Math 310: Homework 2
due in recitation on Sept 20/21

Ex 1. Prove that if the list (v_1, v_2, v_3) spans V then so does the list $(v_1 + 2v_2, v_2 - v_3, v_3)$.

Ex 2. Prove that if the list (v_1, v_2, v_3) is linearly independent in V then so is the list $(v_1 + 2v_2, v_2 - v_3, v_3)$.

Ex 3. Find a basis for the vector space

$$V = \{(x_1, \ldots, x_4) \in \mathbb{F}^4 : x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}.$$

What is the dimension of V?

Ex 4. Suppose that $(v_1, \ldots, v_n)$ is linearly independent in V.

(i) Suppose that for some $w \in V$ the list $(v_1 - w, v_2 - w, \ldots, v_n - w)$ is linearly dependent. Show that $w \in \text{span}(v_1, \ldots, v_n)$.

(ii) Is the converse true? That is, if $w \neq 0$ is in $\text{span}(v_1, \ldots, v_n)$ must it be true that the list $(v_1 - w, v_2 - w, \ldots, v_n - w)$ is linearly dependent?
Hint: What does this say when $n = 1, 2$? Try these cases first.

Ex 5. Let $\mathcal{P}(\mathbb{F})$ be the space of polynomials with coefficients in $\mathbb{F}$.

(i) Find two different 2-dimensional subspaces of $\mathcal{P}(\mathbb{F})$.

(ii) Find an infinite dimensional *proper* subspace of $\mathcal{P}(\mathbb{F})$ (i.e. a subspace that does not equal the whole of $\mathcal{P}(\mathbb{F})$.)

Ex 6. (i) Let U, V be subspaces of $\mathbb{F}^7$ such that $U \oplus V = \mathbb{F}^7$. If $\dim U = 3$ show that $\dim V = 4$.

(ii) Does this statement remain true if all you know is that $U + V = \mathbb{F}^7$? Give a proof or counterexample.

Note: In this question you may use all the results numbered up to and including 2.12. Anything else should be proved. Try to find the most econimical argument that you can.