

## Math 310: Notes, Sept 20

Here is the Lemma that I proved in class.

**Lemma 1.** *Suppose that  $(v_1, \dots, v_n)$  is a linearly independent list of vectors in  $V$  and that  $v_{n+1} \in V$ . Then the list  $(v_1, \dots, v_{n+1})$  is linearly independent iff  $v_{n+1} \notin \text{sp}(v_1, \dots, v_n)$ .*

NOTE: “iff” means “if and only if”

We saw in class that it was hard to give a direct proof of the lemma. – How do you express that  $v_{n+1}$  is NOT in  $\text{sp}(v_1, \dots, v_n)$ ? In class, instead of proving P is equivalent to Q, I proved NOT P is equivalent to NOT Q. (Logically these are the same.) In other words, I proved

**Lemma 2.** *Suppose that  $(v_1, \dots, v_n)$  is a linearly independent list of vectors in  $V$  and that  $v_{n+1} \in V$ . Then the list  $(v_1, \dots, v_{n+1})$  is linearly dependent iff  $v_{n+1} \in \text{sp}(v_1, \dots, v_n)$ .*

**Proof** ( $\Rightarrow$ ) Suppose that  $(v_1, \dots, v_{n+1})$  is linearly dependent. Then there is a relation

$$a_1v_1 + \dots + a_{n+1}v_{n+1} = 0$$

in which the scalars  $a_1, \dots, a_{n+1}$  are not all 0.

If  $a_{n+1} = 0$  we would have a relation  $a_1v_1 + \dots + a_nv_n = 0$ . By the linear independence of  $(v_1, \dots, v_n)$  this means that  $a_i = 0$  for all  $i = 1, \dots, n$ . But the  $a_i$  do not all vanish. So this is impossible. Hence  $a_{n+1} \neq 0$ .

Therefore we can divide the above relation by  $a_{n+1}$  to get

$$v_{n+1} = -\left(\frac{a_1}{a_{n+1}}v_1 + \dots + \frac{a_n}{a_{n+1}}v_n\right) \in \text{sp}(v_1, \dots, v_n).$$

( $\Leftarrow$ ) The proof of the converse implication is left to you. (It’s easier.) You must show that if  $v_{n+1} \in \text{sp}(v_1, \dots, v_n)$  then  $(v_1, \dots, v_{n+1})$  is linearly dependent.  $\square$ .

Chapter 3 contains some very important facts about linear transformations that are not formulated into Propositions. The next result is taken from p 39-40.

**Prop 3** *Let  $(v_1, \dots, v_n)$  be a basis of  $V$  and  $w_1, \dots, w_n$  be ANY elements in  $W$ . Then there is a UNIQUE linear map  $T : V \rightarrow W$  such that*

$$T(v_i) = w_i, \quad \text{for } i = 1, \dots, n.$$

**Proof** First, let us prove that  $T$  exists. To do this, we must define  $T(v)$  for every  $v \in V$  and show that  $T$  has the two properties: **additivity**  $T(u + v) = Tu + Tv$  for all  $v, w \in V$

**homogeneity**  $T(av) = aT(v)$  for all  $a \in \mathbb{F}, v \in V$ .

*Definition of  $Tv$ :* For each  $v \in V$  there are unique scalars  $a_i$  so that  $v = a_1v_1 + \cdots + a_nv_n$ . Define

$$T(v) := a_1w_1 + \cdots + a_nw_n.$$

Since  $v_1 = 1v_1 + 0v_2 + \cdots + 0v_n$ , we find  $Tv_1 = w_1$ . Similarly,  $Tv_i = w_i$  for all  $i$ .

*Check additivity:* Suppose that  $v = a_1v_1 + \cdots + a_nv_n$  and  $w = b_1v_1 + \cdots + b_nv_n$ . Then

$$v + w = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n.$$

Therefore by the definition of  $T$ ,

$$T(v + w) = (a_1 + b_1)w_1 + \cdots + (a_n + b_n)w_n.$$

But by definition

$$Tv = a_1w_1 + \cdots + a_nw_n, \quad Tw = b_1w_1 + \cdots + b_nw_n.$$

Therefore  $T(v + w) = Tv + Tw$ .

*Check homogeneity* This is similar. Write out  $T(av)$  and compare it with  $aTv$ .

We have now defined  $T : V \rightarrow W$  with the required properties. We need to check that it is unique. But this is immediate. The properties (*Additivity*) and (*Homogeneity*) imply that for any vectors  $v_i \in V$  and scalars  $a_i \in \mathbb{F}$  we have

$$T(a_1v_1 + \cdots + a_nv_n) = a_1Tv_1 + \cdots + a_nTv_n.$$

Therefore, if we are told that  $Tv_i = w_i$  for all  $i$ ,  $T(a_1v_1 + \cdots + a_nv_n)$  must equal  $a_1w_1 + \cdots + a_nw_n$ . Therefore if  $v = a_1v_1 + \cdots + a_nv_n$ ,  $Tv$  must be given by the formula

$$Tv = a_1Tv_1 + \cdots + a_nTv_n.$$

Thus there is only one linear map  $T$  that satisfies the given conditions, i.e.  $T$  is unique.

**NOTE:** If  $Tv_i = w_i$  for all  $i$  as above, then the image (or range) of  $T$  is the span of the vectors  $w_1, \dots, w_n$ . Can you prove this?

**Example** Find a nonzero linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with image contained in the subspace  $W = \{(x, y, z) : x + 2z = 0\}$ . What is the range of  $T$ ? Calculate  $T(2, 3)$ .

Let  $v_1 = (1, 0), v_2 = (0, 1)$  be the standard basis of  $\mathbb{R}^2$ . Take  $w_1 = w_2 = w = (2, 0, -1) \in W$ . Consider the map  $T$  such that  $T(1, 0) = (2, 0, -1) \in W$  and  $T(0, 1) = (2, 0, -1) \in W$ . Then  $T(a_1v_1 + a_2v_2) = a_1w + a_2w = (a_1 + a_2)w$ . In particular  $T(2, 3) = 5w = (10, 0, -5)$ . Note that  $Tv \in W$  for all  $v \in \mathbb{R}^2$ , as we would expect. But the vectors  $Tv$  do not span  $W$ . Rather they span the line  $\{aw : a \in \mathbb{R}\}$ . This line is the span of  $w_1, w_2$  in this case (since  $w = w_1 = w_2$ ). Hence this agrees with the NOTE above.