Prob 1

i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a+d=1 \Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

Can we find neutral element? NO, a and $(1-a)$ cannot both be zero $\Rightarrow$ Not a subspace

ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a+d=0 \Rightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

There is a "zero" element, and closed under addition and multiplication $\longrightarrow$ a subspace

Prob 2

i) That means,

$$C = xA_1 + yA_2 + zA_3 = \begin{bmatrix} x & x-y \\ 0 & y+z \end{bmatrix}$$

ii) let,

$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} x & x-y \\ 0 & y+z \end{bmatrix} \Rightarrow x=2,\ y=-1,\ z=5$$

$$= C = (2,-1,5)_B$$

Prob 3

i) $A^T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

ii) let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$

$$TX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_3 \end{bmatrix}$$

$\|TX\| = \sqrt{x_2^2 + x_4^2 + x_1^2 + x_3^2} = \|\underline{x}\| \longrightarrow T$ is orthogonal

since T is orthogonal, $T^{-1} = T^T$

$$\therefore B^{-1} = T^{-1} = T^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

check!

$$B^{-1} \cdot B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prob 4

i) let $(1,1,3)$ and $(0,1,2)$ be two linearly independent vectors on the plane,

$\underline{u}_1 = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \qquad \underline{v}_2^{\perp} = \underline{v}_2 - (\underline{u}_1 \cdot \underline{v}_2)\underline{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{7}{11}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$

$$\hat{u}_2 = \frac{\underline{v}_2^{\perp}}{\|\underline{v}_2^{\perp}\|} = \frac{1}{\sqrt{66}}\begin{pmatrix}-7\\4\\1\end{pmatrix}$$

$\therefore \frac{1}{\sqrt{66}}\begin{pmatrix}-7\\4\\1\end{pmatrix}$ and $\frac{1}{\sqrt{11}}\begin{pmatrix}1\\1\\3\end{pmatrix}$ are the orthonormal basis of the plane $x+2y-z=0$. obviously, this choice is not unique.

ii) $\quad \underline{x} = \begin{pmatrix}1\\1\\2\end{pmatrix} \qquad Y = \mathrm{Proj}(\underline{x}) = (\underline{u}_1 \cdot \underline{x})\underline{u}_1 + (\underline{u}_2 \cdot \underline{x})\underline{u}_2$

$$Y = \frac{8}{11}\begin{pmatrix}1\\1\\3\end{pmatrix} - \frac{1}{66}\begin{pmatrix}-7\\4\\1\end{pmatrix} = \begin{pmatrix}55/66\\44/66\\143/66\end{pmatrix}$$

iii) $\quad X - Y = \begin{pmatrix}11/66\\22/66\\-11/66\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1\\2\\-1\end{pmatrix}$

From the plane eqn., the normal to the plane is, $(1,2,-1) \Longrightarrow (X-Y)$ is parallel to the normal to the plane $\Longrightarrow (X-Y)$ is orthogonal to the plane.

Prob 5

i) $T(f_1+f_2) = f_1' + f_2' - f_1(3) - f_2(3)$

$= f_1' - f_1(3) + f_2' - f_2(3) = T(f_1) + T(f_2)$

$T(kf) = (kf)' - kf(3) = k\left(f' - f(3)\right) = kT(f)$

$\longrightarrow$ T is linear

ii) let $f(t) = at^2 + bt + c$

$T(f) = f' - f(3) = 2at + b - 9a - 3b - c$

$= 2a(t) + (-9a - 2b - c)$

$$\Rightarrow T\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} -9a-2b-c \\ 2a \\ 0 \end{pmatrix}_{1,t,t^2} \quad \text{———} \circledast$$

$$\text{Img } T = (-9a-2b-c)\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{Img basis}} \longrightarrow \text{rank} = 2$$

$\circledast$ leads to $T = \begin{pmatrix} -1 & -2 & -9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}_{1,t,t^2}$

to find the Kernel of T, solve the system,

$$\begin{pmatrix} -1 & -2 & -9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow z = 0,\ y = t,\ x = -2t$$

$$\therefore \text{Kernel } T = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \text{Dim} = 1$$

$$\text{rank} + \text{nullity} = 2+1 = 3 \longrightarrow$$ same dimension of the space P_2

iv) $$T = \begin{pmatrix} -1 & -2 & -9 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$

Prob 6

i) $$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}_A$$

ii) $$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}_A$$

iii) $$a\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + b\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

iv) $$\begin{pmatrix} a \\ b \end{pmatrix}$$

v) $$S_{B\to A} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}, \text{ check, } S_{B\to A}\begin{pmatrix} a \\ b \end{pmatrix}_B = \tfrac{1}{2}\begin{pmatrix} a-b \\ a+b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}_A$$

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