Review Sheet

Prob 1

same det same trace

let's find a matrix that proves the similarity (S)

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+3c & b+3d \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$$

$$\therefore c = 0 \quad , d = \frac{2a}{3} \Rightarrow S = \begin{bmatrix} a & b \\ 0 & \frac{2a}{3} \end{bmatrix}$$

confirm,

$$AS = \begin{bmatrix} a & b+2a \\ 0 & \frac{2a}{3} \end{bmatrix} \quad , SB = \begin{bmatrix} a & b+2a \\ 0 & \frac{3a}{2} \end{bmatrix} \Rightarrow \text{ok.}$$

Prob2 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow (\lambda-1)(1-\lambda^2)=0 \Rightarrow \lambda = 1, \pm 1$

$E_{-1} \longrightarrow$ one dim $\quad E_{-1} = \mathrm{Ker}\begin{bmatrix} +1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & +1 \end{bmatrix} = \mathrm{Span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$E_{+1} = \mathrm{Ker}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \rightarrow 2$ dim

$\because$ A is 3×3 matrix and E_{-1}, E_{+1} add up to 3 dim

$\therefore$ A is diagonalizable

$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \mathrm{Det}\, S = 2$

$S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\therefore D = S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Prob 3

i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \longrightarrow (3-\lambda)\lambda(\lambda-3) = 0$

$\Rightarrow \lambda = 0, 3, 3$

ii) $\lambda = 0 \longrightarrow$ no free term in the characteristic eqn.

$\longrightarrow \det = 0 \longrightarrow A$ is non-invertible

iii) $\lambda = 3$ has algebric multiplicity of 2

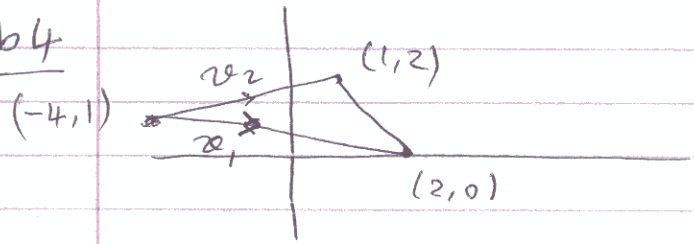
$\longrightarrow \lambda = 3$ " geometric " of 2 or 1

$\longrightarrow$ the matrix might not be diagonalizable

iv) if we have distinct λ's $\longrightarrow$ matrix is diagonalizable

Here, this is not the case $\longrightarrow$ more work is needed.

Prob 4



$v_1 = (2,0) - (-4,1) = (6,-1)$

$v_2 = (1,2) - (-4,1) = (5,1)$

$\therefore \text{Area} = \frac{1}{2} \text{Det} \begin{vmatrix} 6 & 5 \\ -1 & 1 \end{vmatrix} = \frac{1}{2}(11) = 5.5$

Prob 5

i) we have $SA = BS$

$* A^{-1} \longrightarrow SAA^{-1} = BSA^{-1} \longrightarrow S = BSA^{-1}$

$* B^{-1} \longrightarrow \underline{B^{-1}S} = B^{-1}BSA^{-1} = \underline{S\cancel{B}A^{-1}}$

$$B^{-1}S = SA^{-1}$$

$\therefore B^{-1}, A^{-1}$ are similar True

ii) False,

Consider $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

let $S = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

$$AS = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}, \quad SB = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$AS = SB \implies A, B$ are similar

iii) True Fact 7.3.6

iv) True Def. 7.4.2

v) True,

Consider, $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$

$SA = \begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix}$ $\quad BS = \begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix} \implies A, B$ are similar

Prob 6

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \longrightarrow (1-\lambda)^2 - 4 = 0 \Rightarrow \lambda = 3, -1$$

$$E_3 = \operatorname{Ker}\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$E_{-1} = \operatorname{Ker}\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \operatorname{span}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\therefore S = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \longrightarrow S^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$\therefore D = S^{-1}AS = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^{10} = S D^{10} S^{-1} = \cancel{\frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{pmatrix}}$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{pmatrix}\left(\frac{1}{4}\right)\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2\cdot 3^{10} + 2 & 4\cdot 3^{10} - 4 \\ 3^{10} - 1 & 2\cdot 3^{10} + 2 \end{pmatrix}$$

Prob 7

let $f(t) = at^2 + bt + c$

$Tf = (t+1)f'(t) = (t+1)(2at+b) = 2at^2 + (2a+b)t + b$

if the transformation result is $\lambda f \Rightarrow f$ is eigenvector

i.e., $Tf = \lambda f$

$2at^2 + (2a+b)t + b = \lambda(at^2 + bt + c)$

$= \lambda a t^2 + \lambda b t + \lambda c$

Let's study the possible solutions of this equality

Compare λ coefficients on both sides,

(*)

$2a = \lambda a \longrightarrow \lambda = 2$

$2a + b = \lambda b \longrightarrow 2a + b = 2b \longrightarrow b = 2a$

$b = \lambda c \longrightarrow c = \frac{b}{2} = a$

$\Rightarrow f(t) = at^2 + 2at + a \quad , \lambda = 2$

(*)(*) $2a = \lambda a \longrightarrow a = 0$

take $\lambda = 1 \Rightarrow b = c$

$\therefore f(t) = 0t^2 + bt + b \quad , \lambda = 1$

(*)(*)(*) $2a = \lambda a \longrightarrow a = 0$

take $\lambda = 0 \longrightarrow f(t) = 0t^2 + 0t + c \quad , \lambda = 0$

ii) Three distinct eigenvalues $\longrightarrow$ there is an eigenbasis

iii) Let's apply the matrix formalism and see if we can obtain the same results in (i),

$$T(at^2+bt+c) = 2at^2+(2a+b)t+b$$

or

$$T\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} b \\ 2a+b \\ 2a \end{pmatrix}$$

