Math 310: Homework 4 (revised)

due Oct 4,5 2006 in recitation

NOTE: Ex 3 (ii) is changed, and I added a hint to Ex 2.

Ex 1. (i) Let $L: V \to W$ be a linear map. Let w_0 be an element of W. Let v_0 be an element of V such that $L(v_0) = w_0$. Show that any solution of the equation $L(X) = w_0$ is of type $v_0 + u$, where u is an element of the kernel of L.
Hint: You might find it easier to do (ii) and (iii) before (i)!

(ii) Consider the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Find a linear map $L : V \to W$ and element $w_0 \in W$ such that the solution set of this system of equations can be identified with the set of vectors v such that $Lv = w_0$.

(iii) Solve the equations in (ii) and express the solutions in the form $v_0 + u$ as explained in (i).

Ex 2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Define the *trace* of A to be the sum of the diagonal elements, that is

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

(1) What is the dimension of the space of $n \times n$ traceless matrices (i.e., $tr(A) = 0$)?
(2) Show that the trace is a linear map of the space of $n \times n$ matrices into $\mathbb{F}$.
(3) If A, B are $n \times n$ matrices, show that $tr(AB) = tr(BA)$.
(4) Prove that there are no matrices A, B such that

$$AB - BA = I_n$$

.

Hint: Part (3) is an exercise in using the double summation notation: if $A = (a_{ij})$ and $B = (b_{ij})$ then $AB = (c_{ik})$ where $c_{ik} = \sum_j a_{ij} b_{jk}$. If you think of c_{ik} as the dot product of the ith row of A with the kth col of B it is not so easy to see why the trace has this symmetry.

Ex 3. (i) Find the matrix of a nonzero linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L^2 = 0$.

(ii) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $L \neq 0$ but $L^2 = 0$. What are the dimensions of Null$L$ and Range$L$? Is there any relation between these two spaces? (Understanding this will help with the bonus question.)

(iii) Let $L: V \to V$ be a linear mapping such that $L^2 = 0$. Show that $I - L$ is invertible. (I is the identity mapping on V.)

Hint: Show that Null $(I - L) = \{0\}$. (There is another proof that finds an algebraic formula for the inverse. This argument works also when V is infinite dimensional.)

Ex 4. Let $V = \mathbb{R}^3$ with basis $\mathcal{B} := (v_1, v_2, v_3)$ where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The basis $\mathcal{B}$ determines a unique isomorphism $\mathcal{M} : V \to \mathbb{R}^3$ such that $\mathcal{M}(v_1) = e_1, \mathcal{M}(v_2) = e_2$, and $\mathcal{M}(v_3) = e_3$.

(i) Calculate $\mathcal{M}(v)$ for $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the matrix

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Calculate $\mathcal{M}(T)$ where $\mathcal{M} := \mathcal{M}(A, (v_1, v_2, v_3), (v_1, v_2, v_3))$. (cf pp 48–53: I will lecture on this next Tuesday)

Hint: You should be able to use your answer to (i) when doing this.

(iii) Calculate Tv, where v is as in (i) and T is as in (ii). Also calculate $\mathcal{M}(Tv)$.

(iv) Check that $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$.

Bonus question 5. (i) Find a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L^2 \neq 0$ but $L^3 = 0$.

(ii) Is there a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L^3 \neq 0$ but $L^4 = 0$?