Extra HW Solutions

Prob. 1:

i) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. So the basis is , $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. The choice of the basis is not unique but it will be always 2 dimensional

ii) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ Hence, the vector is expressed as $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Prob. 2:

Testing the linearity,

i)

$$T(aM_1 + bM_2) = aM_1 + bM_2 - I$$

$$aT(M_1) + bT(M_2) = aM_1 - aI + bM_2 - bI$$

They are not equal → not linear

ii) $T(aM_1 + bM_2) = aM_1J + bM_2J = aT(M_1) + bT(M_2)$ → linear transformation

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $T(M) = \begin{bmatrix} 0 & a+2b \\ 0 & c+2d \end{bmatrix} = \begin{bmatrix} 0 & x_1 \\ 0 & x_2 \end{bmatrix}$. This is the image

The rank is 2 since we have two basis matrices, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Prob. 3:

i) $T(t^2) = (1-t)2t = 2t - 2t^2$

ii) $T(1) = (1-t)(0) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad T(t) = (1-t) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\qquad T(t^2) = 2t - 2t^2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

$$\Rightarrow TransformationMatrix = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Prob. 4:

i) Check the book.

ii) The normalization condition ; $c^2 + k^2 = 1$

The orthogonality condition ; $-\frac{c}{\sqrt{5}} + \frac{2k}{\sqrt{5}} = 0 \Rightarrow k = \frac{c}{2}$

$c^2 + \frac{c^2}{4} = 1 \Rightarrow c = \pm\frac{2}{\sqrt{5}} and, k = \pm\frac{1}{\sqrt{5}}$

Therefore, the matrix A could be either , $\begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} or \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$

Prob. 5:

a) $v_1.v_2 = 2+1-3 = 0 \Rightarrow orthogonal$

b) The vectors are orthogonal already → enough to normalize them ,

$$u_1 = \frac{(2,1,3)}{\sqrt{14}} and, u_2 = \frac{(1,1,-1)}{\sqrt{3}}$$

c)

$$Orthogonalprojection = (u_1.x)u_1 + (u_2.x)u_2$$

$$= (\frac{(2,1,3)}{\sqrt{14}}.\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})\frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + (\frac{(1,1,-1)}{\sqrt{3}}.\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{4}{14}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 0 = \frac{1}{7}\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

Notice that the vector (0,1,1) is orthogonal to one of the basis vectors (v2), therefore, the projection has no component in this direction

$$\|x\|^2 = 2$$

$$\|y\|^2 = \frac{56}{49}$$

d) $x - y = (0,1,1) - \frac{1}{7}(4,3,6) = \frac{1}{7}(-4,5,1)$

$$\|x - y\|^2 = \frac{16+25+1}{49} = \frac{42}{49}$$

$$\|y\|^2 + \|x - y\|^2 = \frac{56}{49} + \frac{42}{49} = \frac{98}{49} = 2 = \|x\|^2$$