5.5

Problem 10

This integral vanishes for any $(f(t)\cdot g(t))$ = odd function

$\Rightarrow$ if $f(t) = t$ $\Longrightarrow$ $g(t)$ must contain $1, t^2$ components only

$\therefore$ in general, $\boxed{g(t) = a + bt^2}$

This is the space, but we need basis. Could it be a, bt^2?

let's check,

normalization,

$$\langle a, a\rangle = \frac{1}{2}\int_{-1}^{1} a\cdot a\,dt = 1 \Rightarrow a^2 = 1 \rightarrow a = \pm 1$$

$$\langle bt^2, bt^2\rangle = \frac{1}{2}\int_{-1}^{1} bt^2\cdot bt^2\,dt = 1 \Rightarrow \frac{1}{2}b^2\left.\frac{t^5}{5}\right|_{-1}^{1} = 1$$

$$\Rightarrow \frac{b^2}{5} = 1 \Rightarrow b = \pm\sqrt{5}$$

$\therefore$ It seems that $(\pm 1, \pm\sqrt{5}t^2)$ might work as a basis

However, they are not orthogonal to each other. Their product is even function and the integral will not vanish.

There is two ways to find orthonormal basis,

\# Method one, start with $1, t^2$. These are neither orthogonal nor parallel. Use these as starting vectors in "Gram-Schmidt" process

$\underline{u}_1 = \underline{v}_1 = 1$ $\qquad$ $\underline{v}_2^{\perp} = \underline{v}_2 - (\underline{u}_1 \cdot \underline{v}_2)\underline{u}_1$

$$= t^2 - \frac{1}{2}\int_{-1}^{1}(1 \cdot t^2)\,dt = t^2 - \frac{1}{3}$$

$$\|\underline{v}_2^{\perp}\| = \sqrt{\langle t^2 - \tfrac{1}{3}, t^2 - \tfrac{1}{3}\rangle} = \sqrt{\frac{1}{2}\int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt} = \sqrt{\frac{4}{45}}$$

$$\Rightarrow \underline{u}_2 = \frac{t^2 - 1/3}{\sqrt{4/45}}$$

$\therefore$ $\underline{u}_1, \underline{u}_2$ form the orthonormal basis

Method two,

The space is $a + bt^2$. Dimension $= 2$. Assume very general basis, $\left\{ \begin{matrix} a_1 + b_1 t^2 \\ a_2 + b_2 t^2 \end{matrix} \right\}$

we need to satisfy three conditions;

$$\left.\begin{aligned} &1)\ \frac{1}{2}\int_{-1}^{1}(a_1 + b_1 t^2)(a_1 + b_1 t^2)\,dt = 1 \\ &2)\ \frac{1}{2}\int_{-1}^{1}(a_2 + b_2 t^2)(a_2 + b_2 t^2)\,dt = 1 \end{aligned}\right] \rightarrow \text{normalization}$$

$$3)\ \frac{1}{2}\int_{-1}^{1}(a_1 + b_1 t^2)(a_2 + b_2 t^2)\,dt = 0 \longrightarrow \text{orthogonality}$$

$\Rightarrow$ Three equations in Four variables $\rightarrow$ Many solutions

And this makes sense because the basis choice is not unique.

You may choose $\boxed{a_1 = 1}$ and solve the three eqn system to find an orthonormal basis

Sec 6.1

8) $\text{Det} = 1\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = -1 + 4 - 3 = 0$

$\Rightarrow$ Matrix is noninvertible

30) $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$

$$\Rightarrow (3-\lambda)\underbrace{\left[(6-\lambda)(4-\lambda) - 8\right]}_{(\lambda^2 - 10\lambda + 16) = (\lambda-2)(\lambda-8)} = 0$$

$\Rightarrow \lambda = 2, 3, 8$

34) $$\text{Det} = -4\underbrace{\begin{bmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 1 & 8 & 2 \end{bmatrix}}_{2(24-15)} + 3\underbrace{\begin{bmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 2 & 7 & 1 \end{bmatrix}}_{1(24-15)}$$

$= -8 \times 9 + 3 \times 9 = -45$