Math 310: Homework 7 (revised)

due Nov 1,2 2006 in recitation

In this homework V is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. I corrected a typo in Ex 5.

Ex 1: *A question about invertibility.* (i) Recall that $T \in \mathcal{L}(V)$ is said to be invertible if there is $S \in \mathcal{L}(V)$ such that $ST = TS = I$. In fact, it is enough to assume that there is $R \in \mathcal{L}(V)$ satisfying just the first identity: $RT = I$. Prove this, explaining each step of the argument.

(ii) Prove that if $S, T \in \mathcal{L}(V)$ satisfy $ST = I$ then $TS = I$.

(iii) Deduce carefully from (ii) that if A, B are any $n \times n$ matrices over $\mathbb{F}$ such that $AB = I$ then $BA = I$.

Note: This fact always seems to me to be to be surprising. It is an elementary computational fact, but I don't see a way to prove it by a simple calculation.

Ex 2: Prove that if $\mathbf{x}, \mathbf{y}$ are nonzero vectors in $\mathbb{R}^2$ then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. (Think of $\mathbf{x}, \mathbf{y}$ as the sides OX, OY of a triangle, and use the law of cosines.)

Ex 3: (i) Let $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show that for any vector $\mathbf{x} \in \mathbb{R}^2$, $\|A_\theta \mathbf{x}\| = \|\mathbf{x}\|$ and the angle between $\mathbf{x}$ and $A_\theta(\mathbf{x})$ is θ. Thus this matrix A_θ represents the rotation through angle θ. It is a simple example of an *orthogonal* matrix.
(ii) Verify that $A_\theta A_\phi = A_{\theta+\phi}$.
(iii) Find the (complex) eigenvalues and eigenvectors of A_θ. (Here you must think of the linear transformation $v \mapsto A_\theta v$ as an element of $\mathcal{L}(\mathbb{C}^2)$.)

Ex 4: (i) Let $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Show that for any vector $\mathbf{x} \in \mathbb{R}^2$, $\|R_\theta \mathbf{x}\| = \|\mathbf{x}\|$. Find the (real) eigenvectors and eigenvalues for R_θ and describe the linear map $\mathbf{x} \mapsto R_\theta \mathbf{x}$ geometrically.
(ii) Suppose that B is any 2×2 real matrix such that $\|B\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$. Show that B either equals A_θ or R_θ for some θ.

Ex 5: (i) Define $\|(x_1, x_2)\| := |x_1| + |x_2|$ for $(x_1, x_2) \in \mathbb{R}^2$. Check that with this definition the triangle inequality holds, i.e. $\|(x_1+y_1, x_2+y_2)\| \le \|(x_1, x_2)\| + \|(y_1, y_2)\|$.
(ii) Check that the parallelogram rule given as (6.14) in the book does NOT hold.
(iii) Deduce that there is no inner product on $\mathbb{R}^2$ for which this is the associated norm. (Note: Since $\|\cdot\|$ is positive and homogeneous, it satisfies the axioms for a *norm*. Ex 8 on p 123 – which relies on Ex 6 and Ex 7– shows that if the parallelogram rule holds for some norm then it does come from an inner product.)

Ex 6: (i) Given an example of a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ that has only one real eigenvalue. (**Hint:** Use Ex 3.)

(Bonus question). (ii) Give the shortest proof you can that *every* linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ has at least one real eigenvalue. (You can try to adapt the proof of (5.26) to the 3-dimensional case.)