$\therefore$ T can be $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow \lambda = 0, 1, 2$

$E_0 = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad E_1 = \text{span}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$E_2 = \text{span}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Same resullts like those in (i)

Prob 8

Method 1,

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ -2x-3y \\ x+4y \end{pmatrix} \longrightarrow \underbrace{x\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}}_{2\,\text{dim}}$$

Method 2, Dusa's

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

Get the Rref form,

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 5 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4/5 & 3/5 \\ 0 & 1 & -1/5 & -2/5 \end{bmatrix}$$

take z, t as parameters,

$\therefore\ y = \frac{1}{5}z + \frac{2}{5}t \quad , \quad x = -\frac{4}{5}z - \frac{3}{5}t$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} -4/5\, z - 3/5\, t \\ 1/5\, z + 2/5\, t \\ z \\ t \end{pmatrix} \longrightarrow \underbrace{z\begin{pmatrix} -4/5 \\ 1/5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3/5 \\ 2/5 \\ 0 \\ 1 \end{pmatrix}}_{2\,\text{dim}}$$

You may easily check that the space in both methods is the same

Prob 9

i) let $T \equiv \left[\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3 \right]$ $\quad \underline{v}_3 = \underline{v}_1 + \underline{v}_2$

$\therefore$ Img basis = Span $\{ \underline{v}_1, \underline{v}_2 \}$ $\longrightarrow$ 2 dim

ii) $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$

This eqn describes a $(n-1)$ dimensional surface in n dimensional space.

Here, we have 4-dim space, but our surface is only 2 dim (since it has only two basis vectors)

$\Longrightarrow$ It's not possible to write Im(T) in this format.

Prob 10

i) $A^{-1} = \begin{bmatrix} 3 & 1/3 & -1 \\ 2 & 2/3 & -1 \\ -2 & -1/3 & 1 \end{bmatrix}$

ii) remember, $(A^T)^{-1} = (A^{-1})^T$

Prob 11

if $A = (v_1\ v_2\ v_3) \Rightarrow A^T = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$A^T A = \begin{pmatrix} v_1 v_1 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2 v_2 & v_2 v_3 \\ v_3 v_1 & v_3 v_2 & v_3 v_3 \end{pmatrix}$$

in this problem, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are mutually orthogonal

$\Rightarrow$ off diagonal elements vanish

$$\therefore A^T A = \begin{pmatrix} v_1^2 & 0 & 0 \\ 0 & v_2^2 & 0 \\ 0 & 0 & v_3^2 \end{pmatrix}$$

Prob 12

i) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$

$\Rightarrow X_3 = -1 \quad , \quad X_1 + X_2 = 3$

let $x_1 = t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 3-t \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

ii) Direction vector is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

iii) $$\underline{n}_1 \wedge \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

iv) $\therefore$ $\underline{n}_1 \wedge \underline{n}_2$ is parallel to the direction vector

This is logical since the intersection ~~point~~ line is normal to both planes

$\therefore$ Direction vector ~~is~~ $\parallel (\underline{n}_1 \wedge \underline{n}_2)$

Prob 13

i) 2-fold dilation in X-Component + a reflection in Y component

ii) SDS^{-1} has the same geometric effect but with respect to a different basis (basis described by the change of basis matrix S)

Prob 14

i) $$\text{Det} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \text{zero} \Rightarrow$$ The vectors are linearly dependent

ii) 2-dim

iii) Any two vectors form a basis.

Prob (15)

i) v_1 is parallel to the line. v_2 is perpendicular

ii) in page (216) $QQ^T = [u_1\ u_2]$

$$\therefore QQ^T = \begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

iii) $u_1 = \frac{1}{\sqrt5}\begin{bmatrix}2\\1\end{bmatrix} \Rightarrow QQ^T = \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}(2\ \ 1) = \frac{1}{5}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}$

Prob (16)

i) $\begin{pmatrix}x\\y\\z\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\\-2x\end{pmatrix} \longrightarrow x\underbrace{\begin{pmatrix}1\\0\\-2\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{basis (but it's not normalized)}}$

$$\therefore \underline{u}_1 = \frac{1}{\sqrt5}\begin{pmatrix}1\\0\\-2\end{pmatrix}, \quad \underline{u}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

ii) one vector

iii) $\underline{v}_3 = \underline{u}_1 \wedge \underline{u}_2 = \frac{1}{\sqrt5}\begin{vmatrix} i & j & k\\ 1 & 0 & -2\\ 0 & 1 & 0\end{vmatrix} = \frac{1}{\sqrt5}\begin{pmatrix}2\\0\\1\end{pmatrix} \quad \therefore \underline{u}_3 = \frac{1}{\sqrt5}\begin{pmatrix}2\\0\\1\end{pmatrix}$

(*) Another way is to use Gram-Schmidt procedure,

$$\underline{v}_3^{\perp} = \underline{v}_3 - (\underline{u}_1 \cdot \underline{v}_3)\underline{u}_1 - (\underline{u}_2 \cdot \underline{v}_3)\underline{u}_2$$

let $\underline{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \underline{v}_3^{\perp} = \frac{3}{5}\begin{pmatrix}2\\0\\1\end{pmatrix}$ normalize this, $\underline{u}_3 = \begin{pmatrix}2/\sqrt5\\0\\1/\sqrt5\end{pmatrix} = \frac{1}{\sqrt5}\begin{pmatrix}2\\0\\1\end{pmatrix}$

iv) notice that $(2,0,1)$ is normal to the plane,

$\Rightarrow$ it has only one coordinate in the perpendicular direction (i.e. $\underline{u}_3$). And the coordinate value is the vector length $(\sqrt5)$