D in M that we can chose to be nowhere tangent to Z along $\partial D = \gamma$. The normal bundle ν_D of D in M splits along the boundary as a sum $\epsilon^{d-1} \oplus \epsilon^{2n-d-1}$ where ϵ^{d-1} is the trivial normal bundle to γ in Z. If this splitting extends over D so that ν_D is a sum of oriented bundles $\xi^{d-1} \oplus \xi^{2n-d-1}$ then we can kill γ in $\pi_1(Z)$ by doing (possibly twisted) surgery to Z, removing a neighborhood of γ and adding the boundary of a neighborhood of D in ξ^{d-1}. Therefore we need to check that the element α of $\pi_2(B\mathrm{SO}(2n-2))$ represented by the normal bundle to D with its given boundary trivialization lifts to an element of $\pi_2(B\mathrm{SO}(d-1) \times B\mathrm{SO}(2n-d-1))$. The obstruction is the image of $\alpha \in \pi_1(Gr)$, where $Gr := \mathrm{SO}(2n-2)/\mathrm{SO}(d-1)\times\mathrm{SO}(2n-d-1)$ is the Grassmannian of oriented $(d-1)$-planes in $\mathbb{R}^{2n-2}$. Since $\pi_1(Gr) = 0$, the extension is unobstructed. □

We now restate Proposition 1.10 for the convenience of the reader.

Proposition 4.5. *Let $\sigma : Z \to M$ be a smooth and simply connected nontrivial cycle, and let $\widetilde{f}_\epsilon : Z \to B\mathrm{Ham}(\widetilde{M}, \widetilde{\omega}_\epsilon)$ be the corresponding classifying map. Then $(\widetilde{f}_\epsilon)_*[\sigma] \neq 0$ if one of the following conditions hold.*

(i) *There is $\kappa \in \mathcal{C}^*$ such that $\kappa([\sigma]) \neq 0$.*

(ii) *$Z = S^{2k}$ and (M, ω) has the hard Lefschetz property.*

Proof. Case (i) has the same proof as Proposition 4.1. The crucial point is that an analog of Lemma 1.3 holds in this setting. We know that all classes $a \in \mathcal{C}$ extend to M_{Ham} by Lalonde–McDuff–Polterovich [13, 12]. Moreover, because $\pi_1(Z) = 0$ any two extensions differ by a class pulled back from the base. Hence the normalization condition is enough to provide a *unique* extension that satisfies the compatibility condition (4.3).

In case (ii) all classes in $H^*(M;\mathbb{Q})$ extend to M_{Ham} by Blanchard [3]. The same holds for $\widetilde{M}$ since this also has the hard Lefschetz property. The difficulty is to ensure that the

[11]We can assume that Z is embedded by Thom's well known result that rational homology is generated by embedded smooth cycles.

extensions satisfy the analog of (4.3). We do this by imitating the normalization procedure in Lemma 1.3. Given $b \in H^d(M)$ we claim there is $\widehat{b} \in M_{\rm Ham}$ such that

$$\pi_!(\widehat{b}\,\widehat{a}^n) := \int_M \widehat{b}\,\widehat{a}^n = 0 \in H^d(B\mathrm{Ham}), \tag{4.4}$$

where $\widehat{a}$ is the coupling class (i.e. normalized extension of the symplectic class $[\omega]$), and $\int_M$ denotes integration over the fiber. (The proof is as in Lemma 1.3; given any extension $\widehat{b}\,'$ simply subtract from it the pullback of a suitable multiple of $\pi_!(\widehat{b}\,'\,\widehat{a}^n)$.) This normalized extension $\widehat{b}$ may not be unique. However, its pullback to any bundle $P \to Z$ over a sphere $Z = S^d$ is unique. Moreover because the normalization condition on P is simply that $\widehat{b}\,\widehat{a}^n = 0 \in H^{2n+d}(P)$ the analog of (4.3) is also satisfied. Thus, if $Z = S^d$ and $b \in H^*(M)$,

$$\phi_P^*(pr_M^*(b)) = \widehat{b}|_{\widetilde{P}},$$

where $\widehat{b}|_{\widetilde{P}}$ denotes the restriction to $\widetilde{P}$ of *any* extension of $\phi_M^*(b)$ to $\widetilde{M}_{\rm Ham}$ that satisfies (4.4).

Now argue as in Proposition 4.1 using the characteristic class $c(E; \widehat{e}^{\,n-1}, \widehat{e}^{\,n-1}\,\widehat{b})$ instead of $c(E; \widehat{e}^{\,n-1}, \widehat{e}^{\,n-1}\,\widehat{q}_{\widetilde{\kappa}})$, where b is chosen so that $b([\sigma]) \neq 0$ and $\widehat{b}$ is any extension of $\phi_M^*(b)$ that satisfies (4.4). This proves (ii). □

Example 4.6. Consider $M = \mathbb{T}^{2n}$ with a standard form. Because ev : $\pi_1(\mathrm{Symp}\,M) \to \pi_1(M)$ is surjective in this case, the bundle $\widetilde{M} \to \widetilde{P} \to M$ of equation (1.6) is symplectically trivial over the 1-skeleton of the base M. Therefore, because $\pi_1(M)$ has no torsion, this bundle is classified by a map into $B\mathrm{Ham}\,\widetilde{M}$; see [12, Erratum]. Because $\mathbb{T}^{2n}$ has the hard Lefschetz property, one might then expect $H_d(M)$ to inject into $H_*(B\mathrm{Ham}\,\widetilde{M})$ for all $1 < d \le 2n$. But this map is trivial because there is an automorphism of the product bundle $M \times M \to M$ that takes the diagonal to the constant section; in fact the evaluation map ev : $\mathrm{Symp}_0(\mathbb{T}^{2n}) \to \mathbb{T}^{2n}$ is surjective on homology. This example explains why one needs some conditions in part (ii) of Proposition 4.5, though one might well be able to weaken the ones given.

Remark 4.7. Part (ii) of Proposition 4.5 holds whenever $(\widetilde{M}, \widetilde{\omega}_\epsilon)$ has the c-splitting property for all $\epsilon < \epsilon_0$, i.e. whenever the cohomology Leray–Serre spectral sequence of the universal $\widetilde{M}$ bundle $\widetilde{M}_{\rm Ham} \to B\mathrm{Ham}(\widetilde{M}, \widetilde{\omega}_\epsilon)$ degenerates at the E_2 term. We claim that in this case (M, ω) also has the c-splitting property. However, it is not clear whether the c-splitting property for (M, ω) implies that for $(\widetilde{M}, \widetilde{\omega}_\epsilon)$ since there may be symplectomorphisms of $\widetilde{M}$ that have little to do with those of M.

To prove the claim, note that if $(\widetilde{M}, \widetilde{\omega}_\epsilon)$ has the c-splitting property then, because compact subsets of $B\mathrm{Ham}(M, p)$ map compatibly to $B\mathrm{Ham}(\widetilde{M}, \widetilde{\omega}_\epsilon)$, the universal M-bundle with section $M_{\mathrm{Ham}(M,p)} \to B\mathrm{Ham}(M, p)$ also has degenerate spectral sequence. To see that the spectral sequence of $M_{\rm Ham} \to B\mathrm{Ham}(M, \omega)$ also degenerates, it suffices to check that the map $\pi : M_{\rm Ham} \to B\mathrm{Ham}(M, \omega)$ on base spaces induces an injection on cohomology, since then the spectral sequences inject; cf. [12, Lemma 4.1]. But this holds because there is a class $\widehat{c} \in H^{2n}(M_{\rm Ham})$ (namely $\widehat{c} := \widehat{a}^n$ where $\widehat{a}$ is the canonical extension of $a := [\omega]$) that restricts to a generator of $H^{2n}(M; \mathbb{R})$. Since in this case the map

$$H^*(B\mathrm{Ham}; \mathbb{R}) \to H^*(B\mathrm{Ham}; \mathbb{R}), \quad b \mapsto \pi_!(\widehat{c} \cup \pi^*(b))$$

is multiplication by a nonzero constant, π^* is injective.

