HW12 Solutions

Sec 6.2 :

Prob. 2: The matrix can be easily transformed into the following form using Gaussian elimination ,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$ and its determinant equals 1.4.6=24 *No row swap or multiplication was used*

Prob. 10: After successive elimination steps, the matrix becomes,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ So, the determinant equals unity. *No row swap or multiplication was used*

Prob. 12:

$$\begin{bmatrix} V_4 \\ V_2 \\ V_3 \\ V_1 \end{bmatrix}$$ one swap → $$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$ So, del = 8 . -1 = -8

Prob. 14:

$$\begin{bmatrix} V_1 \\ V_2 + 9V_4 \\ V_3 \\ V_4 \end{bmatrix}$$ -9* IV +II @ II → $$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$ So, det 1st = det 2nd

Prob. 30:

a) F(t) = del $\begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}$ To prove that the function is quadratic, enough to show that the coefficient of t^2 is not zero. If we used the third column to evaluate the matrix, we would find that the coefficient of t^2 is (b-a). So, it's quadratic unless (b-a) vanishes.

b) If t = a or t = b , we'll have two identical columns. What is the value of this determinant?

Let's look at the matrix transpose.

It will be something like, $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & a & a^2 \end{bmatrix}$. 1st row is the same like 3rd . So, the system of equations is clearly inconsistent. The determinant is necessarily zero. Using fact 6.2.7, we prove that the f(a) = f(b) = 0

Another way to show that, is to think of the matrix that has two identical columns as if it expresses a volume in 3 dimensions. If two edges of that volume are expressed with the same vector, this means that it is a shape in two dimensions instead of three ; the volume = zero. → determinant is zero.

If f(t) = k(t-a)(t-b)→ k is the coefficient of t^2. From part (a), **k = (b-a)**

c) f(t) = k (t-a)(t-b) . Obviously, the matrix will be **noninvertible** when

Det = f(t) = 0 → t = a or t = b

Prob. 38:

$$Det\,(A^T A) = DetA^T .DetA = (DetA)^2 = 3^2 = 9$$

Sec 6.3 :

Prob. 2:

Form a matrix out of the two vectors as $\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$. The determinant of this matrix is twice the area surrounded by the two vectors. Area = Det/2 = | (6-56) |/2 = 25

Prob. 7:

Form a matrix out of each two successive pair of vectors. Four matrices are produced. Their determinants give twice of the area shaded in the graph

$$\text{Area}= \frac{1}{2}\left\{Det\begin{bmatrix} 5 & -7 \\ 5 & 7 \end{bmatrix} + Det\begin{bmatrix} -7 & -5 \\ 7 & -6 \end{bmatrix} + Det\begin{bmatrix} -5 & 3 \\ -6 & -4 \end{bmatrix} + Det\begin{bmatrix} 3 & 5 \\ -4 & 5 \end{bmatrix}\right\} = 110$$

Prob. 24:

Using Cramer's rule,

$$X = \frac{\begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}} = 1 \qquad Y = \frac{\begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}} = 2 \qquad Z = \frac{\begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}} = -1$$

Check by yourself that these values satisfy the equation system.

Prob. 24:

The classical adjoint should NOT be calculated from the inverse matrix. Actually, most of the time, we calculate the adjoint in order to obtain the inverse.

- ***How to calculate the classical adjoint :***

1) Calculate the *matrix of the minor determinants* of each element in the matrix

i.e.) from the matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ we get, $\begin{bmatrix} 18 & 12 & -2 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ ***take a look at "Minors" at page 250***

2) Multiply each element by the sign in its location from $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

i.e.) our matrix here becomes, $\begin{bmatrix} 18 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 3 \end{bmatrix}$

3) Take the Transpose,

i.e.) $\begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix}$ and that is the final result ; the classical adjoint

Apply this method to any square matrix of any size.

To check that the answer is correct, we know that,

$$A^{inverse} = \frac{A^{adjoint}}{DetA} \Rightarrow A.A^{inverse} = I = \frac{A.A^{adjoint}}{DetA} \Rightarrow A.A^{adjoint} = DetA.I \qquad I \text{ ,is the identity matrix}$$

So, you can multiply the adjoint matrix with the one you started with. The result is expected to be a matrix with equal diagonal elements and zero’s elsewhere (the identity matrix multiplied by the determinant value)