Math 310: Review for the Final, Dec 19, 5–7:30

Final version

The exam will be cumulative. You are expected to know all the previous definitions and also:

multiplicity of an eigenvalue;

generalized eigenvalues;

the characteristic and minimal polynomials;

the definition of trace and determinant of a linear map;

the formula for the determinant of an $n \times n$ matrix (using permutations).

the formula for the characteristic polynomial in terms of the determinant (see Ex 1 below, and the book Thm (10.17).)

All this JUST WHEN $\mathbb{F} = \mathbb{C}$. I will also NOT ask anything about the adjoint; or anything from Ch 7. From Ch 8 we did not do square roots or the Jordan normal form. I will ask you to prove some easy (and short) statements. (as in Ex 2); and also to compute some examples.

The most important theorems: (8.6), (8.9), (8.18), (8.19), (8.23), (8.28), (8.36); (10.3), (10.17), (10.33).

Here are some sample problems.

Ex 1 (i) First, some unfinished business from the lecture on Tuesday. Let V be a finite dimensional vector space over $\mathbb{C}$. I did not have time in class today to point out that $T \in \mathcal{L}(V)$ has 0 as an eigenvalue iff $\det T = 0$. (This is obvious because $\det T$ is the product of the eigenvalues of T.) Using this statement show that λ is an eigenvalue for T iff $\det(\lambda I - T) = 0$. Deduce from this that the characteristic polynomial of T is $\det(zI - T) = 0$. (Show that this is a monic polynomial $q(z)$ that has the same roots as the characteristic polynomial.) This justifies one of the standard definitions of the characteristic polynomial. It also justifies a standard computational method for finding eigenvalues. (Of course I would not ask you to prove this on an exam, but it's good review...)
(ii) Use this determinantal formula to compute the eigenvalues of $T_A : \mathbb{C}^3 \to \mathbb{C}^3$, where $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$.
(iii) Find a basis $\mathcal{B} = v_1, v_2, v_3$ with respect to which T_A is represented by a diagonal matrix $\mathcal{M}(T_A, \mathcal{B})$, and calculate $\mathcal{M}(T_A, \mathcal{B})$.

Ex 2. Define a linearly independent list. Suppose that the list $v_1, \ldots, v_k$ of vectors in V is linearly independent but does NOT span V. Show that there is $v_{k+1} \in V$ such that the list $v_1, \ldots, v_k, v_{k+1}$ is linearly independent.
Note: Prove this just using the basic definitions, DO NOT quote any theorems from the book.

Ex 3. Suppose $T \in \mathcal{L}(V)$, m is a positive integer and $v \in V$ is such that $T^2v \neq 0$ but $T^3v = 0$. Show that (v, Tv, T^2v) is linearly independent. (Hint: Suppose this is false; write down a linear relation and play with it.) This generalizes; cf p 188 ex 3.)

Ex 4 (i) Give an example of an operator on $\mathbb{C}^4$ whose minimal polynomial and characteristic polynomial are equal.
(ii) Give an example of an operator on $\mathbb{C}^4$ with minimal polynomial $(z-1)^2(z-2)$ and characteristic polynomial. $(z-1)^3(z-2)$.

Ex 5 (i) Suppose V is an n-dimensional complex vector space and let $T \in \mathcal{L}(V)$. Show that $(\operatorname{null} T^n) \cap (\operatorname{range} T^n) = \{0\}$. (Use (8.6).)
(ii) Suppose that the eigenvalues of T are $\lambda_0 = 0, \lambda_1, \ldots, \lambda_k$ and write $V = V_0 \oplus \cdots \oplus V_k$ where $V_i = \operatorname{null}(T - \lambda_i I)^n$ as in Thm (8.23). Show that $V_i \subseteq \operatorname{range} T^n$ for all $i > 0$.
Hint: Show by induction on k that $\operatorname{null}(T - \lambda I)^k \subset \operatorname{range} T^n$ whenever $\lambda \neq 0$.
(iii) Deduce from (i) and (ii) that $V = \operatorname{null} T^n \oplus \operatorname{range} T^n$.
(iv) Give an example of a T with eigenvalues 0 and 3 such that $V \neq \operatorname{null} T \oplus \operatorname{range} T$.

Ex 6 (i) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, \ldots, x_5) : x_1 = 2x_4, x_2 = 6x_5\}.$$

Find a basis for U.
(ii) Find a basis for a subspace W such that $\mathbb{R}^5 = U \oplus W$. Hint: first decide what dimension W should have.)

Ex 7. Suppose that e_1, e_2 is an orthonormal list in a real inner product space V. Let $v \in V$. Show that $\|v\|^2 = |\langle v, e_1\rangle|^2 + |\langle v, e_2\rangle|^2$ iff $v \in \operatorname{span}(e_1, e_2)$.

Ex 8. Suppose U is a subspace of a real inner product space V. Show that $U^\perp = \{0\}$ iff $U = V$.
Note: in both ex 6 and ex 7 you should argue from the definition of inner product. Do not just quote results from the book. (eg if you assume that $V = U \oplus U^\perp$ then 7. is obvious, but what's the shortest argument that gives what you want, if assume you do NOT know this?)

Ex 9 Suppose that $T \in \mathcal{L}(V)$ is such that every subspace of V is invariant under T. Show that $T = cI$ for some scalar c.

Ex 10. Suppose $T \in \mathcal{L}(V)$ is such that $\operatorname{null} T \cap \operatorname{range} T = \{0\}$. What can you say about $\operatorname{range} T^2$?