5. Blowing up at many points

Now suppose that we blow up more than once. The first result concerns simultaneous blow ups.

Proposition 5.1. *Let (M, a) be a c-symplectic manifold and denote by $\widetilde{M}_k$ its k-fold blow up. Then:*

(i) *There is a homomorphism $\widetilde{f}_*^k : \big(\mathbb{Z}\oplus\pi_2 M\big)^k \to \pi_2\big(B\mathrm{Diff}\,\widetilde{M}_k\big)$ whose kernel is contained in the torsion subgroup of $\big(\pi_2 M\big)^k$.*

(ii) *If (M,ω) is symplectic then there is $\epsilon_0 > 0$ such that the elements in $\widetilde{f}_*^k\big((\pi_2 M)^k\big)$ can all be realised in $B\mathrm{Symp}(\widetilde{M}_k, \omega_\epsilon)$ whenever the blow up parameter $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ satisfies $\epsilon_i \le \epsilon_0$ for all i.*

Proof. Fix a metric on M. Given a (small) constant $\nu > 0$, define $\Delta_\nu \subset M^k$ by setting

$$\Delta_\nu := \{(x_1, \dots, x_k) \in M^k : d(x_j, x_i) \le \nu, i \ne j\}.$$

Then the product bundle $M \times (M^k \setminus \Delta_\nu) \to (M^k \setminus \Delta_\nu)$ has k symplectic sections of the form $(x_1, \dots, x_k) \mapsto (x_j, x_1, \dots, x_k)$ that are mutually separated by a distance of at least ν. Hence, for sufficiently small $\epsilon_0 = \epsilon_0(\nu) > 0$, it is possible to blow up $M \times (M^k \setminus \Delta_\nu)$ along these sections simultaneously with any weights $\epsilon_i \le \epsilon_0$. This defines a model fibration $(\widetilde{M}_k, \omega_\epsilon) \to (\widetilde{P}_k, \Omega_\epsilon) \to M^k \setminus \Delta_\nu$ that plays the role of the fibration (1.6).

To prove (ii) fix k distinct points y_j in M and choose smooth maps $\sigma_\alpha^j : S^2 \to M$, for $1 \le \alpha \le r$, that represent a basis for $H_2(M)$/torsion and are such that for all α, j the image of σ_α^j does not contain the points $y_i, i \ne j$. Now set

$$\nu := \frac{1}{2}\inf\{d(\sigma_\alpha^j(z), y_i) : i \ne j, z \in S^2, 1 \le \alpha \le r\}.$$

For each j, α as above, consider the map

$$\widehat{\sigma}_\alpha^j : S^2 \to M^k \setminus \Delta_\nu, \quad z \mapsto (y_1, \dots, y_{j-1}, \sigma_\alpha^j(z), y_{j+1}, \dots y_k).$$

The element $\widetilde{f}_*^k([\widehat{\sigma}_\alpha^j]) \in \pi_2(B\mathrm{Symp}(\widetilde{M}_k, \omega_\epsilon))$ is represented by the pullback of the model fibration by $\widehat{\sigma}_\alpha^j$. This defines $\widetilde{f}_*^k$ on a set of generators of $H_2(M^k)$/torsion. To see that it is injective, one calculates volumes as before, considering these as a function of the k blow up parameters $\epsilon_1, \dots, \epsilon_k$. This proves (ii).

The proof of (i) is a straightforward adaptation of that of Corollary 1.5, and is left to the reader. □

The above result is weaker than it need be. Suppose for example that $H_2(M)$ has rank r and that $k = 2$. Then the argument above detects a subgroup of rank $2r + 2$ inside $\pi_2\big(B\mathrm{Diff}\,\widetilde{M}_2\big)$. However this group should contain a subgroup of rank $2r + 4$, where the two extra generators γ_1, γ_2 come from doing one of the blow ups along a line in the exceptional divisor formed by the other blow up. These blow ups can no longer be done simultaneously. Rather, the corresponding fibration $\widetilde{P} \to S^2$ is a fiber sum $\widetilde{P}_1 \# \widetilde{P}_2$ where each $\widetilde{P}_i$ is the ϵ_i-blow up of $\widetilde{M}_1 \times S^2$ along a suitable section. Thus when forming $\widetilde{P}_1$ one starts with a trivial fibration whose fiber has volume $V - v_{\epsilon_2}$, while for $\widetilde{P}_2$ the fiber has volume $V - v_{\epsilon_1}$. More generally, if we blow up k times, then $\widetilde{P}$ decomposes as $\widetilde{P}_1 \# \cdots \# \widetilde{P}_k$, where $\widetilde{P}_i$ is the ϵ_i-blow up of a trivial bundle with fiber of volume $V - \sum_{j\ne i} v_{\epsilon_j}$ along a section s_i whose homology class lies in a group of rank $r+k-1$ and has one parameter ℓ_i describing its normal bundle. Therefore,

$$\operatorname{vol}(\widetilde{P}_{\lambda,\ell},\widetilde{\Omega}_\epsilon) = \sum_i \Big(\big(V - \sum_{j\neq i} v_{\epsilon_j}\big)\mu_i - v_{\epsilon_i}\big(\mu_i + \lambda_i - \frac{\ell_i}{n+1}\epsilon_i\big)\Big) \tag{5.1}$$

where $\lambda_i := I_a(s_i)$ is no longer constant but is a sum $\lambda_i' + \lambda_i''$, where $\lambda_i' = \sum_{j\neq i} m_{ij}\epsilon_j$ for $m_{ij} \in \mathbb{Z}$, and $\lambda_i'' \in \mathbb{R}$. If the bundle were trivial this polynomial would have the form $(V - \sum_j v_{\epsilon_j})(\mu + \sum k_j\epsilon_j)$. Since $V \neq 0$ and $\operatorname{vol}(\widetilde{P}_{\lambda,\ell},\widetilde{\Omega}_\epsilon)$ has no terms that are linear in the ϵ_i it follows as before that all the coefficients $k_i, m_{ij}, \lambda_i, \ell_i$, must vanish and that $\mu = \sum \mu_i$. This proves the following result.

Proposition 5.2. *Let (M,a) be a c-symplectic manifold and denote by $\widetilde{M}_k$ its k-fold blow up. Then the rank of $\pi_2\big(B\mathrm{Diff}\widetilde{M}_k\big)$ is at least $k(r+k)$, where $r := \operatorname{rank}\pi_2(M)$.*

We now address the question of when these new elements in $\pi_2\big(B\mathrm{Diff}\widetilde{M}_k\big)$ can be constructed to be symplectic. Note that if (M,ω) is a blow up, the size of its exceptional divisor does not constrain the maximal size of a single symplectically embedded ball; for example if one blows up $\mathbb{C}P^2$ just a little, one can blow it up again using a much larger ball. However the next argument shows that it does constrain the size of suitable families of embedded balls.

