

## Project 3

(due by 12/14/06 – 5:00pm)

*This project concerns some simple applications of Linear Algebra to Fibonacci Numbers. Discuss the problem below in a concise and precise essay at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.*

Let  $a_n$  denote the basic sequence of Fibonacci numbers defined by the recursive relation

$$a_{n+2} = a_n + a_{n+1}, \quad a_0 = 1, a_1 = 1.$$

Work out the problems below and find an explicit formula for  $a_n$ .

**1.** Consider an operator  $T$  on the vector space  $\mathbb{R}^2$  such that  $T$  maps the vector  $(x, y)$  to the vector  $(y, x + y)$ . Show that  $T$  maps  $(a_{n-2}, a_{n-1})$  to  $(a_{n-1}, a_n)$ , where  $a_n$  is the Fibonacci sequence. Write the matrix  $A$  of  $T$  in the standard basis and prove that

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

**2.** Diagonalize the operator  $T$  by finding its eigenvalues and eigenvectors. Show that the eigenvectors  $v_1, v_2$  form a basis in  $\mathbb{R}^2$ . If  $B$  denotes the diagonal matrix of  $T$  with respect to this basis, verify that  $A = P^{-1}BP$ , where the columns of the transition matrix  $P$  are  $v_1, v_2$  in terms of the coordinates with respect to the standard basis.

**3.** Find  $B^n$  and conclude that  $A^n = P^{-1}B^nP$  from  $A = P^{-1}BP$ . Now easily find  $A^n$  and write an explicit formula for the  $n$ -th Fibonacci number.

**4.** Suppose that the numbers  $b_n, n \geq 0$ , are defined by the same recursive relation, but with  $b_0 = 1, b_2 = 3$ . Thus  $b_2 = 4, b_3 = 7 \dots$ . Find an explicit formula for  $b_n$ . Check your answer for  $b_{10}$  (using a calculator).