

**Math 310: Homework 11**  
**due Dec 14 2006 in recitation or class**

**Ex 1** Let  $T_1, T_2, T_3 \in \mathcal{L}(\mathbb{C}^3)$  be the linear maps given by the following matrices:

$$A_1 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A_3 := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find their minimal and characteristic polynomials.

(ii) Let  $T \in \mathcal{L}(V)$  where  $\dim V = n$ . Suppose that  $q(t)$  is a monic polynomial of degree  $n$  such that  $q(T) = 0$ . What conditions guarantee that  $q(t)$  is the characteristic polynomial of  $T$ ? Discuss this question using the maps in (i) as examples.

**Ex 2** (i) Let  $\dim V = n$ . Suppose that for some vector  $v \in V$  the list  $v_0 := v, v_1 := Tv, \dots, v_{n-1} := T^{n-1}v$  is linearly independent. Why is there a linear relation

$$T^n v = a_0 v_0 + a_1 v_1 + \dots + a_{n-1} v_{n-1}?$$

(ii) Let  $q(t) = t^n - a_{n-1}t^{n-1} - \dots - a_1 t - a_0$ . Show that  $q(T)v_i = 0$  for all  $i$ . (Use the fact that  $v_i = T^i(v_0)$  for all  $i$ .) Hence deduce that  $q(T) = 0$ .

(iii) Show also that there is no polynomial  $m$  of degree  $< n$  such that  $m(T) = 0$ . Hence deduce that  $q(t)$  is the minimal *and* the characteristic polynomial of  $T$ . (Compare Ex 1.)

(iv) Use this method to find the characteristic polynomial of  $T_A : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  where

$$A := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(v) What are the eigenvalues and eigenvectors of  $T_A$ ? (You can use any method here.)

**Ex 3** (i) Consider the permutations on 8 letters given by the arrays

$$\mathbf{m} := [m_1, \dots, m_8] = [3, 5, 1, 6, 7, 2, 8, 4], \quad \mathbf{n} := [n_1, \dots, n_8] = [3, 8, 1, 6, 7, 2, 5, 4].$$

Calculate  $\text{sign } \mathbf{m}$  and  $\text{sign } \mathbf{n}$ . Why do you expect  $\text{sign } \mathbf{m} = -\text{sign } \mathbf{n}$ ?

(ii) A permutation  $\mathbf{m}$  is a map  $\{1, \dots, 8\} \rightarrow \{1, \dots, 8\}$  given by  $\mathbf{m}(1) = m_1, \dots, \mathbf{m}(8) = m_8$ . To emphasize this one can describe  $\mathbf{m}$  by two rows:

$$\mathbf{m} := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 6 & 7 & 2 & 8 & 4 \end{bmatrix}.$$

Another way of describing a permutation is in terms of cycles  $(d_1, d_2, \dots, d_r)$ : this is the map that takes  $d_1$  to  $d_2$ ,  $d_2$  to  $d_3$  and so on, finally taking  $d_r$  back to  $d_1$ . In this notation, when one writes a product one does the right hand one first. Thus:

$$(134)(24) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}.$$

Note that  $(134)(24) = (1342)$ .

(a) Write down  $(15)(245)(34)$  in the double row format.

(b) Hence write  $(15)(245)(34)$  as a product of *disjoint* cycles (ie so that no number occurs in more than one cycle.)

(c) Repeat the above two steps with  $(23)(245)(34)$

(d) Write down the permutation  $\mathbf{m}$  above as a product of disjoint cycles.

(iii) Use (a) (b) above to write down  $(15)(245)(34)$  as a product of transpositions in two different ways. Check that the number of transpositions has the same parity in both cases. (A transposition is a cycle of length 2. We saw in class that any permutation can be written as a product of transpositions. For example  $(1234) = (43)(42)(41)$ .)

(iv) Use (d) to write  $\mathbf{m}$  as a product of  $k$  transpositions. Check that  $(-1)^k = \text{sign } \mathbf{m}$ .

**Ex 4** Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Recall that

$$\det B = \sum_{\mathbf{m} \in \text{perm}_5} \text{sign } \mathbf{m} b_{m_1,1} b_{m_2,2} b_{m_3,3} b_{m_4,4} b_{m_5,5}.$$

Since  $\mathbf{m}$  is a permutation, each term in this sum contains just one element from each row and each column of  $B$ . For  $B$  as above, which permutations give you nontrivial terms in this sum? (Hint: since for example there is just one element in the first row which lies in position  $b_{12}$  we must have  $m_2 = 1$ .) List all these permutations, and hence calculate  $\det B$ .

**Ex 5** (i) The trace  $\text{tr}A$  of a matrix  $A$  is the sum of its diagonal entries. Explain why  $\text{tr}A$  is the sum of the eigenvalues of the linear map  $T_A$ .

(ii) Suppose that  $A$  is a complex  $n \times n$  matrix such that  $A^2 = A$ . Show that  $\text{tr}A$  is a nonnegative integer. (Hint: What can you say about the eigenvalues of  $A$ ?)

**Ex 6** (Bonus) Suppose that in the situation of Ex 1 the span of the  $T^i v$ ,  $i = 0, \dots, n-1$ , has dimension  $< n$ . You can still find a polynomial  $f(t)$  such that  $f(T)(T^i v) = 0$  for all  $i$ . What can you say about its roots? What relation does this have to the minimal or characteristic polynomial? You could experiment starting with the matrix

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$