Proposition 5.3. *Let (X,ω) be a symplectic manifold that is itself a blow up of size α, that is α is the integral of ω over a line in the exceptional divisor Σ_X. Suppose that $\sigma : S^2 \to X$ represents a class such that $[\sigma]\cdot[\Sigma_X] \in H_0(X)$ is nonzero and construct $\pi : (\widetilde{P},\widetilde{\Omega}_\epsilon) \to S^2$ by blowing up $X \times S^2$ along gr_σ. Then:*

(i) *$\epsilon < \alpha$.*

(ii) *If $\widetilde{\rho}_t, 0 \le t \le 1$, is any family of symplectic forms on $\widetilde{P}$ that start at some blow up form $\widetilde{\rho}_0 := \widetilde{\Omega}_\epsilon$ then $\int_{E_\sigma}\widetilde{\rho}_1 < \alpha$, where E_σ is the class of a line in the exceptional divisor in the blow up $\widetilde{X}$.*

Proof. Let $P := X \times S^2$. We first claim

$$\mathrm{GW}^{P,E_X}_{0,2}([gr_\sigma], E_X \times [S^2]) = -[\sigma]\cdot[\Sigma_X] \neq 0$$

where E_X is the class of a line in the exceptional divisor Σ_X. To prove this, choose an ω-tame almost complex structure J on X that is integrable near Σ_X, Choose an embedded 2-sphere ℓ_X in the class E_X that meets Σ_X once transversally, and perturb σ so that gr_σ, $\Sigma_X \times S^2$ and $\ell_X \times S^2$ are in general position. Then, to each intersection point $x \in gr_\sigma \cap (\Sigma_X \times S^2)$ there is precisely one $(J \times j)$-holomorphic curve in $X \times S^2$ that meets $\ell_X \times [S^2]$. It is regular and, because $\ell_X \cap \Sigma_X = -1$, it contributes to the Gromov–Witten invariant with sign opposite to that of the orientation of the intersection at x. This proves the claim.

Now consider $(\widetilde{P},\widetilde{\Omega}_\epsilon)$. We claim that

$$\mathrm{GW}^{P,E_X}_{0,2}([gr_\sigma], E_X \times [S^2]) = \mathrm{GW}^{\widetilde{P},E_X - E_\sigma}_{0,2}(\widetilde{\Sigma}, E_X \times [S^2]),$$

where $\widetilde{\Sigma}$ denotes the subbundle of $\widetilde{P}$ formed by the exceptional divisors.[12] This is proved by direct construction. Denote by Ω a product symplectic form $\omega + \pi^*(\beta)$ on P, and choose an Ω-tame almost complex structure J_P on P so that gr_σ is J_P-holomorphic and so that J_P is integrable near gr_σ. If ϵ is sufficiently small we can perform the blow up of P along gr_σ in such a way that J_P lifts to an $\widetilde{\Omega}_\epsilon$-tame almost complex structure $\widetilde{J}_P$ on $\widetilde{P}$, i.e. so that the blow down map is $(\widetilde{J}_P, J_P)$-holomorphic. It then follows that there is a bijective correspondence between the curves counted by each of the above Gromov–Witten invariants. (For details of a similar proof see [23, Ch 9.3].)

It follows that the integral of $\widetilde{\Omega}_\epsilon$ over $E_X - E_\sigma$ must be positive. This proves (i). (ii) follows because the Gromov–Witten invariants are unchanged by a deformation of the symplectic structure. □

Now suppose that B_1, B_2 are two disjoint symplectically embedded balls in (M, ω). Denote by $X_i, i = 1, 2$, the blow up of M by B_i, and by X_{12} the blow up by both balls. Denote by E_i the class of a line in the exceptional divisor corresponding to B_i. For $i = 1, 2$ let $\Lambda_i \in \pi_2(B\mathrm{Diff}(X_{12}))$ be the element formed by blowing up $X_i \times S^2$ along the graph of E_i. We would like to be able to conclude from the above proposition that the element $\Lambda_1 + \Lambda_2$, for example, is not homotopic to an element of $\pi_2\big(B\mathrm{Symp}(X_{12}, \widetilde{\omega})\big)$ for any blow up form $\widetilde{\omega}$ on X_{12} because we need $\widetilde{\omega}(E_1) < \widetilde{\omega}(E_2)$ to represent Λ_2 and the reverse inequality to represent Λ_1.[13] The problem is that Proposition 5.3 applies only to bundles $(\widetilde{P}, \widetilde{\Omega})$ such that $\widetilde{\Omega}$ can be deformed into a blow up form, while the elements of $\pi_2(\mathrm{Ham}\widetilde{M_k})$ correspond to bundles $(\widetilde{P}, \widetilde{\Omega})$ where $\widetilde{\Omega}$ is an arbitrary extension of a blow up form on $\widetilde{M_k}$ that may not have such a deformation. In the next section we show that this does not happen in dimension 4.

6. The 4-dimensional case.

This section discusses k-fold blow ups of a 4-dimensional manifold M. First a preliminary lemma. Though well known, we include a brief proof for completeness. (See [23] for further details.)

Lemma 6.1. *Let (M, ω) be a symplectic 4-manifold and E the class of a symplectically embedded sphere of self intersection -1. Then if $J_z, z \in Z$, is any generic compact 2-parameter family of ω-tame almost complex structures, there is a finite subset $\{z_1, \ldots, z_k\} \subset Z$ such that E is represented by a unique embedded J_z-holomorphic sphere for all $z \in Z \setminus \{z_1, \ldots, z_k\}$. Moreover, for $z = z_i$ the class E is represented by a nodal J_z-holomorphic curve consisting of two embedded spheres, intersecting once transversally and with self-intersections -2 and -1.*

[12] This GW invariant (which can be considered as a parametric GW invariant on the bundle $\widetilde{P}$) need not come from a characteristic class on $B\mathrm{Symp}_0$ since it is not clear that the submanifold $\widetilde{\Sigma}$ extends to a codimension 2 homology class in $\widetilde{M}_{\mathrm{Symp}_0}$. For instance, $\widetilde{\Sigma}$ is not the Poincaré dual of the normalized extension $\widehat{e}$.

[13] By "blow up form" on a k-fold blow up $\widetilde{M_k}$ we mean a symplectic form on $\widetilde{M_k}$ that is constructed from (M, ω) by cutting out k disjoint standard balls. Similarly, a symplectic form on the $\widetilde{M}$-bundle $\widetilde{P}$ is called a blow up form if it is obtained from a form on an M-bundle over S^2 by cutting out a suitable neighborhood of a section. This is equivalent to requiring that $\widetilde{P}$ has a subbundle with the exceptional divisor as fiber. One feature of such forms is that they can always be deformed through symplectic forms so as to decrease the size of the balls.

