Math 310: Review for Midterm I, Oct 4

You are expected to know the following definitions, know some relevant examples, and to be able to do simple calculations using them.

subspace; sum of two subspaces $U + W$; direct sum $U \oplus W$;

linear (in)dependence of a list $(v_1, \ldots, v_n)$; span of a list $(v_1, \ldots, v_n)$;

finite and infinite dimensional vector spaces;

basis and dimension of a finite dimensional vector space;

linear map $T : V \to W$; null space and range of T; surjective, injective, invertible;

the matrix $\mathcal{M}(T)$ of a linear map $T : V \to W$ with respect to given bases of V and W.

In the midterm I will ask you to give some definitions. I will also ask you to prove some statements. In your answers you may quote any theorem from the list, unless you are given instructions not to.

Here are some sample problems.

Ex 1: Let V be a finite dimensional space.
(i) Define $\dim V$.
(ii) Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. Give a careful proof that $v_1, \ldots, v_n$ is a basis for V.

You may use any theorem on the list. Any other statement must be proved.

Ex 2: (i) Find a basis for the subspace

$$V := \{(x_1, \ldots, x_4) \in \mathbb{F}^4 : x_1 - x_2 + x_4 = 0, x_2 = x_3\}.$$

(ii) Extend this to a basis for $\mathbb{F}^4$.
(iii) Define a linear map $T : \mathbb{F}^4 \to \mathbb{F}^3$ with null $T = V$. Justify all your claims.

Ex 3: (i) Define an infinite dimensional space.
(ii) Give an example of an infinite dimensional space V and a linear map $T : V \to V$ that is

(1) injective but not surjective
(2) surjective but not injective.

(iii) Do such linear maps exist when V is finite dimensional? Give examples or a careful proof that such examples cannot exist.

Ex 4: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $T(x, y) = (x, 3x + 2y)$. Let $\mathcal{B} = (v_1, v_2)$ be the basis with $v_1 = (1, -3), v_2 = (0, 1)$. Find the matrix $\mathcal{M}(T)$ that represents T with respect to this basis.

Ex 5: Let $V = \mathcal{P}_3(\mathbb{R})$ the vector space of real polynomials with degree ≤ 3. Let $\mathcal{B}$ be the basis:

$$f_1 = 1, \quad f_2 = z, \quad f_3 = z^2, \quad f_4 = z^3.$$

Define $T : V \to V$ by $T(f) = zf'(z)$, where f' denotes the derivative of f.
(i) Show that T is linear.

(ii) Give an example of a map $S : V \to V$ that is NOT linear, explaining your answer.

(iii) Find the matrix $\mathcal{M}(T)$ that represents T with respect to the basis $\mathcal{B}$.

Note: I added an extra one of these for practice; there certainly will NOT be more than one question on the exam about this.

Ex 6: Let V be as in Ex 5.

(i) What is a subspace of the vector space V?.

(ii) Show that the subset $W := \{f \in V : f(1) = 0\}$ is a subspace of V.

(iii) Find a basis for W.

(iv) Let $g = 1 - z + z^2 - z^3$. Write g as a linear combination of the elements of the basis that you found in (iii).

Ex 7: (i) What does it mean to say that the list $v_1, \ldots, v_n$ spans the vector space V?

(ii) Suppose that the list v_1, v_2, v_3 spans V. Show that the list $v_1, v_2 + v_1, v_3 + v_1$ also spans V.

(iii) Show that your argument in (i) fails with the list $v_1 + v_2, v_2 + v_3, v_3 - v_1$.

(iv) Is it possible to find 3 vectors v_1, v_2, v_3 that span $\mathbb{R}^2$ but are such that $v_1 + v_2, v_2 + v_3, v_3 - v_1$ do not?

Note: I wouldn't put a question like (iv) on the exam because it is too open ended. But it is a good review question.

Ex 8: (i) What does it mean to say that the list $v_1, \ldots, v_n$ is linearly dependent?

(ii) Use this definition to prove (2.4):

If $(v_1, \ldots, v_m)$ *is linearly dependent in* V *and* $v_1 \neq 0$ *then there exists* $j \in \{2, \ldots, m\}$ *such that the following hold:*

(1) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(2) *if the* j*th term is removed from* $(v_1, \ldots, v_m)$*, the span of the remaining list equals* $\operatorname{span}(v_1, \ldots, v_m)$.

(iii) Give a list $v_1, v_2, v_3 \in \mathbb{R}^2$ that is linearly dependent, but is such that any pair of vectors from this list is linearly independent.

Ex 9: (i) Let U, W be subspaces of V. Define $U + W$.

(ii) Suppose that U, V are 2-dimensional subspaces of $\mathbb{R}^4$ such that $U \cap W = \{0\}$. Show that $U + W = \mathbb{R}^4$.

Ex 10: (i) Let $T : V \to W$ be a linear map. Define null T and range T.

(ii) Let v_1, v_2, v_3 be a basis for a vector space V and define a linear map $T : V \to \mathbb{R}^3$ by setting

$$T(v_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad T(v_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad T(v_3) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Find a basis for null (T) and a basis for range T.

(iii) Is T injective?