Math 310: Homework 10 (revised)

due Dec 6,7 2006 in recitation

Ex 1 (a review question) (i) Define a linearly independent list $(v_1, \dots, v_n)$.

(ii) Using this definition, show that if the lists $(v_1, \dots, v_m, u_1, \dots, u_k)$ and $(v_1, \dots, v_m, w_1 \dots, w_\ell)$ are linearly independent and if also

$$span(u_1, \dots, u_k, v_1, \dots, v_m) \cap span(w_1, \dots, w_\ell) = \{0\}$$

then $(v_1, \dots, v_m, u_1, \dots, u_k, w_1 \dots, w_\ell)$ is also linearly independent.

(iii) Deduce from (ii) that for any subspaces U, W of a finite dimensional vector space V,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Ex 2. (i) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation $T(x, y, z) = (2x + y, 3y + z, 2z)$. Write down the matrix that represents T with respect to the standard basis.

(ii) Find bases $\mathcal{B}_1 := v_1, v_2, v_3$ and $\mathcal{B}_2 := w_1, w_2, w_3$ such that the matrices $\mathcal{M}(T, \mathcal{B}_i)$ that represent T with respect to these bases are:

$$\mathcal{M}(T, \mathcal{B}_1) := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathcal{M}(T, \mathcal{B}_2) := \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(iii) Is there a basis $\mathcal{B}_3$ such that $\mathcal{M}(T, \mathcal{B}_3) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$?

Ex 3 (i) Let $\mathcal{B} = v_1, \dots, v_4$ be a basis for V and $T \in \mathcal{L}(V)$. Suppose that $\mathcal{M}(T, \mathcal{B}) = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Let $R = T - I$ and $S = T - 2I$. Write down $\mathcal{M}(R, \mathcal{B})$ and $\mathcal{M}(S, \mathcal{B})$.

(ii) If $U = span(v_1, v_2, v_3)$ then the proof of (8.10) in the book implies that $\operatorname{Null} R^4 \subseteq U$ and $\operatorname{Null} S^4 \subsetneq U$. Why? Prove these statements by calculating $\operatorname{Null} R^4$ and $\operatorname{Null} S^4$. **Hint:** What are the dimensions of these spaces? The calculation will be easier if you remember that if $\dim \operatorname{Null} R^4 = k < 4$ then $\dim \operatorname{Null} R^4 = \dim \operatorname{Null} R^k$.

(iii) Find a basis for $\operatorname{Null} R^4$ and $\operatorname{Null} S^4$. You should get 4 vectors in all that form a basis for V. Which theorem in the book does this follow from?

(iv) Let $k = \dim \operatorname{Null} R^4$. Then (8.5) and the proof of (8.9) imply that $\operatorname{Range} R^k = \operatorname{Range} R^{k+1}$. Calculate these two spaces and check this. Do you notice anything about this space (e.g. is it equal to any other space you have recently calculated?)

Ex 4 (i) Let $A := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. Find the eigenvalues of A (using any convenient method) and hence the characteristic polynomial $P(z)$ of A.

(ii) Calculate A^2 and check that $p(A) = 0$.

Ex 5 Let $A := \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$. Calculate A^2 and A^3. Find a linear relation between A^3, A^2, A and I. Hence find a polynomial $p(z)$ such that $p(A) = 0$.
Hint: look at $A^3 + 2A$.

(ii) If $p(z)$ is the characteristic polynomial of A, what does that tell you about the eigenvalues of A? Find these eigenvalues by some other method and compare answers.

(iii) (Bonus) Why must $p(z)$ be the characteristic polynomial of A?