Proof. Because $GW_{0,0}^{M,E} = 1$, E always has a J-holomorphic representative. We must investigate the possible nodal curves representing E. These are finite unions $C_0 \cup \cdots \cup C_k$ of spheres in classes $A_0, \dots, A_k$. The index $\operatorname{ind} D_u$ of a sphere $u : S^2 \to M$ in class A in a 4-manifold is $4 + 2c_1(A)$, where D_u denotes the linearized Cauchy–Riemann operator. In a generic 2-parameter family the cokernel of D_u has dimension at most 2. Hence if u is simple so that there is a 6-parameter reparametrization group this index must be at least 4, i.e. we must have $c_1(A) \ge 0$. If u is multiply covered then the reparametrization group has dimension at least 10 which means that the index must be at least 8. In other words $c_1(A) \ge 2$. Since $\sum_i c_1(A_i) = 1$, it follows that all components are simple and that all but one (say A_0) have $c_1(A_i) = 0$. Since the components are simple, they satisfy the adjunction inequality: $c_1(A_i) = 2 + (A_i)^2 - 2d_i$ where $d_i \ge 0$ is the defect.[14] Hence $(A_i)^2 \ge -2$ for $i > 0$ and $(A_0)^2 \ge -1$. Since the nodal curve is connected, there are at least k pairs of components C_i, C_j that intersect. Hence, by positivity of intersections, $A_i \cdot A_j \ge 1$ for $i \ne j$, and we find

$$-1 = (A_0 + \cdots + A_k)^2 = \sum A_i^2 + \sum_{i \ne j} 2A_i \cdot A_j \ge -1 - 2k + 2k.$$

Hence all inequalities are equalities. Thus the components are embedded and $A_i \cdot A_j$ is 0 or 1. This means that the classes A_i are all distinct. But at most one class A_i with $c_1(A_i) = 0$ is represented for any J_z. Thus E has at most two components, and these intersect transversally because $A_0 \cdot A_1 = 1$.

Since the energy $\omega(A_i)$ is bounded above by $\omega(E)$ and J_z varies in a compact set, the usual compactness argument implies that there are only finitely many possibilities for the classes A_i. Moreover, because the family J_z is generic, each point z at which a class A_i with $c_1(A_i) = 0$ is represented is isolated. The result follows. □

Remark 6.2. A similar argument shows that in a generic 3-parameter family of J_z no new nodal curves appear, though now of course degenerations appear for a 1-parameter family of z.

The first (well known) corollary shows that in 4 dimensions any form on a k-fold blow up that can be deformed into a blow up form must itself be a blow up form.

Corollary 6.3. *Consider a family $\widetilde{\omega}_t, t \in I$, of symplectic forms on a 4-dimensional manifold $\widetilde{M}_k$ such that $\widetilde{\omega}_0$ is a blow up form. Then $\widetilde{\omega}_1$ is a blow up form.*

Proof. By assumption there are k disjoint $\widetilde{\omega}_0$-symplectically embedded -1 spheres. Denote their classes by $E_1, \dots, E_k$, and choose an $\widetilde{\omega}_0$-tame almost complex structure for which they are holomorphic. Extend J_0 to a generic family $J_t, t \in I$, of almost complex structures such that J_t is $\widetilde{\omega}_t$-tame. By the lemma each class E_i has a unique embedded J_t holomorphic representative. Hence result. □

We now show that many of the new elements constructed in Proposition 5.2 do not lie in $\pi_2(B\mathrm{Symp}(\widetilde{M}_k, \widetilde{\omega}))$ for any blow up form. For simplicity, we restrict to simply connected M (so that every M-bundle over S^2 has a symplectic form) and will first formulate our argument in a special case. We shall denote by $\mathcal{E} := \mathcal{E}(M, \omega) \subset H_2(M; \mathbb{Z})$ the set of classes that are represented by symplectically embedded spheres of self-intersection -1

[14] If C_i is immersed, d_i is the number of double points; $d_i = 0$ iff C_i is embedded.

(or symplectic -1 spheres, for short). Recall from [20] that the following conditions on a symplectic 4-manifold are equivalent:

- (M, ω) is not a blow up of $\mathbb{C}P^2$ or of a ruled surface (manifold with a fibering by symplectically embedded 2-spheres);
- $E \cdot E' = 0$ for any two distinct elements $E, E' \in \mathcal{E}$.

If these conditions hold we shall say that (M, ω) has Kodaira dimension $\kappa(M) \geq 0$. Such manifolds have a unique minimal reduction; equivalently, if $C_1, \dots, C_\ell$ is a maximal set of disjoint symplectic -1 spheres then $\{[C_i] : 1 \leq i \leq \ell\} = \mathcal{E}$. Finally, (M, ω) is said to be minimal if $\mathcal{E}(M) = \emptyset$.

Proposition 6.4. *Let (M, ω) be a minimal symplectic 4-manifold with $\kappa(M) \geq 0$ and let $(\widetilde{M}_k, \widetilde{\omega}) \to \widetilde{P} \to S^2$ be a symplectic bundle with fiber the k-fold blow up of (M, ω) with weights $\epsilon_1 \geq \dots \geq \epsilon_k$. Then $\widetilde{P}$ is the k-fold blow up of a symplectic bundle $(M, \omega) \to P \to S^2$ along sections $s_i, 1 \leq i \leq k$, whose homotopy class varies in a group spanned by $\pi_2(M; \mathbb{Z})$ and the classes $E_j, j < i$, of the exceptional divisors with $\epsilon_j > \epsilon_i$.*

Proof. Denote by $E_1, \dots, E_k \in H_2(\widetilde{M}_k)$ the classes of the exceptional divisors; set $\epsilon_i := \widetilde{\omega}(E_i)$. By the minimality of (M, ω), the only classes in $\mathcal{E}(\widetilde{M}_k)$ are the E_i. Let $J_z, z \in S^2$, be a generic family of $\widetilde{\omega}_z$-tame almost complex structures on the fibers of $\widetilde{P} \to S^2$, where $\widetilde{\omega}_z$ is the symplectic form in the fiber $M_z := \pi^{-1}(z)$ of $P \to S^2$. Then we claim that the smallest class E_k has an embedded J_z-holomorphic representative for every $z \in S^2$. Otherwise, by an obvious generalization of Lemma 6.1, we may write $E_k = A_0 + A_1$ where $A_0 \in \mathcal{E}(\widetilde{M}_k) = \{E_1, \dots, E_k\}$. Since $\widetilde{\omega}(A_1) < \widetilde{\omega}(E_k) = \epsilon_k \leq \epsilon_i$ for all i, this is impossible.

Let $\widetilde{\Sigma}$ be the submanifold of $\widetilde{P}$ formed by the union of these E_1-curves. Then $\widetilde{\Sigma}$ is an S^2 bundle over S^2 with fibers that are $\widetilde{\omega}_z$-symplectically embedded. Choose a closed extension $\widetilde{\Omega}$ of the $\widetilde{\omega}_z$ defined in some neighborhood U of $\widetilde{\Sigma}$ and such that $\widetilde{\Sigma}$ is a symplectic submanifold of U.[15] By [20] the restriction of $\widetilde{\Omega}$ to $\widetilde{\Sigma}$ is standard, and so, by the symplectic neighborhood theorem, $\widetilde{\Sigma}$ may be blown down. Hence $\widetilde{P}$ is formed by blowing up some bundle $\widetilde{Q} \to S^2$ with fiber $\widetilde{M}_{k-1}$ along some section s_k. Repeating this argument, we find that $\widetilde{P}$ is a k-fold blow up along sections s_i whose homotopy class depends on $\pi_2(M)$ and the $E_j, j < i$.

It remains to prove the last statement. If $\epsilon_i > \epsilon_{i+1}$ for all i there is nothing to prove. Otherwise the result follows from Lemma 6.5 below. □

We need to adapt Proposition 5.3 to the case when the initial bundle $M \to P \to S^2$ is nontrivial.

