

Math 310: Review for Midterm II, Nov 16

The exam will concentrate on Ch 5 and 6, except for p 91-93, and p 113-117 (about orthogonal projections and minimization problems). Obviously you also need to know how to use most of the concepts of Ch 1-3, but there will be no specific questions on this material. You are expected to know the following definitions, know some relevant examples, and to be able to do simple calculations using them. You are also expected to know the statements of the main theorems so that you can quote them in your answers. I will NOT distribute a list since there aren't so many this time.

invariant subspace for operator T ;

eigenvalue, eigenvector for T ;

upper triangular matrix;

inner product; orthogonal vectors, orthonormal basis;

orthogonal complement; orthogonal projection of a vector onto a subspace

adjoint T^* of $T \in \mathcal{L}(V, W)$.

Most important results:

existence and basic properties of eigenvalues/vectors: (5.6), (5.10), (5.13), (5.16), (5.20)

the properties of inner product: (6.3), (6.6), (6.9), and Gram-Schmidt. Orthogonal projection of a vector on a subspace. Elementary properties of adjoint.

Here are some sample problems. Note also that on the syllabus I suggested some problems from the book.

Ex 1: (i) Let V be a vector space over \mathbb{C} with subspace U . What does it mean to say that U is invariant under $T \in \mathcal{L}(V)$?

(ii) Let $V = \mathbb{C}^\infty$ and $T : V \rightarrow V$ be the right shift: $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Find a T -invariant subspace U that is NOT equal to $\{0\}$ or V .

(iii) What are the eigenvalues and eigenvectors of T ?

Ex 2: Let

$$A := \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad B := \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $T_A, T_B \in \mathcal{L}(\mathbb{R}^3)$ be the linear maps given by multiplication by the matrices A, B .

(i) Questions on T_A :

(1) Find all eigenvalues and eigenvectors for T_A .

(2) Find TWO different bases $\mathcal{B} := (v_1, v_2, v_3)$ of \mathbb{R}^3 such that the matrix $\mathcal{M}(T)$ representing T with respect to \mathcal{B} is upper triangular. (Here $\mathcal{B} := (v_1, v_2, v_3)$ is DIFFERENT from $\mathcal{B}' := (v'_1, v'_2, v'_3)$ if there do not exist scalars a_1, a_2, a_3 so that $v'_i = a_i v_i$ for all i .)

(3) How much choice do you have in choosing such a basis \mathcal{B} ? (I wouldn't ask this in an exam ... but it is good to help you understand.)

(ii) Questions on T_B :

- (1) Find a subspace U of \mathbb{R}^3 that is not $\{0\}$ or \mathbb{R}^3 and is invariant by T_B . **Hint:** look at the zero entries of B – what does B do to the elements of the standard basis??
- (2) Find the eigenvalues and eigenvectors for $T_B : U \rightarrow U$.
- (3) Find all eigenvalues and eigenvectors for $T_B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.
- (4) Find an invertible matrix Q such that $Q^{-1}AQ$ is diagonal.

Ex 3: (slightly revised) This exercise asks you to find an eigenvector for $T : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ using the method in the proof of (5.10). Let $T = T_A$ where

$$A := \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & -6 \\ 1 & -2 & 1 \end{bmatrix}.$$

- (1) Let $v = (1, 0, 0)$. Calculate Tv, T^2v, T^3v .
- (2) Find $a, b, c, d \in \mathbb{C}$ so that $(aT^3 + bT^2 + cT + dI)v = 0$.
- (3) Find an eigenvector and eigenvalue for T . Check your answer. **Note:** the calculations here should not be too hard. If they are, you've made an arithmetic error. Also: in fact you should be able to find at least two eigenvalues and eigenvectors from this calculation.

Ex 4: (i) Let U be a subspace of an inner product space (V, \langle, \rangle) . What is its orthogonal complement U^\perp ? Given any subspaces U, W of V show that $U \subseteq W$ iff $W^\perp \subseteq U^\perp$.

(ii) Let U be the subspace of \mathbb{C}^4 given by the equations $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 - x_2 + 2x_3 + x_4 = 0$.

- (1) Find a basis for U^\perp . **Note:** this is an easy question that requires NO calculation! You just need to understand what the equations for U tell you. But this subspace is the one considered in Ex1 on HW8, so all calculations are done anyway.
- (2) Find an orthonormal basis for U^\perp .
- (3) Let $w = (1, 2, 3, 0)$. Write $w = u + v$ where $u \in U$ and $v \in U^\perp$.
- (4) Check that $\|w\|^2 = \|u\|^2 + \|v\|^2$.

Ex 5: (i) Let (V, \langle, \rangle) and (W, \langle, \rangle) be finite dimensional inner product spaces. Let $T \in \mathcal{L}(V, W)$. Define the adjoint T^* of T .

(ii) Define $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $T(z, w) = (iz - w, 2z + (1 + i)w)$. Use the definition in (i) to calculate T^* .

(iii) Now go back to the general case with V, W, T as in (i).

- (1) Show that $(aT)^* = \bar{a}T^*$.
- (2) Show that $\text{range } T^* = (\text{null } T)^\perp$. Note: it is easy to show that $\text{range } T^* \subseteq (\text{null } T)^\perp$ (DO THIS), but not so easy to give a direct argument why $(\text{null } T)^\perp \subseteq \text{range } T^*$. But note that $U \subseteq W$ iff $W^\perp \subseteq U^\perp$. Therefore instead of proving $(\text{null } T)^\perp \subseteq \text{range } T^*$ it is enough to show that $(\text{range } T^*)^\perp \subseteq \text{null } T$. For some (slightly mysterious) reason, this is easier! (And you note that this what the book does in Prop 6.46.)
- (3) Show that $\text{range } T$ and $\text{range } T^*$ have the same dimension. Give an example where $\text{null } T$ and $\text{null } T^*$ do NOT have the same dimension.