Math 539 Homework 5

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Given a map $f : X \to Y$ the **mapping path space** E_f is

$$E_f = \{(x, \tilde{y}) \in X \times M(I, Y) \ : \ \tilde{y}(1) = x\}.$$

There is a map $\pi : E_f \to Y$ where $(x, \tilde{y}) \mapsto \tilde{y}(0)$ and the **homotopy fiber** $P_f := \pi^{-1}(y_0)$.

A map $p : E \to B$ has the HLP if for every $f : Z \to E$ and $F : Z \times I \to B$ such that $p \circ f = F(\cdot, 0)$ there is a lift $\widehat{F} : Z \times I \to E$ such that $p \circ \widehat{F} = F$ and $\widehat{F}(\cdot, 0) = f$.

Problem 1: Comparing fiber and homotopy fiber (i) Suppose that Y is path connected. Show that the homotopy type of the homotopy fiber P_f is independent of the choice of point y_0.

(ii) Suppose that B is path connected and $p : E \to B$ has the HLP. Fix $b_0 \in B$ and set $F := p^{-1}(b_0)$, $P_p := \pi^{-1}(b_0)$, where $\pi : E_p \to B$ as above. Define a map $F \to P_p$ and show that it is a homotopy equivalence.

(iii) (i) and (ii) imply that all the fibers $F_b := p^{-1}(b)$ of p are homotopy equivalent. Prove this directly using the HLP.

Problem 2 Examples (i) Show that any projection map $p : F \times B \to B$ has the HLP.

(ii) Let $E = \{(a, b) \in [0, 2] \times (0, 2) \ : b \le 1 \text{ if } a \le 1\}$, and let $p : E \to [0, 2]$ be the projection $(a, b) \mapsto a$. Show that $p : E \to A$ has the HLP.

(iii) Let $E' := E \cup (\{1\} \times (0, 2))$. Show that projection $p : E' \to A$ does NOT have the HLP, even though its fibers are all homotopy equivalent.

Problem 3 Pullbacks Let $p' : g^*(E) \to A$ be the pullback of $p : E \to B$ by the map $g : A \to B$. So

$$g^*(E) = \{(a, e) : g(a) = p(e)\}.$$

Show that $p' : g^*(E) \to A$ has the HLP if $p : E \to B$ does.

Problem 4 Barrett–Puppe fibration sequence Suppose that X and Y are path connected. In class I sketched a proof that for any based map $f : X \to Y$ with homotopy fiber P_f the homotopy fiber P_q of the map $q : P_f \to X$ is homotopy equivalent to the based loop space ΩY. (i) gives another proof.

(i) Show that $q : P_f \to X$ has the HLP. Use Problem 1(ii) to conclude that there is a homotopy equivalence $\Omega Y \to P_q$. Describe the induced map $r : \Omega Y \to P_f$

It follows that any W there is a long exact sequence

$$\cdots \to [W, \Omega X]_* \to [W, \Omega Y]_* \to [W, P_f]_* \to [W, X]_* \to [W, Y]_*.$$

Therefore, taking $W = S^n, n \ge 0$, we get the long exact sequence

$$\ldots \pi_{n+1}(X) \to \pi_{n+1}(Y) \to \pi_n(P_f) \to \pi_n(X) \to \pi_n(Y) \to \ldots.$$

Here all the maps except for $\pi_{n+1}(Y) \to \pi_n(F)$ are are the obvious ones, induced by the maps $q : P_f \to X$ and $f : X \to Y$.

(ii) Suppose that $f : X \to Y$ has the HLP. Then by Problem 1 we can replace P_f by the fiber F, so that there is a map $\delta : \pi_{n+1}(Y) \to \pi_n(F)$. Work out a nice description of this map using the commutative diagram

$$\begin{array}{ccc} \pi_{n+1}(Y) & \xrightarrow{\delta} & \pi_n(F) \\ \downarrow & & \downarrow \\ \pi_n(\Omega Y) & \xrightarrow{r} & \pi_n(P_f), \end{array}$$

where r is as in (i).