Lemma 6.5. *Let $(X_1, \omega_1) \to \widetilde{P}_1 \to S^2$ be the ϵ_1-blow up of a symplectic bundle $(X_0, \omega_0) \to P \to S^2$ along a section s_1. Denote the exceptional class in X_1 by E_1 and the bundle of exceptional divisors in $\widetilde{P}_1$ by $\widetilde{\Sigma}_1$. Let $s_2 : S^2 \to \widetilde{P}_1$ be a section such that $[s_2] \cdot [\widetilde{\Sigma}_1] \neq 0$, and construct $\pi : (\widetilde{P}_2, \widetilde{\Omega}_\epsilon) \to S^2$ as the ϵ_2-blow up of $\widetilde{P}_1$ along s_2. Then $\epsilon_1 > \epsilon_2$.*

Proof. The proof is essentially the same as that of Proposition 5.3. We first show that

$$\mathrm{GW}_{0,2}^{\widetilde{P}_1, E_1}([s_2], [\widetilde{\Sigma}_1]) = -[s_2] \cdot [\widetilde{\Sigma}_1] \neq 0,$$

[15]Note that we do not assume that $H^1(M; \mathbb{R}) = 0$ or that $\widetilde{P}$ supports a closed extension of the $\widetilde{\omega}_z$, i.e. the bundle $\widetilde{P} \to S^2$ need not be Hamiltonian. The present arguments are geometric and do not use the coupling form.

and then prove that

$$\mathrm{GW}_{0,2}^{\widetilde{P}_1,E_1}([s_2],[\widetilde{\Sigma}_1]) = \mathrm{GW}_{0,2}^{\widetilde{P}_2,E_1-E_2}([\widetilde{\Sigma}_2],[\widetilde{\Sigma}_1]),$$

where $\widetilde{\Sigma}_2$ is the bundle of exceptional divisors in $\widetilde{P}_2$. Further details are left to the reader. □

It follows from Remark 6.2 that each blow down above is unique, i.e. it is independent of the choice of the family J_z. Moreover, it depends on the section only up to homotopy. Hence we find:

Corollary 6.6. *With $(\widetilde{M}_k, \widetilde{\omega}_\epsilon)$ as above, suppose that $\epsilon_1 > \cdots > \epsilon_k > 0$. Then the rank $\pi_2\big(B\mathrm{Symp}(\widetilde{M}_k, \widetilde{\omega}_\epsilon)\big)$ exceeds that of $\pi_2\big(B\mathrm{Symp}(M,\omega)\big)$ by at most $rk + k(k-1)/2$, where $r = \mathrm{rk}\,\pi_2(M)$.*

We cannot give a more precise answer here because there is a **realization problem**: it is not obvious that the condition in Lemma 6.5 is the only obstruction to being able to blow up a section with weight ϵ, even if we assume that $\kappa(M) \ge 0$ and that M itself has such a blow up. Therefore some work is still needed before we can fully understand the relation between $\pi_2\big(B\mathrm{Symp}(\widetilde{M}_k, \widetilde{\omega}_\epsilon)\big)$ and $\pi_2\big(B\mathrm{Symp}(M,\omega)\big)$.[16] Nevertheless, comparing with Proposition 5.2 it is clear that when $k \ge 2$ there are many elements in $\pi_2\big(B\mathrm{Diff}(\widetilde{M}_k)\big)$ that are constructed using standard Hermitian structures but yet cannot be realized symplectically. Here is a sample result.

Corollary 6.7. *Let B_1, B_2 be two disjoint embedded balls in a symplectic 4-manifold (M,ω) with $\kappa(M) \ge 0$. Denote by $X_i, i = 1,2$, the blow up of M by B_i, and by X_{12} the blow up by both balls. Denote by E_i the class of the exceptional divisor corresponding to B_i. For $i = 1,2$ let $\Lambda_i \in \pi_2(B\mathrm{Diff}(X_{12}))$ be the element formed by blowing up $\widetilde{M}_i \times S^2$ along the graph of E_i with the standard Hermitian structure. Then $\Lambda_1 + \Lambda_2$ is not homotopic to an element of $\pi_2\big(B\mathrm{Symp}(X_{12}, \widetilde{\omega})\big)$ for any blow up form on X_{12}.*

Proof. For $i = 1,2$ let $X_{12} \to \widetilde{P}_i \to S^2$ be the bundle corresponding to Λ_i. Then the bundle corresponding to $\Lambda_1 + \Lambda_2$ is the fiber[17] sum $\widetilde{Q} : \widetilde{P}_1 \# \widetilde{P}_2 \to S^2 \# S^2 = S^2$; see [13]. But $\widetilde{Q} \to S^2$ is not a symplectic bundle since it does not satisfy the conclusion of Proposition 6.4. □

Finally we investigate the analog of Proposition 6.4 for blow ups of $\mathbb{C}P^2$. (We leave the case of ruled surfaces to the reader.)

We need a preliminary lemma. Denote by X_k the k-fold blow up of $\mathbb{C}P^2$, and fix a basis $L, E_1, \dots, E_k$ of $H_2(X_k;\mathbb{Z})$ where L is the class of a line and $E_1, \dots E_k$ are represented by the obvious -1 spheres. Let ω^μ be the standard (Fubini–Study) symplectic form on $X := \mathbb{C}P^2$ normalized so that $\omega^\mu(L) = \mu$. Define $\epsilon := (\epsilon_1, \dots, \epsilon_k)$ where $\epsilon_1 \ge \epsilon_2 \ge \cdots$, and let ω^λ_ϵ be the symplectic form on X_k obtained by cutting out k disjoint balls from (X, ω^λ) of sizes $\epsilon_1, \dots, \epsilon_k$ (assuming that this exists). By [21] all such forms are isotopic. We say that ω^λ_ϵ is **reduced** if $\epsilon_1 + \epsilon_2 + \epsilon_3 \le \mu$.

Lemma 6.8. *Every symplectic form ω on X_k is diffeomorphic to some reduced form ω^λ_δ.*

[16] It is likely that by using appropriate local models one can show that given (M,ω) and k one can find ϵ_0 so that Lemma 6.5 does give the only obstructions when all $\epsilon_i \le \epsilon_0$.

[17] This is the Gompf (or connect) sum of $\widetilde{P}_1, \widetilde{P}_2$ along a fiber.

Proof. By [11] ω is diffeomorphic to some form $\omega := \omega_\epsilon^\lambda$ with $\epsilon_1 \geq \epsilon_2 \cdots$ and so we just need to make sure we can arrange that $\epsilon_1 + \epsilon_2 + \epsilon_3 \leq \lambda$. If $[\omega]$ is rational then this is proved by B.-H.Li [17].

His argument is as follows. It suffices to suppose that $[\omega]$ is integral. Suppose that $[\omega]$ is not reduced and consider the form $\omega_{\epsilon'}^{\lambda'}$ obtained from ω by doing the Cremona transformation in the first three E_i. This is a diffeomorphism ϕ_C that acts on homology by:

$$\begin{array}{rcl} L & \mapsto & L' := 2L - E_1 - E_2 - E_3, \\ E_i & \mapsto & E_i' := L - E_1 - E_2 - E_3 + E_i, \quad i = 1,2,3 \\ E_i & \mapsto & E_i' := E_i, \quad i > 3. \end{array}$$

Then $\lambda' := \omega(L') < \lambda$, and the sequence ϵ' is obtained by rearranging the numbers $\omega(E_i')$. If $\omega' := \omega_{\epsilon'}^{\lambda'}$ is not reduced, then one repeats this process to obtain a new form with $\lambda'' < \lambda'$. Since these coefficients $\lambda', \lambda'', \dots$ are positive integers, this process must stop after a finite number of steps.

When $[\omega]$ is not integral, we can argue as follows. Fix a generic complex structure J on X_k, i.e. (X_k, J) is the blow up of the complex manifold $\mathbb{C}P^2$ at k generic points. It suffices to prove that the numbers $\lambda', \lambda'', \dots$ lie in the finite set

$$S := [0, \lambda] \cap \{\omega(A) : A \text{ has a } J\text{-holomorphic representative}\}.$$

Since there is a complex line in $\mathbb{C}P^2$ through 3 generic points, L' has a J-holomorphic representative. Hence $\lambda' \in S$. But we can choose ϕ_C to be J-holomorphic. Then $\lambda'' := \omega'(L'') = (\phi_C^{-1})^*\omega(L'') = \omega((\phi_C^{-1})_* L'')$ lies in S because the class $L'' = L' - E_1' - E_2' - E_3'$ and hence also $(\phi_C^{-1})_* L''$ have J-holomorphic representatives. Similar arguments apply to the subsequent numbers $\lambda''', \dots$. □

Proposition 6.9. *Proposition 6.4 and Corollary 6.7 hold when $M = \mathbb{C}P^2$.*

Proof. By the previous lemma we may suppose that the form ω_ϵ^μ on $\widetilde{M}_k$ is reduced. Hence, because $\epsilon_i \geq \epsilon_{i+1}$ for all i,

$$\sum_{i \in I} \epsilon_i \leq 3\mu\, d, \tag{6.1}$$

for every $I \subset \{1, \dots, k\}$ with $3d$ elements.

Let $\mathcal{E}$ be the set of classes in $H_2(X_k)$ that can be represented by symplectically embedded -1 spheres. Then $\mathcal{E}$ contains the E_i, plus many other classes such as $L - E_1 - E_2$. However, because every class in $\mathcal{E}$ has an embedded representative for generic J, positivity of intersections implies that every class in $\mathcal{E}$ except the E_i has the form $dL - \sum m_i E_i$, where $m_i \geq 0$ and hence $d > 0$. To make the previous argument work we just need to show that in a generic 2-dimensional family J_z the class E_k with minimal energy never decomposes into a nodal J_z-curve of type $(E, E_k - E)$, where $E \in \mathcal{E}$.

Suppose E_k does decompose for J_z. By minimality, $E \neq E_j$ and so we can write $E = dL - \sum m_i E_i$. Now observe that $E_k \cdot E = 0$ because $E \cdot (E_k - E) = 1$. Hence $m_k = 0$. Let $F := E_k - E$ be the -2-curve. Choose i so that $m_i > 0$. Then $E_i \cdot E = -m_i < 0$ which means that E_i also decomposes for J_z. Since by genericity there is at most one -2 class with a J_z-holomorphic representative, E_i must decompose as (E', F) where $E' \in \mathcal{E}$ and F is as above. As before $E_i \cdot E' = 0$ and so $E_i \cdot F = m_i = 1$. Repeating this argument, we see that $F = \sum m_i E_i - dL$, where m_i is 0 or 1. Since $c_1(F) = 0$, there must be precisely $3d$ nonzero m_i. But then $\widetilde{\omega}(F) \leq 0$ by (6.1), which is impossible. □

Remark 6.10. For example, if X_2 is the 2-fold blow up of $\mathbb{C}P^2$, the above results show that $\pi_1(\mathrm{Symp}\, X_2)$ has rank at most 2 when the blowups are of equal size and rank at most 3 otherwise. In this case, the work of Lalonde and Pinsonnault [14, 24] shows that these are the precise ranks.

The above results give examples of elements in $\pi_2(B\mathrm{Diff})$ that are not themselves in $\pi_2(B\mathrm{Symp})$ but are sums of such elements. We now return to the question considered in Proposition 1.7; are there elements of $\pi_2(B\mathrm{Diff})$ that are not in the subgroup generated by the different $\pi_2(B\mathrm{Symp})$? The next result illustrates what our current methods can say about this.

Proposition 6.11. *Let M be a simply connected minimal symplectic 4-manifold with $\kappa(M) \ge 0$ and a unique Seiberg–Witten basic class K. Let $\widetilde{M}_k$ be its k-fold blow up for some $k>0$. Then $\pi_2(B\mathrm{Diff}\,\widetilde{M}_k)$ is not generated by the images of $\pi_2\big(B\mathrm{Symp}(\widetilde{M}_k,\widetilde{\omega})\big)$ as $\widetilde{\omega}$ ranges over all symplectic forms on $\widetilde{M}_k$.*

Proof. T.-J. Li proved in [19, Cor. 3] that when $\kappa(M) \ge 0$ every symplectic form on $\widetilde{M}_k$ is a blow up form with exceptional divisors in the same classes $E_1, \dots, E_k$. In particular its minimal reduction M is unique up to diffeomorphism. By work of Taubes [28], the assumption that there is a unique basic class implies that every symplectic structure on M has the same homotopy class of almost complex structures. Hence every symplectic bundle $(M,\omega) \to P \to S^2$ has a well defined complex vertical tangent bundle whose first Chern class, denoted c_1^V, is independent of ω.

Step 1: *Volume calculations.* By Proposition 6.4, every symplectic bundle $(\widetilde{M}_k,\widetilde{\omega}) \to (\widetilde{P},\widetilde{\Omega}) \stackrel{\pi}{\to} S^2$ is the k-fold blow up of a possibly nontrivial bundle $(M,\omega) \to P \stackrel{\pi}{\to} S^2$ along certain sections s_i with weights ϵ_i, where $\epsilon_1 \ge \epsilon_2 \ge \dots$. Thus there are bundles $\widetilde{M}_{i-1} \to \widetilde{P}_i \to S^2$ for $i = 1, \dots, k$ with $\widetilde{P}_k = \widetilde{P}$. Choose a reference section s_0 for $P \to S^2$. Since we may assume this is disjoint from the s_i there is a unique corresponding reference section, also called s_0, in each bundle $\widetilde{P}_i \to S^2$, and in particular for $\widetilde{P} := \widetilde{P}_k$.

To calculate the volume of P, choose a basis $B_\alpha, 1 \le \alpha \le r$, for $H_2(M;\mathbb{Q})$ and dual basis $b_\alpha, 1 \le \alpha \le r$, for $H^2(M)$. For each $a \in \mathcal{C}_M$, write $a = \sum \lambda_\alpha b_\alpha$. Let $\widehat{a} \in H^2(P)$ be its canonical extension and set $\widehat{a}_\mu := \widehat{a} + \mu\pi^*(\beta)$ where $\beta \in H^2(S^2)$ has total area 1. Then $\mathrm{Vol}(P, \widehat{a}_\mu) = \mu\, \mathrm{Vol}(M,a) =: \mu V_a$, where $V_a := \mathrm{Vol}(M,a)$ is a polynomial function of the λ_α.

To calculate the volume of $\widetilde{P}$, let $\widehat{a}_{\mu,i,\epsilon}$ be the class on $\widetilde{P}_i$ obtained from $\widehat{a}_\mu$ by blowing up along the sections s_j with weights ϵ_j for all $j \le i$. (Note that $\widehat{a}_{\mu,i,\epsilon}$ is *not* a coupling class.) Similarly, denote by $a_{i,\epsilon}$ its restriction to the fiber, i.e. the class on $\widetilde{M}_i$ obtained by doing the first i blow ups. Then, by (3.2),

$$\mathrm{vol}(\widetilde{P}, \widehat{a}_{\mu,k,\epsilon}) = \mu V_a - \sum_i v_{\epsilon_i}\Big(\int_{s_i} \widehat{a}_{\mu,i-1,\epsilon} - \frac{\ell_i}{n+1}\epsilon_i\Big), \quad \text{where } \ell_i := c_1^V(s_i).$$

If we write $s_i = s_0 + B_i$ where $B_i = \sum n_{i\alpha} B_\alpha + \sum_{j<i} m_{ij} E_j$, then

$$\int_{s_i} \widehat{a}_{\mu,i-1,\epsilon} = \mu + I_a(s_0) + \sum_\alpha n_{i\alpha}\lambda_\alpha + \sum_{j<i} m_{ij}\epsilon_j,$$

where I_a is as in equation (1.5). Thus $I_a(s_0)$ is a homogenous rational function of the λ_α of degree 1. Thus,

$$\mathrm{vol}(\widetilde{P}, \widehat{a}_{\mu,k,\epsilon}) = \mu\big(V_a - \sum_i v_{\epsilon_i}\big) - \sum_i \Big(I_a(s_0) + \sum_\alpha n_{i\alpha}\lambda_\alpha + \sum_{j<i} m_{ij}\epsilon_j - \frac{\ell_i}{n+1}\epsilon_i\Big) v_{\epsilon_i}. \tag{6.2}$$

Further, since this is a symplectic blow up,

$$\ell_i := c_1^V(s_i) = I_c(s_0) + \sum_\alpha n_{i\alpha} c_1(B_\alpha) + \sum_{j<i} m_{ij}. \tag{6.3}$$

Note that equation (6.2) holds for all classes $a \in \mathcal{C}$, not just those represented by a symplectic form.

Step 2: *The bundle* $\widetilde{M_k} \to \widetilde{Q} \to S^2$. This bundle is obtained from the trivial bundle $(Q := M \times S^2, \Omega)$ by $k-1$ trivial blowups with weight ϵ_i and trivial Hermitian structure and with the last blow up along $s_k := gr_\sigma$ where $\sigma : S^1 \to \widetilde{M}_{k-1}$ represents E_1 and has a nonstandard Hermitian structure (i.e. $m_{k1} = 1$ and $\ell_k \neq 1$.) Choose s_0 to be the trivial section. Then, $I_a(s_0) = 0 = I_c(s_0)$ and, by (6.2),

$$\mathrm{vol}(\widetilde{Q}, \widehat{a}_{\mu,k,\epsilon}) = \big(V_a - \sum_i v_{\epsilon_i}\big)\mu - v_{\epsilon_k}\big(\epsilon_1 - \frac{\ell_k}{n+1}\epsilon_k\big), \tag{6.4}$$

for every $a \in \mathcal{C}$.

Step 3: *Sums of bundles.* Since the sum in $\pi_2(B\mathrm{Diff})$ corresponds to the fiber sum of bundles, it suffices to show that $\widetilde{Q}$ is not a finite fiber sum of symplectic bundles. Suppose to the contrary that $\widetilde{Q} = \widetilde{Q}^{\gamma_1} \# \dots \# \widetilde{Q}^{\gamma_m}$ where for each $\gamma := \gamma_i$ the bundle $\widetilde{Q}^\gamma \to S^2$ is the k-fold blow up of an ω^γ-symplectic M-bundle $Q^\gamma \to S^2$. By the uniqueness of the blowdown, the sum $Q^{\gamma_1} \# \dots \# Q^{\gamma_m}$ is trivial. The section s_0 splits into the sum $s_0^{\gamma_1} \# \dots \# s_0^{\gamma_m}$ of sections of the Q^γ. Note that this splitting is not unique. However, for all $a \in \mathcal{C}_M$,

$$\sum_\gamma I_a(s_0^\gamma) = I_a(s_0) = 0, \quad \sum_\gamma I_c(s_0^\gamma) = I_c(s_0) = 0.$$

As in Step 1, there are unique corresponding sections s_0^γ of the intermediate bundles $\widetilde{Q}_i^\gamma$. Let s_i^γ, $1 \le i \le k$, be the section along which one does the ith blow up of Q^γ. Write $s_i^\gamma = s_0^\gamma + \sum n_{i\alpha}^\gamma B_\alpha + \sum m_{ij}^\gamma E_j$.

Step 4: *Completion of the proof.* For each γ there is a well defined class $\widehat{a}_{\mu^\gamma} \in H^2(Q^\gamma)$ that blows up to $\widehat{a}_{\mu^\gamma,k,\epsilon} \in H^2(\widetilde{Q}^\gamma)$. Then, if $\mu := \sum_\gamma \mu^\gamma$,

$$\mathrm{Vol}(\widetilde{Q}, \widehat{a}_{\mu,k,\epsilon}) = \sum_\gamma \mathrm{Vol}(\widetilde{Q}^\gamma, \widehat{a}_{\mu^\gamma,k,\epsilon_\gamma}). \tag{6.5}$$

By (6.2) the part of the coefficient of v_{ϵ_i} in $\sum_\gamma \mathrm{Vol}(\widetilde{Q}^\gamma, \widehat{a}_{\mu^\gamma,k,\epsilon})$ that depends on the λ_α is

$$\sum_\gamma I_a(s_0^\gamma) + \sum_{\gamma,\alpha} n_{i\alpha}^\gamma \lambda_\alpha = \sum_{\gamma,\alpha} n_{i\alpha}^\gamma \lambda_\alpha.$$

Comparing with (6.5) and (6.4), we find that $\sum_\gamma n_{i\alpha}^\gamma = 0$ for all i, α. Similarly, looking at the coefficient of $\nu_{\epsilon_i}\epsilon_j$ for $i \neq j$ we find that $\sum_\gamma m_{ij}^\gamma = 1$ if $(i,j) = (k,1)$ and $= 0$ otherwise. Because $\sum_\gamma I_c(s_0^\gamma) = 0$, it now follows from equation (6.3) that $\sum_\gamma \ell_i^\gamma = 1$

when $i = k$ and $= 0$ otherwise. Therefore by (6.5) the coefficient of $v_{\epsilon_k}\epsilon_k$ in $\mathrm{Vol}(\widetilde{Q}, \widehat{a}_{\mu,k,\epsilon})$ is $1/(n+1)$. Since $\ell_k \neq 1$ this contradicts equation (6.4). □

Corollary 6.12. *Suppose that $\pi_1(M) \neq 0$ but that otherwise M satisfies all the conditions in Proposition 6.11. Suppose further that $\cup a : H^1(M;\mathbb{R}) \to H^3(M;\mathbb{R})$ is an isomorphism for all $a \in \mathcal{C}$. Then the conclusion of Proposition 6.11 holds for the k-fold blow up $\widetilde{M}_k$.*

Proof. The condition $\pi_1(M) = 0$ is used once in Step 4 when we apply Lemma 1.3 to assert that every class $a \in \mathcal{C}_M$ extends to the total space of the symplectic M-bundles $Q^\gamma \to S^2$, a necessary step before we calculate volumes. We now show that each of these bundles admits a section and that therefore these extensions do exist. The rest of the proof then goes through as before.

The fact that $Q^\gamma \to S^2$ has a section is immediate from its construction as a blow down. It remains to prove that the classes $a \in \mathcal{C}_M$ extend. The obstruction to extending the symplectic class $[\omega]$ to the total space of a symplectic bundle $(M,\omega) \to Q \to S^2$ is $\mathrm{Flux}^{[\omega]}([\phi])$, where the loop $\phi_t, t \in S^1$, in $\mathrm{Symp}(M,\omega)$ is the clutching function of the bundle $Q = Q_\phi \to S^2$. Therefore the extension exists iff $\mathrm{Flux}^{[\omega]}([\phi]) = 0$. By our assumption on M, this is equivalent to requiring that the volume flux

$$[\omega] \wedge \mathrm{Flux}^{[\omega]}([\phi]) =: \mathrm{Flux}^{[\omega]^2/2}([\phi]) \in H^3(M;\mathbb{R})$$

vanishes. The latter class is well known to be Poincaré dual to the image $\mathrm{ev}([\phi])$ of $[\phi]$ under the homology evaluation map $\mathrm{ev}_* : \pi_1(\mathrm{Diff}(M)) \to H_1(M)$. But $Q_\phi \to S^2$ has a section iff the loop $t \mapsto \mathrm{ev}(\phi_t) := \phi_t(x_0)$ is contractible in M. Therefore, if there is a section $\mathrm{ev}_*([\phi]) = 0$. This shows that the symplectic class $[\omega^\gamma]$ extends to each Q^γ. Then every class $a \in \mathcal{C}_M$ has such an extension by [13]. □

Example 6.13. Taubes showed that every symplectic form on the 4-torus $\mathbb{T}^4$ gives rise to an almost complex structure with $c_1 = 0$. Using this, or the fact that $\mathbb{T}^4$ has a unique Seiberg–Witten basic class (namely $K = 0$), we can conclude that, when $M = \mathbb{T}^4$ and $k > 0$, $\pi_2(B\mathrm{Diff}\,\widetilde{M}_k)$ is not generated by the images of $\pi_2\big(B\mathrm{Symp}(\widetilde{M}_k, \widetilde{\omega})\big)$ as $\widetilde{\omega}$ ranges over all symplectic forms on $\widetilde{M}_k$.

Remark 6.14. We have concentrated in this section on results on $\pi_2(B\mathrm{Diff})$ since, by Lemma 6.1, Proposition 6.4 is valid for quite general choices of $[\omega]$ and ϵ when the base is 2-dimensional. However, if $[\omega]$ is integral, there are special values for ϵ for which the blow up classes E_i never degenerate. For example, one can either take the ϵ_i all equal to $1/N$, or, given positive integers N_i one could take $\epsilon_1 = 1/N_1, \epsilon_2 = \epsilon_1/N_2, \dots, \epsilon_k = \epsilon_{k-1}/N_k$. For such ϵ every symplectic bundle $(\widetilde{M}_k, \widetilde{\omega}_k) \to \widetilde{P} \to B$ is a k-fold blow up. This can be proved either by adapting the proof of Proposition 6.4 or by noting that the corresponding spaces of (unparametrized) exceptional divisors $\mathcal{E}_i/U$ are all contractible. (Here $\mathcal{E}_i$ is the space of symplectically embedded E_i-spheres in $\widetilde{M}_{i-1}$; cf. the discussion at the beginning of §2.) In these cases our methods would give results about the higher homotopy and homology of $B\mathrm{Symp}\,\widetilde{M}_k$ and $B\mathrm{Diff}\,\widetilde{M}_k$.

References

[1] M. Abreu and D. McDuff, Topology of symplectomorphism groups of rational ruled surfaces, SG/9910057, *Journ. of Amer. Math. Soc.* **13**, (2000) 971–1009.

[2] S. Baldridge, Seiberg–Witten vanishing theorem for S^1-manifolds with fixed points, GT/0201034, *Pac. J. Math.* **217**, (2004) 1–10..

[3] A. Blanchard, Sur les variétés analytiques complexes, *Ann. Sci. Ec. Norm. Sup.* , **(3) 73**, (1956), 157–202.

[4] O. Buse, Relative family Gromov–Witten invariants and symplectomorphisms, SG/01110313, *Pac. J. Math.* **218** (2005), 315–341.

[5] O. Buse, Whitehead products in symplectomorphism groups and Gromov-Witten invariants, SG/0411108.

[6] D. Gottlieb, A certain subgroup of the fundamental group, *Amer. J. Math* **87** (1965), 840-856.

[7] D. Gottlieb, Self coincidence numbers and the fundamental group, AT/0702236.

[8] J. Kędra, Evaluation fibrations and topology of symplectomorphisms, *Proc. Amer. Math. Soc* **133** (2005), 305-312.

[9] J. Kędra, Fundamental group of $\mathrm{Symp}(M,\omega)$ with no circle action, SG/0502210.

[10] J. Kędra and D. McDuff, Homotopy properties of Hamiltonian group actions, *Geometry and Topology*, **9** (2005) 121–162.

[11] F. Lalonde and D McDuff, The classification of ruled symplectic 4-manifolds, *Math. Research Letters* **3**, (1996), 769–778.

[12] F. Lalonde and D. McDuff, Symplectic structures on fiber bundles, SG/0010275, *Topology* **42** (2003), 309–347. Erratum *Topology* **44** (2005), 1301-1303.

[13] F. Lalonde, D. McDuff and L. Polterovich, Topological rigidity of Hamiltonian loops and quantum homology, *Invent. Math* **135** (1999), 369–385.

[14] F. Lalonde and M. Pinsonnault, The topology of the space of symplectic balls in rational 4-manifolds, SG/0207096, *Duke Math. J.* **122** (2004), 347–397.

[15] H. V. Le and K. Ono, Parameterized Gromov–Witten invariants and topology of symplectomorphism groups, preprint #28, MPIM Leipzig (2001), revised version SG/0704.3899.

[16] E. Lerman, Symplectic Cuts, *Mathematical Research Letters* **2** (1995), 247-58.

[17] Bang-He Li, Representing nonnegative homology classes of $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ by minimal genus smooth embeddings, *Trans. Amer. Math. Soc.* **352** (1999), 4155–4169.

[18] Jun Li and Gang Tian, Virtual moduli cycles and Gromov–Witten invariants for general symplectic manifolds, *Topics in Symplectic 4-manifolds (Irvine CA 1996),* Internat. Press, Cambridge, MA (1998), pp 47–83.

[19] Tian-Jun Li, Smoothly embedded spheres in symplectic 4-manifolds, *Proc. Amer. Math. Soc* **127** (1999), 609–613.

[20] D. McDuff, The structure of rational and ruled symplectic 4-manifolds, *Journ. Amer. Math. Soc.* **3** (1990), 679–712; Erratum: *Journ. Amer. Math. Soc* **5** (1992), 987–988.

[21] D. McDuff, From symplectic deformation to isotopy, *Topics in Symplectic 4-manifolds (Irvine CA 1996),* Internat. Press, Cambridge, MA (1998), pp 85-99.

[22] D. McDuff, Enlarging the Hamiltonian group, SG/0503268, *Journal of Symplectic Geometry* **3** (2005), 481–530.

[23] D. McDuff and D.A. Salamon, *J-holomorphic curves and symplectic topology.* Colloquium Publications **52**, American Mathematical Society, Providence, RI, (2004).

[24] M. Pinsonnault, Symplectomorphism groups and embeddings of balls into rational ruled surfaces, SG/0603310,

[25] Yongbin Ruan, Symplectic topology and extremal rays, Geom and Funkt Anal, 3, (1993) 395–430.

[26] P. Seidel, On the group of symplectic automorphisms of $\mathbb{C}P^m\times\mathbb{C}P^n$, *Amer. Math. Soc. Transl.* (2) **196** (1999), 237–250.

[27] P. Seidel, Lectures on four dimensional Dehn twists, SG/0309012.

[28] C. H. Taubes, The Seiberg–Witten invariants and symplectic forms, *Math. Res. Letters* **1** (1994), 809–822

[29] A. Zinger, Pseudocycles and Integral Homology, AT/0605535.

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
E-mail address: dusa@math.sunysb.edu